Exercises
26 September 2018

1. For each of the following sets, tell whether they are closed or not, convex or not

(a) \((-\infty, 7]\). Closed and convex
(b) \((-5, 7]\). Not closed and convex
(c) \([0, 3]\). Closed and convex
(d) \((-2, 23]\). Not closed and convex
(e) \([4, +\infty)\). Closed and convex
(f) \([0, 3] \cup [5, 10]\). Closed and not convex
(g) \([0, 3] \cup [2, 10]\). Closed and convex
(h) The empty set (tricky!). Closed and convex (The empty set does satisfy the definitions since there are no points in this set. Indeed, consider closedness; a set is closed if any limit point of a sequence of points in the set also belongs to the set. Since there are no such sequences in the case of the empty set, the statement is formally true and the set is closed. If you prefer, you can take the fact the empty set is closed and convex as a convention).
(i) The non-negative orthant: \(\mathbb{R}_+ = \{x \in \mathbb{R}^n : x \geq 0\}\) (When \(x\) is a vector and we write \(x \geq 0\) we mean that every component of the vector greater or equal to 0; similarly for \(>, \leq, <, \leq\); note also that in Italian we use the word "quadrante" in \(\mathbb{R}^2\) and "ortante" in \(\mathbb{R}^n\) with \(n > 2\)). Closed and convex
(j) The positive orthant \(\mathbb{R}_{++} = \{x \in \mathbb{R}^n : x > 0\}\). Not closed and convex
(k) The line, in \(\mathbb{R}^2\), \(x-y=0\) (to get geometric feel in this and in the following questions up to (s), draw a picture). Closed and convex
(l) The set \(\{(x, y) : y \leq x^2\}\). Closed and not convex
(m) The set \(\{(x, y) : y = x^2\}\). Closed and not convex
(n) The set \(\{(x, y) : y \geq x^2\}\). Closed and convex
(o) The set \(\{(x, y) : x^2 + y^2 < 1\}\). Not closed and convex
(p) The set \(\{(x, y) : x^2 + y^2 \geq 1\}\). Closed and not convex
(q) A triangle (including all the area inside the triangle) Closed and convex
(r) A triangle without its three vertices. Not closed and convex
(s) A triangle without its barycenter. Not closed and not convex

Remarks.

- In \(\mathbb{R}\) the only convex sets are intervals, of any kind, therefore all examples in (a)-(e) are convex.
- The intersection of two closed sets is always closed, the intersection of two convex sets is always convex. For example, the non-negative orthant considered in example (i) can be seen as the intersection of the \(n\) half-spaces defined by \(x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0\) and since these half-spaces are closed and convex, their intersection is also closed and convex.
- Sets are usually defined analytically as the solutions of an equality or an inequality. For example \(\{(x, y) : x - y = 0\}\) or, for short, \(x - y = 0\) is the bisector of the first and third orthant; \(x \leq 1\) is a closed half-plane and \(x - y < -1\) is an open half-plane (draw them!). The intersection of two sets defined by \(g_1(x) \leq 0\) and \(g_2(x) \leq 0\) is the set of points that satisfies both inequalities at the same time. More in general the solution of a system of equalities or inequalities is, geometrically, the intersection of the sets defined by the single equalities and inequalities.
• If a set is defined by \( g(x) \leq 0 \), \( g(x) = 0 \), or \( g(x) \geq 0 \) and \( g \) is continuous the set is closed. A set defined by \( g(x) < 0 \), or \( g(x) > 0 \) is not necessarily closed (actually, most of the times it is open: can you give an example of a set defined by \( g(x) < 0 \) with \( g \) continuous that is open and one example where the set is closed?)

• If the function \( g(x) \) is convex, the set defined by \( g(x) \leq 0 \) is convex. Note that in general, unless \( g(x) \) is linear, the set defined by \( g(x) \geq 0 \) or \( g(x) = 0 \), with \( g \) convex, is not convex.

• In general the union of two closed sets is closed, but the union of two convex sets is not convex in general, see example (f). There are however particular cases, for example in (g) the two convex intervals are overlapping and so their union is \([0, 10]\) which, being an interval, is convex.

• Remember that open is not the opposite of closed when talking about sets! That is, if a set is not closed, it is not necessarily open, although it can be open. For example \((-\infty, 7]\) is neither closed nor open, while \((-5, 7)\) is not closed and open.

2. Give the definition of variational inequality \( VI(K, F) \) and of solution of a \( VI(K, F) \) (without looking at your notes!).

A variational inequality problem \( VI(K, F) \) is defined once a closed, convex set \( K \subseteq \mathbb{R}^n \) is given together with a function \( F : K \to \mathbb{R}^n \). The set \( K \) is termed feasible set and note that the function \( F \) need not be defined outside \( K \). A solution of the \( VI(K, F) \) is a feasible point \( \bar{x} \in K \) such that

\[
F(\bar{x})^T(y - \bar{x}) \geq 0, \quad \forall y \in K
\]

In geometric terms this means that the vector \( F(\bar{x}) \) must form an angle that is less or equal to \( 90^\circ \) with all vectors of the form \( y - \bar{x} \).

3. Consider the following \( VI(K, F) \) with

\[
F = \begin{pmatrix} 3x - 2y - 2 \\ x + y - 2 \end{pmatrix}, \quad K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, x + y \geq 0, x \leq 1\}
\]

Check graphically which of the following points are solutions: \((0, 0)\), \((1, -1)\), \((1, 1)\), \((0, 1)\)

\((0, 0)\) is not a solution.
\((1, -1)\) is not a solution.
\((1, 1)\) is a solution.
\((0, 1)\) is not a solution.

4. Consider the sets

• \( K_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \),

• \( K_2 = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\} \),

• \( K_3 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\} \),

• \( K_4 = \{(x, y) \in \mathbb{R}^2 : x = 0\} \),

and the constant function \( F(x, y) = (1, 1)^T \). For the four variational inequalities \( VI(K_i, F) \), \( i = 1, 2, 3, 4 \) find the solution set \( SOL(K_i, F) \) (by \( SOL(K, F) \) we denote the solution set of the \( VI(K, F) \)).

\[
SOL(K_1, F) = \{(-1/\sqrt{2}, -1/\sqrt{2})^T\}
\]

\[
SOL(K_2, F) = \{(0, 0)^T\}
\]

\[
SOL(K_3, F) = K_3 \text{ (all feasible points are solutions)}
\]

\[
SOL(K_4, F) = \emptyset \text{ (there are no solutions)}
\]

Remark. The above examples clearly show that a VI can have no solutions, one solutions or more than one solution.
Exercises
3 October 2018

1. Consider the VI\((K, F)\), where

\[
K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0\}, \quad F = \begin{pmatrix} x^2 + y^2 + 1 \\ xy - 1 \end{pmatrix}
\]

Determine SOL\((K, F)\) by inspection.

The feasible region \(K\) is depicted in the figure below.

1. In the interior of this region a point is a solution if the value of \(F\) is \((0, 0)^T\). But this cannot happen because the first component of \(F\) is \(x^2 + y^2 + 1\) and is always greater or equal to 1. So there are no solutions in the interior of \(K\).

2. Point A is a solution if the value of \(F\) has both components non negative. But in A the second component of \(F\) is -1 and therefore A is not a solution.

3. On the part of the y axis between A and B (extremes excluded) a point is a solution if and only if the first component is non negative and the second one is 0. But since for any point in this segment \(x = 0\), it is readily seen that the second component of \(F\) is -1 and therefore there are no solutions on this segment.

4. Point B is a solution if in that point the first component of \(F\) is non negative and the second non positive. Since \(F(B) = (2, -1)^T\), \(B = (0, 1)^T\) is a solution.

5. If a point P in the half circle that has extremes A and B (A and B excluded) is a solution it is readily seen that the first component of \(F\) must be negative. Since the first component of \(F\) is always greater or equal to 1, there are no solutions on this half circle.

We conclude that the only solution of this VI is the point \((0, 1)^T\).

2. Give the definition of Nash Equilibrium Problem (NEP) and Nash Equilibrium point. Show that under some adequate conditions an NEP is equivalent to a VI.

Check the Lecture Notes.
3. Consider the following Nash Equilibrium Problem, where the first player controls $x$ and $y$ while the second player controls $z$.

$$\min_{x,y} \quad x^4 + 2x^2 + xy + y^2 - xz + yz$$
$$x \geq 0, \quad y \geq 0$$
$$x + y \leq 1$$

$$\min_{z} \quad \frac{1}{4}(z + x)^2 + yz$$
$$1 \leq z \leq 2$$

Is it possible to rewrite this problem as a VI? If the answer is yes, write the corresponding VI.

In order to say whether this Game can be reformulated as a VI, we must check whether three conditions are satisfied: 1. The feasible sets of the two problems must be closed and convex; 2. The objective functions must be $C^1$ and 3. Each objective function, for fixed values of the other player variables, must be convex. Let’s check.

1. The two feasible regions are defined by linear (non strict) inequalities, therefore they are polyhedra, that are certainly convex and closed.
2. The two objective functions are polynomials and therefore certainly $C^1$.
3. Consider the first objective function. Since $x$ must be positive (see the constraints for the second player), the term $x^4$ is obviously convex. The term $-xz + yz$ is linear in $x$ and $y$ for fixed $z$ and therefore convex. So we are left with the term $2x^2 + xy + y^2$. This is a quadratic function in $x$ and $y$ and its Jacobian is

$$\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

This matrix is easily seen to be positive definite (the two north-west minors are 4 and 7) and therefore also this term is convex (and actually strictly convex, can you tell why?)

Let’s now look at the second objective function. The term $yz$ is linear in $x$ for fixed $y$ and therefore convex. The term $\frac{1}{4}(z + x)^2$ is also obviously convex for any fixed value of $x$ (it is a parabola with upward concavity) and therefore also the second objective function is convex for any given values of $x$ and $y$.

We can conclude that this Game can be converted to a VI $(K, F)$, with

$$K = \{(x, y, z) : x \geq 0, \quad y \geq 0, \quad x + y \leq 1, \quad 1 \leq z \leq 2\}$$

$$F = \begin{pmatrix} 4x^2z + 4x + y - z \\ x + 2y + z \\ x + y + z \end{pmatrix}$$

**Remark.** In this example we checked condition 3 about convexity by checking that the objective functions are the sum of convex functions. Since it is well-known and easy to verify that the sum of convex functions is convex, this gives us the desired convexity. But please, note that the vice versa is not true in general, i.e. a function can be convex being the sum of functions that are not all convex. As a simple example consider $f(x) = -x^2 + 2x^2$. This function is the sum of two terms, $-x^2$ and $2x^2$, with the first, $-x^2$, clearly non convex. But this function, being equal to $x^2$ is actually convex! Therefore always remember that if a function is the sum of convex functions then it is convex, but if a function is the sum of some functions, with some of them being non convex, we cannot immediately conclude form this fact that the function is non convex, we have to check some other way.
4. Consider two firms that produce the same good. Each firm must decide the amount of good (x for the first firm, y for the second) to produce, and each of them can produce at most 70 tons. The firms know that whatever they produce will be sold, but of course the price will depend on the total amount in the market (i.e. on the total amount of good produced by the two firms together). The inverse demand law, giving the price in function of the quantity is \( p = 150 - (x + y) \). Producing has costs and for the first firm the cost of producing \( x \) is \( 2x + x^2 \) while for the second firm it is \( y + 3y^2 \). Of course, the aim of the two firms is to minimize costs or, equivalently, to maximize revenues.

(a) Formulate this Nash-Cournot equilibrium problem, specifying the optimization problems of the two players.

(b) Show that the equilibrium problem can be reformulated as a VI and derive this VI, specifying the set \( K \) and the function \( F \).

The two firms want two maximize their revenues and their problems are therefore

\[
\begin{align*}
\max_x \ & (150 - x - y)x - 2x - x^2 \\
0 \leq x & \leq 70 \\
\max_y \ & (150 - x - y)y - x - 3y^2 \\
0 \leq y & \leq 70
\end{align*}
\]

Let us check whether we can reduce this game to a VI. We proceed similarly to what done in the previous exercise. The feasible regions are clearly closed and convex and the two objective functions are obviously continuously differentiable. Before checking the convexity requirement, we note that the conditions for convergence of a game to a VI where given for problems where the players minimize a function, while here we have maximization. We then have to rewrite the problem in minimization form, which can readily be done by changing the sign to the objective functions in the optimization problems:

\[
\begin{align*}
\min_x \ & (-150 + x + y)x + 2x + x^2 \\
0 \leq x & \leq 70 \\
\min_y \ & (-150 + x + y)y + y + 3y^2 \\
0 \leq y & \leq 70
\end{align*}
\]

It is really trivial to check now that these two functions are convex in the players' variables once the other players' variables are fixed. Note that in order to check the convexity of the feasible regions or the differentiability of the objective functions there was no need to change sign to the objective functions, but for the convexity requirement the change of sign is essential.

It is now easy to give the VI \((K, F)\) equivalent to this game, and this is obtained by setting

\[
K = \{(x, y) : 0 \leq x \leq 70, \ 0 \leq y \leq 70\}, \quad F = \begin{pmatrix} 6x + y - 148 \\ 3y + x - 149 \end{pmatrix}.
\]

**Remark.** It is interesting to note that in this example the Jacobian of \( F \) is

\[
JF = \begin{pmatrix} 6 & 1 \\ 1 & 8 \end{pmatrix}
\]

and is symmetric. This means that the VI\((K, F)\) equivalent to the game is actually in turn equivalent to an optimization problem. Since it is easy to check that \( F \) is equal to the gradient of

\[
f = \frac{1}{2}(x, y) \begin{pmatrix} 12 & 1 \\ 1 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (-148, -149) \begin{pmatrix} x \\ y \end{pmatrix},
\]

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it turns out that we can compute the Nash equilibrium of the Nash-Cournot problem considered in this exercise by solving the minimization problem

\[
\min_{x,y} f(x,y) \\
0 \leq x \leq 70 \\
0 \leq y \leq 70
\]

Note that this problem is coercive and strictly convex (both properties derive from the fact that the Hessian of \( f \), coinciding with the Jacobian of \( F \), is positive definite, as we showed above) and therefore has one and only one solution that we can easily compute using standard optimization techniques. Since the solutions of this optimization problem coincide the solutions of the game, we can conclude that the game considered in this exercise has one and only one Nash equilibrium that can be computed using standard optimization techniques.

5. Define what an NCP(\( F \)) is. Find out which among the following points

\[
x^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

is a solution of the NCP(\( F \)) with

\[
F = \begin{pmatrix}
e^{x_1^2 - x_2^2 + x_3^2} - 1 \\
e^{x_1 + x_2 + x_3} - 1 \\
x_1 + x_2 + x_3
\end{pmatrix}
\]

What is the VI equivalent to this NCP(\( F \))? For the definition see the Lecture Notes. Let's consider the three points. \( x^2 \) has a negative component, so, being infeasible, cannot be a solution of NCP(\( F \)). We have

\[
x^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad F(x^1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \geq 0.
\]

Since complementarity is obviously satisfied, \( x^1 \) is a solution. We also have

\[
x^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad F(x^3) = \begin{pmatrix} e - 1 \\ e - 1 \\ 1 \end{pmatrix} \geq 0.
\]

We see that \( F(x^3) \geq 0 \), but we do not have complementarity satisfied for the first component since both the first component of \( x^3 \) and of \( F(x^3) \) are positive and therefore \( x^3 \) is not a solution.

6. Consider the NCP(\( F \)), with

\[
F = \begin{pmatrix} 4x^2 + y - 1 \\ x(1 + y) \end{pmatrix}.
\]
Compute by inspection all solutions of this NCP.

We can partition the feasible region, which is \( \mathbb{R}_+^2 \), in 4 regions.

1. The interior of the non negative orthant, i.e. \( \mathbb{R}_+^2 \), i.e. the set of points with both components positive. In this region a solution is a point where both components of \( F \) are zero, but it is easy to see that the second component of \( F \), \( x(1+y) \), is never 0 if \( x > 0 \) and \( y > 0 \). Therefore there are no solution in \( \mathbb{R}_+^2 \).

2. The second region we consider is the positive part of the \( x \) axis, i.e. points of the type \((x,0), \) with \( x > 0 \). If such a point is a solution the first component of \( F \) must be 0 and the second one greater or equal to 0. We have \( F(x,0) = (4x^2 - 1, x)^T \). If the first component must be zero, we have \( 4x^2 - 1 = 0 \) which gives \( x = \pm \frac{1}{2} \). Disregarding the negative \( x \), since \( x > 0 \) must hold, the only candidate to be a solution in this region is \( (\frac{1}{2}, 0)^T \). Since we have \( F(\frac{1}{2}, 0) = (0, \frac{1}{2})^T \), i.e. the second component of \( f \) is positive, we have that \( (\frac{1}{2}, 0)^T \) is indeed a solution.

3. The third region we consider is the positive part of the \( y \) axis, i.e. points of the type \((0,y), \) with \( y > 0 \). If such a point is a solution the first component of \( F \) must be non negative and the second one 0. We have \( F(0,y) = (y - 1, 0)^T \). Therefore it is clear that all points \( (0,y)^T \) with \( y \geq 0 \) are solutions.

4. The last region we consider is the origin. If the origin is a solution, the two components of \( F \) must be non negative. It is immediate to check that in the origin the first component of \( F \) is -1 and therefore the origin is not a solution.

7. Consider the NCP(\( F \)) defined by

\[
   f = \begin{pmatrix}
      3xy + \alpha y \\
      x^2 - y + \alpha
   \end{pmatrix}
\]

where \( \alpha \) is a parameter. For each of the following points, determine the values of \( \alpha \) (if any) for which the given point is a solution of NCP(\( F \)):

\[
   (a) \begin{pmatrix}
      0 \\
      0
   \end{pmatrix}, \quad (b) \begin{pmatrix}
      1 \\
      0
   \end{pmatrix}, \quad (c) \begin{pmatrix}
      0 \\
      1
   \end{pmatrix}, \quad (d) \begin{pmatrix}
      1 \\
      1
   \end{pmatrix}.
\]

(a) This point is a solution if \( F \) has both components that are non negative. We have \( F(0,0) = (0,0)^T \) a therefore \( (0,0)^T \) is a solution for all \( \alpha \geq 0 \).

(b) This point is a solution if the first component of \( F \) is 0 and the second is non negative. We have \( F(1,0) = (0,1 + \alpha)^T \) and therefore \( (0,0)^T \) is a solution for all \( \alpha \geq -1 \).

(c) This point is a solution if the first component of \( F \) is non negative and the second is 0. We have \( F(0,1) = (\alpha, -1 + \alpha)^T \) and therefore \( (0,1)^T \) is a solution for \( \alpha = 1 \).

(d) This point is a solution if both components of \( F \) are zero. Since we have \( F(1,1) = (3 + \alpha, \alpha)^T \) it is clear that there is no value of \( \alpha \) that makes both components of \( F \) 0 at the same time and therefore \( (1,1)^T \) never is a solution.
8. Consider the optimization problem

\[
\min \quad (x_1, x_2, x_3) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (1, 0, 3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

\[x \geq 0.\]

Is this problem convex? Is it strictly convex? Write down the minimum principle for this problem; show it is equivalent to an NCP and write down the corresponding \(F\).

The constraints of this function are linear and therefore convex (the feasible region is the non-negative orthant). The objective function is quadratic and therefore this problem is convex if the Hessian of the objective functions (which is constant) positive semidefinite and strictly convex if the Hessian is positive definite. We have

\[
Jf = \begin{pmatrix} 8 & -2 & 0 \\ -2 & 10 & 2 \\ 0 & 2 & 6 \end{pmatrix}
\]

The northwest principal minors are 8, 76 and 424 and therefore this matrix is positive definite and the function (and therefore the problem) strictly convex.

The minimum principle for this problem is

\[
\begin{pmatrix} 8 & -2 & 0 \\ -2 & 10 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \\ y_3 - x_3 \end{pmatrix} \geq 0, \quad \forall y_1, y_2, y_3 \geq 0.
\]

Of course, this is a VI where the feasible set is \(\mathbb{R}^3_+\) and therefore the minimum principle is equivalent to NCP\((F)\), where \(F\) is the gradient of \(f\) and is given in the displayed formula above.
1. Consider the following Nash equilibrium problem where the first player controls the variable $x$, the second the variable $y$ and $\alpha$ is a parameter,

$$\min_{x,y} \alpha x^2 + xy \quad \min_y (y - x)^2 + \alpha y$$

Note that the players' problems are unconstrained.

(i) For which values of the parameter $\alpha$ can we transform this game into a VI?

(ii) For the values of $\alpha$ determined in the previous point, write down the VI equivalent to the game.

(iii) For the values of $\alpha$ determined in (i) compute the Nash equilibria (this can be done in several ways, pick up your favourite method....)

(i) As usual, we have to check that (a) the feasible regions are closed and convex, (b) the objective functions are continuously differentiable and (c) for any player and for any given feasible values of the other player's variables the objective function of the player is convex. (a) is clearly satisfied because, being unconstrained, the feasible regions of both players are the whole space, which is certainly closed and convex. (b) is also clearly satisfied because the objective functions are polynomials. Regarding (c) we may observe that $(y - x)^2 + \alpha y$ is convex in $y$ for any given $x$ and $\alpha$. Indeed, the term $(y - x)^2$ is an upward parabola whatever $x$ and $\alpha y$ is a linear, and therefore convex, term for any value of $\alpha$. The situation is different for the first player objective function $\alpha x^2 + xy$. In fact, for any given $y$ the term $xy$ is linear in $x$ but the term $\alpha x^2$ is convex if an only if $\alpha \geq 0$. From another point of view, for any given $y \alpha x^2 + xy$ is an upward parabola in $x$ for any positive $\alpha$, it is a line for $\alpha = 0$ and it is a downward parabola if $\alpha$ is negative. We therefore see that this objective function is convex if and only if $\alpha \geq 0$. As a consequence the conditions to transform this game in a VI are met if and only if $\alpha \geq 0$.

(ii) VI $(K, F)$, with

$$K = \mathbb{R}^2, \quad F(x, y) = \left( \begin{array}{c} 2\alpha x + y \\ 2(y - x) + \alpha \end{array} \right)$$

(iii) Possibly the easiest way to compute the solutions is to solve the VI $(\mathbb{R}^2, F)$. Since the feasible set is the whole $\mathbb{R}^2$, this VI is equivalent to solving the equations $F(x, y) = 0$ which is a simple linear system with two equations and two variables, with $\alpha$ a parameter. Its direct solutions gives

$$x = \frac{\alpha}{2(2\alpha + 1)}, \quad y = -\frac{\alpha^2}{(2\alpha + 1)}$$
2. Define what an NCP$(F)$ is. Check which of the following points is a solution:

\[ x^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]

when

\[ F = \begin{pmatrix} e^{x_1^2 - x_2^2 + x_3^2} - 1 \\ e^{x_1 + x_2 + x_3} - 1 \\ x_1 + x_2 + x_3 \end{pmatrix} \]

What is the VI equivalent to this NCP$(F)$?

A solution of NCP$(F)$ is a point $x$ such that $0 \leq x \perp F(x) \geq 0$.

Point $x^2$ is not feasible, since the second component is negative. We have

\[ F(x^1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad F(x^2) = \begin{pmatrix} e - 1 \\ e - 1 \\ 1 \end{pmatrix}. \]

Both $F(x^1)$ and $F(x^2)$ are non-negative and therefore satisfy the condition $F(x) \geq 0$. However $(x^1)^T F(x^1) = 0$ while $(x^3)^T F(x^3) = e - 1 > 0$. Therefore we see that the only solution among the three points is $x^1$.

The VI equivalent to the NCP$(F)$ is, in this case, VI $(\mathbb{R}^2, F)$.

3. Consider the following NCP $(F)$:

\[ F = \begin{pmatrix} -x^2 + y + 1/2 \\ 2x - y + 1/2 \end{pmatrix} \]

(i) Explain why $(0, 0)$ is a solution

(ii) Check that the point $(0, 0)$ satisfies the KKT conditions

(iii) Check that the point $(1/2, 0)$ is not a solution.

(i) $(0, 0)$ is obviously feasible and $F(0, 0) = (1/2, 1/2)^T$. Since this vector has positive components and obviously $(0, 0)(1/2, 1/2)^T = 0$, $(0, 0)$ clearly is a solution.

(ii) The KKT are the KKT of the VI $(\mathbb{R}^2, F)$ and read

\[ \begin{pmatrix} -x^2 + y + 1/2 \\ 2x - y + 1/2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \lambda_2 = 0 \]

\[ 0 \leq \lambda_1 \perp x_1 \geq 0 \]

\[ 0 \leq \lambda_2 \perp x_2 \geq 0 \]

It can easily be checked that $(0, 0)$ satisfies this KKT system with $\lambda_1 = 0 = \lambda_2$. 

2
(iii) The point \((1/2, 0)\) is feasible and \(F(1/2, 0) = (1/4, 3/2)^T \geq 0\). However complementarity is not satisfied because the first component of \((1/2, 0)\) and \(F(1/2, 0)\) are both positive and we therefore have \((1/2, 0)(1/4, 3/2)^T = 1/8 \neq 0\). We can then conclude that \((1/2, 0)\) is not a solution. We could also use the KKT to check this fact. Indeed, the VI \((\mathbb{R}^2, F)\) equivalent to the NCP\((F)\) we are analyzing has linear constraints and therefore a point is a solution of NCP\((F)\) if and only if \((1/2, 0)\) solves the KKT system in (ii) for some suitable \(\lambda_1\) and \(\lambda_2\).

Writing this system in \((1/2, 0)\) we get

\[
\begin{pmatrix}
1/4 \\
3/2
\end{pmatrix} + \begin{pmatrix}
-1 \\
0
\end{pmatrix} \lambda_1 + \begin{pmatrix}
0 \\
-1
\end{pmatrix} \lambda_2 = 0
\]

\[
0 \leq \lambda_1 \perp 1/2 \geq 0 \implies \lambda_1 = 0
\]

\[
0 \leq \lambda_2 \perp 0 \geq 0 \implies \lambda_2 \geq 0
\]

It is then clear that since \(\lambda_1 = 0\) the first line of the KKT system can never be satisfied and the point is not a KKT point, thus confirming that \((1/2, 0)\) is not a solution of NCP\((F)\).

4. Consider the following Mixed Complementarity Problem

\[
F(x, y, z) = \begin{pmatrix}
3xy - z - 1 \\
x + 2y + x^2 \\
2xy^2z
\end{pmatrix}, \quad K = \{(x, y, z) : x \geq 0, y \geq 0\}
\]

(i) Check whether any of the following points is a solution: (a) \((0,0,0)\); (b) \((1,1,1)\); (c) \((1, 0, -1)\).

(ii) Write down the KKT system for this MiCP

(i) The first two variables are those constrained to be non negative, while the last is free. It is then clear that the three points are feasible. Let’s check them one by one, recalling that a point \((x, y, z)\) is a solution if

\[
0 \leq x \perp F_1(x, y, z) \geq 0
\]

\[
0 \leq y \perp F_2(x, y, z) \geq 0
\]

\[
F_3(x, y, z) = 0.
\]

\(F(0, 0, 0) = (-1, 0, 0)\), since \(F_1(0, 0, 0) = -1 < 0\), the point \((0,0,0)\) is not a solution.

\(F(1, 1, 1) = (1, 4, 2)\), since, for example, \(F_3(1, 1, 1) = 2 \neq 0\), the point \((1,1,1)\) is not a solution.

\(F(1, 0, -1) = (0, 2, 0)\). Since all the conditions listed above are satisfied, this is a solution.

(ii) This MiCP is equivalent to the VI\((\mathbb{R}_+^2 \times \mathbb{R}, F)\), where the feasible set is \(K = \{(x, y, z) : x \geq 0, y \geq 0\}\). The KKT system for this VI is

\[
3xy - z - \lambda_1 = 0
\]

\[
x + 2y + x^2 - \lambda_2 = 0
\]

\[
2xy^2z = 0
\]

\[
0 \leq \lambda_1 \perp x \leq 0
\]

\[
0 \leq \lambda_2 \perp y \leq 0
\]
5. Consider the VI \((K, F)\) where

\[
F = \begin{pmatrix} 4x - y - 5 \\ 2x + y \end{pmatrix},
\]

\[K = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ x + y \leq 1\}\]

(i) Write down the KKT conditions
(ii) A point that satisfies the KKT conditions is surely a solution of the VI for this problem?
(iii) A solution of the VI surely satisfies the KKT conditions for this problem?
(iv) Which, among the points \((0,0), (1,0) \in (0,1)\), satisfies the KKT conditions?

(i) The KKT conditions are

\[
\begin{align*}
4x - y - 5 - \lambda_1 + \lambda_3 &= 0 \\
2x + y - \lambda_2 + \lambda_3 &= 0 \\
0 &\leq \lambda_1 \perp x \leq 0 \\
0 &\leq \lambda_2 \perp y \leq 0 \\
0 &\leq \lambda_3 \perp x + y - 1 \leq 0.
\end{align*}
\]

(ii) Yes, this is always true, there is a theorem stating so, see the Lectures Notes.
(iii) For this to be surely true a CQ must be satisfied. In this case it is easy to observe, for example, that all constraints are linear (CQ) and therefore every solution of the VI does satisfy the KKT conditions.
(iv) The point \((0,0)\) is feasible and \(F(0,0) = (-5,0)\). In \((0,0)\) the first two constraints are active, while the third is not, thus implying that \(\lambda_3 = 0\). Taking all this into account, the KKT systems in \((0,0)\) boils down to

\[
\begin{align*}
-5 - \lambda_1 &= 0 \\
-\lambda_2 &= 0 \\
0 &\leq \lambda_1 \\
0 &\leq \lambda_2.
\end{align*}
\]

Since from the first equation we get \(\lambda_1 = -5 < 0\) it is clear that \((0,0)\) does not satisfy the KKT conditions and therefore it is not a solution of the VI (see point (iii)).

The point \((1,0)\) is feasible and \(F(1,0) = (-1,2)\). In \((1,0)\) the last two constraints are active, while the first one is not, thus implying that \(\lambda_1 = 0\). Taking all this into account, the KKT systems in \((0,0)\) boils down to

\[
\begin{align*}
-1 + \lambda_3 &= 0 \\
2 - \lambda_2 + \lambda_3 &= 0 \\
0 &\leq \lambda_2 \\
0 &\leq \lambda_3.
\end{align*}
\]
From the first equation we get $\lambda_3 = 1$ and therefore $\lambda_2 = 3$ from the second. It is then clear that $(1,0)$, with $(\lambda_1, \lambda_2, \lambda_3) = (0, 3, 1)$, satisfies the KKT conditions and therefore it is a solution of the VI (see point (ii)).

The point $(0,1)$ is feasible and $F(0,1) = (-6, 1)$. In $(0,1)$ the first and third constraints are active, while the second one is not, thus implying that $\lambda_2 = 0$. Taking all this into account, the KKT systems in $(0,0)$ boils down to

$$
-6 - \lambda_1 + \lambda_3 = 0 \\
1 + \lambda_3 = 0 \\
0 \leq \lambda_1 \\
0 \leq \lambda_3.
$$

Since from the second equation we get $\lambda_3 = -1 < 0$ it is clear that $(0,1)$ does not satisfy the KKT conditions and therefore it is not a solution of the VI (see point (iii)).

6. Consider the a MiNCP in 3 variables, in which the first variable, $x$, is free, while the second and third ones, $y$ e $z$, are non negative and the function $F$ is

$$
F = \begin{pmatrix}
x^2 + yz - 1 \\
x y + e^y + x^2 + \alpha \\
y(x + y - z)
\end{pmatrix},
$$

Determine for which values of the parameter $\alpha$ the point $(-1, 0, 1)$ is a solution.

A point $(x, y, z)$ is a solution if

$$
F_1(x, y, z) = 0 \\
0 \leq y \perp F_2(x, y, z) \geq 0 \\
0 \leq z \perp F_3(x, y, z) \geq 0.
$$

The point $(-1, 0, 1)$ is feasible and we have $F(-1,0,1) = (0, \alpha, 0)$. It is then clear the the conditions in the first and third line above are satisfied, and that the conditions in the second line, in particular $F(-1,0,1) \geq 0$ is satisfied if and only if $\alpha \geq 0$. We then conclude that the point $(-1,0,1)$ is a solution if and only if $\alpha \geq 0$.

7. Consider the following VI $(K, F)$ dove

$$
F = \begin{pmatrix}
x^2 y + z - 4 \\
x + 2y - z - 4 \\
x^2 - xyz - 6
\end{pmatrix}, \quad K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 3z^2 \leq 5, x, y, z \geq 0\}
$$

(i) Write down the KKT conditions for this VI

(ii) Can we be sure that at a solution of the VI the KKT are satisfied? Why?
(iii) Using the KKT conditions, check whether any of the following points is a solution: (a) (0,0,0); (b) (1,1,1); (c) (2,2,0).

(i) The KKT system for this VI is

\[
\begin{align*}
x^2y + x - 4 + 2x\lambda_1 - \lambda_2 &= 0 \\
x + 2y - z - 4 + 2y\lambda_1 - \lambda_3 &= 0 \\
x^2 - xyz - 6 + 6x\lambda_1 - \lambda_4 &= 0 \\
0 &\leq \lambda_1 \perp x^2 + y^2 + 3z^2 - 5 \leq 0 \\
0 &\leq \lambda_2 \perp x \geq 0 \\
0 &\leq \lambda_3 \perp y \geq 0 \\
0 &\leq \lambda_4 \perp z \geq 0
\end{align*}
\]

(ii) In order to be sure that at a solution of the VI the KKT system is satisfied we must that some CQ is satisfied. It is easy to see, in this case, that Slater's condition holds. In fact the constraints are all convex and the point (1/2, 1/2, 1/2) is such that all constraints are satisfied as strict inequalities:

\[
\frac{1}{4} + \frac{1}{4} + \frac{3}{4} - 5 < 0
\]

\[
\frac{1}{2} > 0 \\
\frac{1}{2} > 0 \\
\frac{1}{2} > 0.
\]

It should be noted that that the fact that Slater's CQ holds does not mean that other conditions are or are not satisfied. It is a good exercise, for example, to check that in this case the linear independence of the gradients of the active constraints is satisfied at every feasible point and therefore so is the Mangasarian-Fromovitz condition, given that this latter condition is implied by the linear independence condition. Of course, the linearity of the constraints is not satisfied, instead, since the first constraint is nonlinear.

(iii) The point (0, 0, 0) is feasible and \(F(0, 0, 0) = (-4, -4, -6)\). In (0, 0, 0) the first constraint is not active while the non negative constraints are all active, therefore \(\lambda_1 = 0\). Taking all this into account, the KKT system in (0,0,0) boils down to

\[
\begin{align*}
-4 - \lambda_2 &= 0 \\
-4 - \lambda_3 &= 0 \\
-6 - \lambda_4 &= 0 \\
0 &\leq \lambda_2 \\
0 &\leq \lambda_3 \\
0 &\leq \lambda_4.
\end{align*}
\]
Since, for example, from the first equation we get $\lambda_2 = -4 < 0$ it is clear that $(0,0,0)$ does not satisfy the KKT conditions and therefore it is not a solution of the VI.

The point $(1,1,1)$ is feasible and $F(1,1,1) = (-2, -2, -6)$. In $(1,1,1)$ the first constraint is active while all the remaining constraints are not, thus implying $\lambda_2 = \lambda_3 = \lambda_4 = 0$. Taking all this into account, the KKT system in $(1,1,1)$ boils down to

\[
\begin{align*}
-2 + 2\lambda_1 &= 0 \\
-2 + 2\lambda_1 &= 0 \\
-6 + 6\lambda_1 &= 0 \\
0 &\leq \lambda_1.
\end{align*}
\]

It is easy to check that $\lambda_1 = 1$ with $(1,1,1)$ does satisfy this system and consequently that the point $(1,1,1)$, together with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 0, 0, 0)$ satisfies the KKT system form which we see that $(1,1,1)$ is a solution of the VI.

The point $(2,2,0)$ is not feasible since it violates the first constraint and therefore it is not a solution of the VI.

8. Consider the following VI $(K, F)$:

\[
F = \left( \begin{array}{c}
-x_1 + \frac{1}{2}x_2 - 3 \\
-x_1^2 - 2x_2
\end{array} \right)
\]

\[K = \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, \ x_1 - x_2 \leq 0, \ x \geq 0 \}\]

(i) Write down the KKT conditions for this VI

(ii) Check graphically whether any of the points $(0,0)$ and $(\sqrt{2}/2, \sqrt{2}/2)$ is a solution

(iii) Verify your answers to (ii) by using the KKT conditions

(i) The KKT system is

\[
\begin{align*}
-x_1 + \frac{1}{2}x_2 - 3 + 2x_1\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\
-x_1^2 - 2x_2 + 2x_2\lambda_1 - \lambda_2 - \lambda_4 &= 0 \\
0 \leq \lambda_1 &\perp x_1^2 + x_2^2 - 1 \leq 0 \\
0 &\leq \lambda_2 \perp x_1 - x_2 \leq 0 \\
0 &\leq \lambda_3 \perp x_1 \geq 0 \\
0 &\leq \lambda_4 \perp x_2 \geq 0
\end{align*}
\]

(ii) The point $(0,0)$ is feasible and $F(0,0) = (-3,0)$ and, from the figure, it is clear it is not a solution.
The point \((\sqrt{2}/2, \sqrt{2}/2)\) is also feasible; it is easy to check by substitution, that it is the point in the first quadrant at the intersection of the circle and the line. We have \(F(\sqrt{2}/2, \sqrt{2}/2) = (-\sqrt{2}/4, -\sqrt{2} - 1/2)\) and, from the figure, it is clear it is a solution.

(iii) In \((0, 0)\) the last three constraints are active while the first one is not, thus implying \(\lambda_1 = 0\). Taking all this into account, the KKT system in \((0, 0)\) reduces to

\[-3 + \lambda_2 - \lambda_3 = 0
-\lambda_2 - \lambda_4 = 0
0 \leq \lambda_2
0 \leq \lambda_3
0 \leq \lambda_4\]

From the second equation we get that \(\lambda_2 = -\lambda_4\) and since they must be both nonnegative, the only possibility is to have \(\lambda_2 = 0 = \lambda_4\). But then we get from the first equation \(\lambda_3 = -3\), thus showing that \((0, 0)\) does not satisfy the KKT conditions and thus confirming what found in (ii) by geometrical reasonings.

In \((\sqrt{2}/2, \sqrt{2}/2)\) the first two constraints are active while the last two are not not, thus implying \(\lambda_3 = 0 = \lambda_4\). Taking all this into account, the KKT system in \((0, 0)\) reduces to

\[-\frac{\sqrt{2}}{4} - 3 + \sqrt{2}\lambda_1 + \lambda_2 = 0
\sqrt{2} - \frac{1}{2} + \sqrt{2}\lambda_1 - \lambda_2 = 0
0 \leq \lambda_1
0 \leq \lambda_2\]
Solving the linear system given by the first two equations, we get

\[ \lambda_1 = \frac{5}{8} + \frac{7}{8} \sqrt{2} > 0, \quad \lambda_2 = \frac{5}{4} - \frac{3}{8} \sqrt{2} > 0 \]

showing that the KKT conditions are satisfied at \((\sqrt{2}/2, \sqrt{2}/2)\) and therefore confirming that this point is a solution of the VI.

9. Consider the following VI \((K, F)\):

\[ F = \begin{pmatrix} 3x_1 - x_2 + \frac{1}{4} \\ x_1 + x_2 + \frac{1}{2} \end{pmatrix}, \quad K = \{ x \in \mathbb{R}^2 : x_1^2 + (x_2 - 1)^2 \leq 1, x_1 + x_2 \leq 1, x_1 \geq 0 \} \]

(i) Write down the KKT conditions for this VI

(ii) Check graphically whether any of the points \((0,0), (0, 1/2), (0,1), (0,2)\) is a solution

(iii) Verify your answers to (ii) by using the KKT conditions

(iv) What CQs are satisfied for this problem and, in particular at the solutions?

(i) The KKT system is

\[ 3x_1 - x_2 + \frac{1}{4} + 2x_1 \lambda_1 + \lambda_2 - \lambda_3 = 0 \]
\[ x_1 + x_2 + \frac{1}{2} + 2(x_2 - 1) \lambda_1 + \lambda_2 = 0 \]
\[ 0 \leq \lambda_1 \perp x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \]
\[ 0 \leq \lambda_2 \perp x_1 + x_2 - 1 \leq 0 \]
\[ 0 \leq \lambda_3 \perp x_1 \geq 0. \]

(ii) The point \((0,0)\) is feasible and \(F(0, 0) = (1/4, 1/2)\) and, from the figure, it is clear it is a solution.

The point \((0,1/2)\) is feasible and \(F(0, 1/2) = (-1/4, 1)\) and, from the figure, it is clear it is not a solution.

The point \((0,1)\) is feasible and \(F(0, 1) = (-3/4, 3/2)\) and, from the figure, it is clear it is not a solution.

9
The point (0,2) is not feasible (it violates the second constraint, and therefore it cannot be a solution.

(iii) In (0,0) the first and third constraints are active while the second one is not, thus implying $\lambda_2 = 0$. Taking all this into account, the KKT system in (0,0) reduces to

\[
\begin{align*}
\frac{1}{4} - \lambda_3 &= 0 \\
\frac{1}{2} - 2\lambda_1 &= 0 \\
0 &\leq \lambda_1 \\
0 &\leq \lambda_3.
\end{align*}
\]

From the first two equations we get $\lambda_1 = \lambda_3 = 1/4$ thus showing that the point (0,0), together with the multipliers $(\lambda_1, \lambda_2, \lambda_3) = (1/4, 0, 1/4)$ satisfies the KKT conditions so that (0,0) is a solution of the VI and thus confirming what found in (ii) by geometrical reasonings.

In (0,1/2) only the third constraint is active while the first and second one are not, thus implying $\lambda_1 = \lambda_2 = 0$. Taking all this into account, the KKT system in (0,1/2) reduces to

\[
\begin{align*}
-\frac{1}{4} - \lambda_3 &= 0 \\
1 &= 0 \\
0 &\leq \lambda_3.
\end{align*}
\]

The second line clearly shows that the KKT system is not satisfied in (0,1/2) and thus confirming that it is not a solution of the VI. Indeed, since Slater’s CQ holds for this problem (see next point), if (0,1/2) were a solution, it should satisfy the KKT conditions.
In (0,1) the second and third constraints are active while the first one is not, thus implying \( \lambda_1 = 0 \). Taking all this into account, the KKT system in (0,1/2) reduces to

\[
\begin{align*}
\frac{3}{4} + \lambda_2 - \lambda_3 &= 0 \\
\frac{3}{2} + \lambda_2 &= 0 \\
0 &\leq \lambda_2 \\
0 &\leq \lambda_3.
\end{align*}
\]

The second equation clearly gives \( \lambda_2 = -3/2 < 0 \) thus showing that the KKT system is not satisfied in (0,1/2) and therefore confirming that this point is not a solution of the VI. Indeed, since Slater’s CQ holds for this problem (see next point), if (0,1) were a solution, it should satisfy the KKT conditions.

The point (0,2) is not feasible and therefore it cannot satisfy the KKT system.

(iv) The CQ stating that all constraints are linear is obviously not satisfied since the first constraint is quadratic.

Slater’s CQ is instead met because all constraints are convex (the first one has a the Hessian equal to 2/ which is positive definite while the other two constraints are linear) and it is easy to see that, for example, the point (1/4, 1/4) makes all constraints satisfied as strict inequalities:

\[
\frac{1}{16} + \frac{9}{16} < 1, \quad \frac{1}{4} + \frac{1}{4} < 1, \quad \frac{1}{4} > 0.
\]

This means that at all solution the KKT conditions are satisfied.

Let’s check that in the solution (0,0) also the Linear Independence (of the gradients of the active constraints) condition is met. Note that while Slater’s condition is a condition on the constraints that is either satisfied or not, with no reference to a specific point, the LICQ might be checked at a specific point and could be satisfied at a certain point and not satisfied at another point. In (0,0) there are two active constraint, the first and the third, whose gradients, in (0,0) are

\[
\begin{pmatrix}
0 \\
-2
\end{pmatrix}, \quad \begin{pmatrix}
-1 \\
0
\end{pmatrix}.
\]

Since these two vectors are clearly linear independent (for example it is enough to check that the matrix that has the two gradients as columns has a nonzero determinant), the LICQ is satisfied at the origin. Since the LICQ implies the MFCQ, also this latter condition is satisfied.

Remark. It is worth noting that the CQ depend heavily on the analytic representation of the feasible set. But this we mean that the same set may be represented in several ways as a solution of a system of equalities and inequalities and, depending on the chosen representation, some CQs may or may not be satisfied. As a first example, consider the set \( K \) of this exercise and add to the list of the constraints the constraint \( x_2 \geq 0 \). It is clear that we did not change the set \( K \) because the new constraint is redundant (the nonnegativity of the second variable
is already forced by the first constraint, see the figures). Therefore, if we now look at (0,0) and check the LICQ we see that there are now three active constraints, the two considered in part (iv) above and the new constraint $x_2 \geq 0$. The gradient of the active constraints are therefore now
\[
\begin{pmatrix}
0 \\
-2
\end{pmatrix}, \quad \begin{pmatrix}
-1 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
-1
\end{pmatrix}.
\]
and the LICQ cannot be satisfied because three vectors in $\mathbb{R}^2$ can never be linear independent. So, while the set did not change, by changing its representation we see that the LICQ is no longer satisfied. However, you can check that the MFCQ is still satisfied at (0,0) and that also Slater’s CQ is still satisfied. As another example consider in $\mathbb{R}$ the set $K = [1, 2]$. This set can be described as $K = \{x : x \geq 0, x \leq 1\}$. For this representation we immediately see that all four CQ we studied are met. Indeed, (a) the constraints are linear; (b) the point $x = 1/2$ obviously makes all the (convex) constraints satisfied as strict inequalities and so also Slater’s conditions holds; (c) LICQ also holds at any feasible point (if $0 < x < 1$ there are no active constraints and LICQ holds vacuously, if $x = 0$ only the first constraint is active and the gradient (i.e. the derivative, since this is a function of one variable) is (-1) which shows LICQ holds and, similarly, if $x = 1$ only the second constraint is active and the gradient is (1), which again shows that LICQ holds; (d) since the MFCQ is implied by the LICQ also this condition is satisfied. But now define the following function
\[
g(x) = \begin{cases} 
x^2 & \text{if } x \leq 0 \\
0 & \text{if } 0 < x < 1 \\
(x - 1)^2 & \text{if } x \geq 1
\end{cases}
\]
It is easy to check (see also the figure), that $g$ is continuously differentiable and that its gradient in $[0, 1]$ is always (0). It is clear that the feasible set $K = \{x : x \geq 0, x \leq 1\}$ can also be written as $K = \{x : g(x) \leq 0\}$. But it is easy to see that with this representation all the four CQ we considered may fail! Indeed (a) $g$ is non linear so that CQ that all constraints be linear is not met; (b) Since $g(x) \geq 0$ for any $x$, also Slater’s condition cannot be satisfied; (c) since the gradient of $g$ is (0) at any feasible point, the LICQ and the MFCQ are not satisfied for $x = 0$ or for $x = 1$.

10. Consider the VI $(K, F)$, where
\[
F = \begin{pmatrix}
3x - 2y - 2 \\
x + y - 2
\end{pmatrix}, \quad K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, x + y \geq 0, x \leq 1\}
\]

(i) Check graphically which if the following points is a solution: (0,0), (1,-1), (1,1), (0,1)

(ii) Write down the KKT system for this VI and check, using the KKT conditions, which of the points listed in point (i) satisfies these conditions. Are the results consistent with what you found in (i)?

(iii) Justify the following assertion: satisfactions of the KKT conditions in (1,-1) is a necessary and sufficient condition for this point to be a solution of the VI
11. Let the VI\((K,F)\) be given, where

\[K = \{(x,y) : x^2 + y^2 \leq 4, \ x \leq 1, \ y \geq 1\}, \quad F = \begin{pmatrix} 4x + 2y - 2 \\ -2x + 5y + 1 \end{pmatrix} \]

(a) Determine graphically which of the following points is a solution of the VI:

\[
\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Why can you be sure that a solution will satisfy the KKT system?

(b) Write down the full KKT system for this VI (attention, do now write generic formulas, but the KKT system for this specific VI).

(c) For the point you found are solutions, check that the KKT system is satisfied and compute the Lagrange multipliers.

12. In class we gave we illustrated Braess' paradox by using a simple network reported below.

\[
\begin{align*}
\mathcal{W} &= \{(0,0)\} \\
\mathcal{P}_{\infty} &= \{(0,1), (1,0)\}, \quad \{(0,2), (2,0)\} \\
\begin{pmatrix} f_{\infty} \\ f_{\infty} \end{pmatrix} & \text{ is the flow on arc } (a,b) \\
\text{The cost of a path is given by the sum on the arcs in the path} \\
d^{\infty} &= 6
\end{align*}
\]

We checked in class that \((3,3)\) is an equilibrium for this problem and we stated that it possible to verify, since there are only a finite number of possibilities, that any other flow is not an
equilibrium (it would be a good exercise to check this statement). Are you able prove the same fact in a shorter and more straightforward way?

The Wardrop first principle states that at equilibrium only paths with the same, minimal cost can be used. In this example is clear that $f_{O1} = f_{1D}$ and that $f_{O2} = f_{2D}$. Calling the two flows on the upper and lower path $f_1$ and $f_2$ respectively, i.e. setting $f_1 \triangleq f_{O1} = f_{1D}$ and $f_2 \triangleq f_{O2} = f_{2D}$, we see that the cost of the two paths are $C_1(h) = 11f_1 + 50$ and $C_2(h) = 11f_2 + 50$. It is then clear that to have $C_1(h) = C_2(x)$ it must be $f_1 = f_2$, i.e., recalling that $f_1 = f_2 = 6$, we can be at an equilibrium where both paths are used if and only if $f_1 = 3 = f_2$. We are only left with the possibility that one path is not used and all the flow of 6 goes on the other one. Let's say the upper pass is not used: $f_1 = 0$ and $f_2 = 6$. But then this cannot be an equilibrium because the cost on the lower path is higher than the cost on the unused upper path (more precisely, of the cost that an user would experience switching from the lower path to the upper one). We therefore see that the only possible equilibrium is $f_1 = 3 = f_2$. 
1. Consider the function \( f(x) = \frac{1}{5}x^4 + \ln\left(\frac{3}{2} + \frac{x}{2}\right) \)

   (a) Show that \( f \) is a contraction on \([0, 1]\)

   (b) Starting with \( x^0 = 1 \) compute the first two iterations of the fixed-point algorithm (it is not necessary to perform all computations relative to \( x^2 \))

   (a) We first check that the \( f \) maps points of \([0, 1]\) in \([0, 1]\). To this end we show that \( f \) is an increasing function in \([0, 1]\) since both \((1/5)x^4\) and the logarithmic increasing function in that interval. Therefore it is enough to check that both \( f(0) \) and \( f(1) \) belong to \([0, 1]\), since, if so, all others function values will be between \( f(0) \) and \( f(1) \) and therefore belong to \([0, 1]\). We have \( f(0) = \ln(1.5) \approx 0.4 \) and \( f(1) = \frac{1}{5} + \ln(1.625) \approx 0.69 \). Therefore \( f \) maps points of \([0, 1]\) in \([0, 1]\). In order to check that this is contraction, then, we may observe that the function is differentiable in \([0, 1]\) and therefore it is enough to check that the absolute value of the derivative is less than 1 and actually bounded away from 1.\(^1\) We have, in \([0, 1]\)

   \[
   f' = \frac{4}{5}x^3 + \frac{1}{8} + \frac{1}{8} \leq \frac{4}{5} + \frac{12}{8} = 0.9.
   \]

   Therefore \( f \) is a contraction on \([0, 1]\)

   (b) \( x^1 - f(x^0) = f(1) = \frac{1}{5} + \ln(1.625) \).

   \( x^2 = f(1) \approx f(0.69) \) (more precise calculations were not required).

2. Consider the function \( f(x) = \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{2} \) in \([-1, 1]\)

   (a) Explain why this is a contraction

   (b) Apply the fixed-point algorithm starting with \( x_0 = 0 \) and compute \( x_1 \) and \( x_2 \).

   (c) Estimate the maximum distance of \( x_2 \) to the fixed point.

   (a) We first check that the \( f \) maps points of \([-1, 1]\) in \([-1, 1]\). To this end we first observe that \( f \) is neither increasing nor decreasing in \([-1, 1]\) and therefore we cannot simply look at the values in the extreme points of the interval as in the previous exercise. We then compute the maximum and minimum value of the \( f \) in \([-1, 1]\) and check whether these two values are in \([-1, 1]\): if so, it is clear that all other values taken by \( f \) in \([-1, 1]\) also belong to that interval. The function is an upward parabola and its vertex (i.e. the minimum point) is

\(^1\)Indeed, we can write, for any \( x \) and \( y \) in \([0, 1]\), \( f(y) = f(x) + f'(\xi)(y - x) \) for some \( \xi \in (0, 1) \) (this is the Mean Value Theorem). Therefore we have \( |f(y) - f(x)| = |f'(\xi)||y - x| \). If \( |f'(\xi)| \leq \rho < 1 \) we conclude that \( |f(y) - f(x)| \leq \rho |y - x| \), showing that \( f \) is a contraction.

\(^2\)we obtained this majorization by majorizing separately the two terms in \([0, 1]\), therefore taking \( x = 1 \) in \( \frac{1}{2}x^2 \) and \( x = 0 \) in \( \frac{1}{8} + \frac{1}{8} \).
\( v = -b/2a = 1/2 \). We have \( f(1/2) = 7/16 \in [-1,1] \). It is now clear that the maximum is reached in \( x = -1 \) (draw a simple picture) and we have \( f(-1) = 1 \in [-1,1] \). We can conclude that \( f \) maps \([-1,1]\) into \([-1,1]\). To check that \( f \) is a contraction we now must check that \( |f(y) - f(x)| \leq \eta |y - x| \) for some \( \eta \in [0, 1) \) and for all \( x \) and \( y \) in \([-1,1]\). Reasoning as in the previous exercise, it is enough to verify that \( |f'(x)| \leq \eta \in [0, 1) \) for some \( \eta \) and for all \( x \) in \([-1,1]\). We have \( f'(x) = \frac{3}{2}x - \frac{1}{2} \) and it is simple to see that the maximum of the modulus is achieved in \( x = -1 \), a point for which we get \( |f'(-1)| = 3/4 \). We can therefore set \( \eta = 3/4 \).

(b) Set \( x_0 = 0 \). We have \( x_1 = f(0) = 1/2 \) and \( x_2 = f(1/2) = 7/16 \).

(c) If \( x^* \) is the fixed point, we know form Banach's fixed-point theorem that, for iteration \( k \)

\[
|x_k - x^*| \leq \frac{\eta^k}{1 - \eta} |x_1 - x_0|
\]

which, in our case, gives the estimate

\[
|x_2 - x^*| \leq \frac{\eta^2}{1 - \eta} |x_1 - x_0| = \frac{9}{8}
\]

Note that, taking into account that the fixed point must anyway belong to \([-1,1]\), this means the fixed point actually belongs to the interval \([1/2, 7/8, 1] = [-\frac{3}{8}, 1]\).

3. Consider the VI \((K, F)\) with

\[
F = \begin{pmatrix}
-4x - y^2 + xz \\
xy^2 + 3z - 1 \\
6xyz
\end{pmatrix}, \quad K = \{ (x, y, z) \in \mathbb{R}^3 : 2x^2 + 3y^2 + z^2 \leq 6, \; y + \frac{4}{3}z \leq \frac{1}{3} \}
\]

Answer the following questions

(a) Does a solution to this VI exist? Why?

(b) Write down the KKT system for this VI
(c) Show that Slater's CQ holds

(d) Check which of the following points satisfies the KKT system: (a) (0, 0, 0),
    (b) (1, -1, 1), (c) (1, 1, 1)

(e) Consider the following variant of $F$ (where we changed the -1 to the parameter $\alpha$)

$$F = \begin{pmatrix}
-4x - y^2 + zx \\
xy^2 + 3z + \alpha \\
6xyz
\end{pmatrix}$$

(the set $K$ is the same as before). For which value(s) of $\alpha$ is the point (0, 0, 0) a solution?

(a) $F$ is defined by polynomials and is therefore continuous. The feasible set $K$ is closed
because it is defined by $\leq$ inequalities with continuous functions and convex (because the
$g$s are convex). Furthermore, and more to the point, $K$ is bounded (and therefore compact)
because of the constraint $2x^2 + 3y^2 + z^2 \leq 6$ (an ellipsoid) which clearly shows that no single
variable can take "too large" values without violating this constraint. Therefore, by the basic
existence theorem for Variational Inequalities, we have that a solution (at least) exists.

(b) The KKT system for the VI is

$$-4x - y^2 + zx + 4x\lambda_1 = 0$$
$$xy^2 + 3z - 1 + 6y\lambda_1 + \lambda_2 = 0$$
$$6xyz + 2z\lambda_1 + \frac{2}{3}\lambda_2 = 0$$
$$0 \leq \lambda_1 \perp 2x^2 + 3y^2 + z^2 - 6 \leq 0$$
$$0 \leq \lambda_2 \perp y + \frac{2}{3}z - \frac{1}{3} \leq 0$$

(c) As already observed in (a), the constraints are convex (of course, we are dealing with a VI,
whose feasible set must be convex by definition) and therefore it is enough to exhibit a point
such that the two constraints are satisfied as strict inequalities. It is easily seen that the point
(0,0,0) satisfies the two inequalities as strictly and therefore Slater's constraint qualification
holds.

(d) Since Slater's conditions holds, being a solution of the VI or satisfying the KKT system
are equivalent problems. We therefore check whether the three given points satisfy the KKT
system.

Substitute (0,0,0) in the KKT system, thus obtaining

$$0 = 0$$
$$-1 + \lambda_2 = 0$$
$$\frac{4}{3}\lambda_2 = 0$$
$$0 \leq \lambda_1 \perp -6 \leq 0$$
$$0 \leq \lambda_2 \perp -\frac{1}{3} \leq 0$$

3
This system is clearly impossible (the second equality gives $\lambda_2 = 1$ while the third $\lambda_2 = 0$, for example), and therefore $(0,0,0)$ is not a solution of the VI.

We now substitute $(1, -1, 1)$ in the KKT system, thus obtaining:

\[
\begin{align*}
-4 + 4\lambda_1 &= 0 \\
3 - 6y\lambda_1 + \lambda_2 &= 0 \\
-6 + 2\lambda_1 + \frac{2}{3}\lambda_2 &= 0 \\
0 \leq \lambda_1 &\perp 0 \leq 0 \\
0 \leq \lambda_2 &\perp 0 \leq 0
\end{align*}
\]

It is easy to see that $(1, -1, 1)$, together with $\lambda_1 = 1$ and $\lambda_2 = 3$ satisfies the system and therefore $(1,-1,1)$ is a solution of the VI.

Finally, the point $(1,1,1)$ is not feasible since it violates the second constraint, and therefore it is neither a solution of the VI nor can it satisfy the KKT system.

(e) If we rewrite the KKT system for the new $F$, the one with the $\alpha$ parameter, and evaluate it in $(0,0,0)$, we obtain:

\[
\begin{align*}
0 &= 0 \\
\alpha + \lambda_2 &= 0 \\
\frac{2}{3}\lambda_2 &= 0 \\
0 \leq \lambda_1 &\perp -6 \leq 0 \\
0 \leq \lambda_2 &\perp -\frac{1}{3} \leq 0
\end{align*}
\]

From the last two lines we see that $\lambda_1$ and $\lambda_2$ must be zero by complementarity. Therefore, it is readily seen that by setting $\alpha = 0$, the point $(0,0,0)$ satisfies the KKT system and is therefore a solution of the VI.

4. Consider the NCP($F$) with

\[ F = \begin{pmatrix} 2x - y + 2 \\ x + 2y - 1 \end{pmatrix} \]

(a) Verify that the assumptions for the application of the (basic) projection method are satisfied and estimate the maximum value of $\tau$ for which the algorithm converges.

(b) Compute then $z^1$ and $z^2$ using the basic projection algorithm, with $z^0 = (0,0)$ and using $\tau = 0.01$.

(c) Can you estimate the maximum distance of $z^2$ from the solution?

(d) Repeat points (b) and (c), but use $\tau = 0.1$ this time.

(a) $F$ is linear, so we know it is surely Lipschtiz continuous on the whole space (and therefore, in particular, on the non negative orthant which is of interest here). To see this we first note that we can write:

\[ F(x,y) = Az + b, \quad \text{with} \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}. \]
Then we have
\[ \|F(z_1) - F(z_2)\| = \|(A z_1 + b) - (A z_2 + b)\| = \|A(z_1 - z_2)\| \leq \|A\| \|z_1 - z_2\|, \]
from which we see that a linear $F$ is always Lipschitz continuous on the whole space, with Lipschitz constant $L = \|A\|_F$. Going back to the $F$ considered in this exercise we can take $L = \sqrt{2^2 + 1^2 + (-1)^2 + 2^2} = \sqrt{10} \ . \ 
3$
In order to estimate $\alpha$ we preliminary observe that $F$ is actually strongly monotone, since
\[ JF(z) = A; \quad JF_{\text{Sym}}(z) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \]
with the matrix $A$ positive definite. For any strongly monotone linear function $F(z) = Az + b$ we can write
\[
(F(z_1) - F(z_2))^T (z_1 - z_2) = [(Az_1 + b) - (Az_2 + b)]^T (z_1 - z_2) \\
= [A(z_1 - z_2)]^T (z_1 - z_2) \\
= (z_1 - z_2)^T A_{\text{Sym}} (z_1 - z_2) \\
\geq \lambda_{\min}(A_{\text{Sym}}) \|z_1 - z_2\|^2,
\]
where the third equality follows from the very definition of symmetric part of a matrix, $\lambda_{\min}(M)$ denotes to smallest eigenvalue of the matrix $M$ and the last inequality follows from well-known properties of positive definite matrices. We thus conclude that we can take $\alpha = \lambda_{\min}(A_{\text{Sym}})$, in our particular case we have $A_{\text{Sym}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and therefore we can take $\alpha = \lambda_{\min}(A_{\text{Sym}}) = 2.$

\[ \text{We recall that we are using the euclidean norm for vectors and that this inequality holds because we are using the Frobenius norm for the matrix, but in general we do not necessarily have } \|A z\| \leq \|A\| \|z\| \text{ for any matrix norm. We also recall that the Frobenius norm of a square } n \text{ matrix is } \|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2}, \]

Another norm for which the inequality $\|A z\| \leq \|A\| \|z\|$ holds is $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$, where $\lambda_{\max}(M)$ denotes the largest eigenvalues of the matrix $M$.

$3$By reasoning made and considering in particular also the previous footnote, it should be clear that we can take $L$ to be equal to the norm of $A$ for any matrix norm for which it holds $\|A z\| \leq \|A\| \|z\|$. Therefore, in the case of the $F$ of this exercise, we could also set $L = \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$, which, in this case, would give $L = \sqrt{5}$, as simple calculations show. This value is more favourable than the previously obtain $\sqrt{10}$. In fact, we should remember that we are estimating $L$ and $\alpha$, the monotonicity constant of $F$, because we want to compute an upper bound for the maximum step $\tau$ we can take in the basic projection algorithm: $\tau \leq \frac{\alpha}{L}$. The larger this estimate the better, since we have a wider choice for $\tau$ and the possibility to take “larger” stepsizes, thus potentially permitting a faster convergence speed. It is then clear form the expression $\frac{\alpha}{L}$ that in order to obtain larger values we should aim at obtaining the largest possible estimate for $\alpha$ and the smallest possible estimate for $L$. In this sense, we prefer the estimate $\sqrt{5}$ to $\sqrt{10}$. 5
We therefore conclude that we should take

\[ \tau < \frac{2\alpha}{L^2} = \frac{2 \cdot 2}{(\sqrt{10})^2} = \frac{2}{5} = 0.4. \]

(b) We now perform two steps of the basic projection algorithm, starting with \( z^0 = (0, 0)^T \) and \( \tau = 0.01 \). Note that by the results so far we know that this value of \( \tau \) guarantees convergence of the algorithm to the unique solution of the NCP. Recalling the explicit expression of projection on the nonnegative orthant we have

\[ z^1 = \Pi_{\mathbb{R}_+^2}(z^0 - \tau F(z^0)) \]

\[ \Pi_{\mathbb{R}_+^2} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 0.01 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right) \]

\[ = \begin{pmatrix} \max(0, -0.02) \\ \max(0, 0.01) \end{pmatrix} \]

\[ = \begin{pmatrix} 0 \\ 0.01 \end{pmatrix}. \]

Repeating the same passages for the second iteration we get

\[ z^2 = \Pi_{\mathbb{R}_+^2}(z^1 - \tau F(z^1)) \]

\[ \Pi_{\mathbb{R}_+^2} \left( \begin{pmatrix} 0 \\ 0.01 \end{pmatrix} - 0.01 \begin{pmatrix} 1.99 \\ -0.98 \end{pmatrix} \right) \]

\[ = \begin{pmatrix} \max(0, -0.0199) \\ \max(0, 0.0198) \end{pmatrix} \]

\[ = \begin{pmatrix} 0 \\ 0.0198 \end{pmatrix}. \]

(c) In order to estimate the distance of \( z^2 \) to the unique solution \( z^* \) of the NCP (recall that in order to apply the projection algorithm we preliminarily established that \( F \) is strongly monotone, a fact that implies that the NCP(\( F \)) has one and only one solution) we can observe that the proof of convergence of the method (Theorem 1.7.5 in the Lectures Notes) hinges on the fact that \( z^* \) is a fixed point of \( \Pi_K(z - \tau F(z)) \) and therefore on the application of Banach's fixed point iteration method to \( G(z) = \Pi_K(z - \tau F(z)) \). In the course of the proof

\[ \tau < \frac{2\alpha}{L^2} = \frac{2 \cdot 2}{(\sqrt{5})^2} = \frac{2}{5} = 0.8 \]

which is twice as large as the one obtained using the Frobenius norm and therefore allows us to take larger steps.

\[ \text{\footnotesize (footnote continued)} \]

Had we used the better estimate obtained in the previous footnote using the \( \| \cdot \|_2 \) matrix norm we would have obtained

\[ \tau < \frac{2\alpha}{L^2} = \frac{2 \cdot 2}{(\sqrt{5})^2} = \frac{4}{5} = 0.8 \]

which is twice as large as the one obtained using the Frobenius norm and therefore allows us to take larger steps.
it is shown that this function, under the conditions of the theorem, is actually a contraction, more precisely we show that\footnote{Below we use $z$ and $w$ to denote two generic points and not $x$ and $y$ as in the proof of Theorem 1.7.5, because in this exercise we used the symbols $x$ and $y$ to denote the components of a generic point $z$ and so we are no longer free to use them with a different meaning.}

$$\|\Pi_K(z - \tau F(z)) - \Pi_K(w - \tau F(w))\|^2 \leq (1 + \tau^2 L^2 - 2\tau \alpha) \|z - w\|^2$$

Therefore, by the definition of contraction, we see that $\Pi_K(z - \tau F(z))$ is a contraction (on $K$, in the specific case of this exercise on $\mathbb{R}^2$) with constant $\eta = \sqrt{1 + \tau^2 L^2 - 2\tau \alpha}$. But then we can use the estimate established in Banach's fixed-point theorem 1.7.2 to give a bound on the distance of a generic $z^k$ to $z^*$, obtaining

$$\|z^k - z^*\| \leq \frac{\eta^k}{1 - \eta} \|G(z^0) - z^0\| = \frac{\eta^k}{1 - \eta} \|z^0 - z^0\|$$

With our estimates, we have

$$\eta = \sqrt{1 + \tau^2 L^2 - 2\tau \alpha} = \sqrt{1 + 0.0001 \cdot 10 - 2 \cdot 0.01 \cdot 2} = \sqrt{0.961}$$

and therefore

$$\|z^2 - z^*\| \leq \frac{\eta^2}{1 - \eta} \|z^1 - z^0\| = \frac{0.961}{1 - \sqrt{0.961}} \left\| \begin{pmatrix} 0 \\ 0.01 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \approx 0.4805$$

It is actually easy to check that $z^*$ is $(0, 1/2)^T$ (indeed, $F(0, 1/2) = (3/2, 0)^T$, confirming this point is the solution) and therefore the real distance of $z^2$ to $z^*$ is

$$\|z^2 - z^*\| = \left\| \begin{pmatrix} 0 \\ 0.0198 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \right\| = \sqrt{0.0198 + 0.5^2} \approx 0.4802 \leq 0.4805.$$ (d) If we use $\tau = 0.1$, by repeating exactly the same calculation done in (b) and (c) and changing only the value of $\tau$ we obtain

$$z^1 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad z^2 = \begin{pmatrix} 0 \\ 0.18 \end{pmatrix}, \quad \|z^2 - z^*\| \leq \frac{\eta^2}{1 - \eta} \|z^1 - z^0\| = 4.805$$

Note that the upper bound to the distance of $z^2$ to the solution $z^*$ is 10 times the previous one, even if the distance of the new $z^2$ to the solution, $\|z^1 - z^2\| = 0.32$, is smaller than in the previous case, when it was 0.4802. This demonstrates well the fact that what Banach’s fixed point distance estimate gives is an upper bound and the real distance; in our example, in the first case this bound was rather tight, in the second case less so. But it also shows, even if only in one simple example that, usually, larger values of $\tau$ (always respecting the theoretical condition $\tau < 2\alpha / L^2$) tend to give faster convergence.

5. Consider the VI $(K, F)$ with

$$K = \{(x, y) : y \geq x^2, -x + y \leq 1, y \leq \alpha\}, \quad F(x, y) = \begin{pmatrix} x^3 + \alpha x + y - 1 \\ x + 4y - 5 \end{pmatrix}$$

where $\alpha \geq 0$ is a parameter.
(a) Determine for which value(s) of $\alpha$ the point $(0,1)^T$ is a solution of the VI (it is most convenient to reason geometrically, drawing pictures).

(b) Determine for which value(s) of $\alpha$ the VI is monotone or strongly monotone.

(c) If for some given $\alpha$ $(0,1)^T$ is a solution of the VI, can there be other solutions?

(a) It is clear that if $\alpha < 1$ the point $(0,1)$ is not feasible (see also Figure (a) below) and therefore cannot be a solution. We are therefore left with the case $\alpha = 1$ and $\alpha > 1$ to analyze. Note that we distinguish these two cases since it is clear geometrically (see Figures (b) and (c) below) that these are structurally different situations. So assume $\alpha = 1$. We have $F(0,1)^T = (0,-1)$ and it clear geometrically, see Figure (b) that $(0,1)^T$ is a solution. (c) If $\alpha > 1$ we still have $F(0,1)^T = (0,-1)$ but in this case, the point is no longer a solution, see Figure (c).
(b) We analyze the symmetric part of the Jacobian of $F$

$$JF(x, y) = \begin{pmatrix} 3x^2 + \alpha & 1 \\ 1 & 4 \end{pmatrix} = JF_{\text{sym}}(x, y).$$

We look first at the north-west minors

$$3x^2 + \alpha \quad \text{and} \quad \begin{vmatrix} 3x^2 + \alpha & 1 \\ 1 & 4 \end{vmatrix} = 12x^2 + 4\alpha - 1.$$

We have $3x^2 + \alpha \geq \alpha$. If $\alpha = 0$ this quantity is non-negative but can be zero, because there are feasible points where $x = 0$. The determinant of the matrix is $12x^2 + 4\alpha - 1 \geq 4\alpha - 1$ with equality if $x = 0$ (as already discussed, a possible case). Therefore the determinant is always positive if $\alpha > \frac{1}{4}$ and nonnegative if $\alpha = \frac{1}{4}$. Therefore if $\alpha > \frac{1}{4}$ both north-west minors are positive and $JF_{\text{sym}}$ is positive definite on the feasible set. This implies that $F$ is strictly monotone and since the feasible region is compact, also that $F$ is actually strongly monotone. If $\alpha = \frac{1}{4}$ the north-west minors are one positive (the first one) and one 0. Therefore, to check whether $F$ is monotone or not, we must look at all principal minors. In this case there is only one left, corresponding to the second row and column. This is $4 > 0$. Therefore we can conclude that, if $\alpha = \frac{1}{4}$, $F$ is monotone. Of course if $\alpha < \frac{1}{4}$ the determinant can be negative at feasible points (for sure at all points for which $x = 0$ and therefore $F$ cannot be monotone on the feasible region.

(c) We find in (a) that $(0, 1)^T$ is a solution only if $\alpha = 1$. In (b) we also found that for $\alpha = 1$ $F$ is strongly monotone and therefore $(0, 1)^T$ is the unique solution.

6. Consider the VI $(K, F)$ with

$$K = \{(x, y) : y \geq 0, -x + y \leq 1, x + y \leq 1\}, \quad F(x, y) = \begin{pmatrix} 3x - y - 1 \\ x + 2y + 1 \end{pmatrix}.$$

(a) Can you be sure that a solution exists?

(b) Can you tell whether this VI can have more than one solution?

(c) Find the solution(s) of this VI by inspection.

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(d) Write down the KKT system for the VI

(e) Why can you be sure that the KKT conditions are satisfied at a solution of this VI?

(f) Check that for the solution(s) found in (c) the KKT conditions are satisfied.

(g) Is the MFCQ satisfied at the solution(s) found in (c)?

(a) Yes, a solution surely exists because \( K \) is bounded and \( F \) is continuous

(b) We analyze the symmetric part of the Jacobian of \( F \)

\[
JF(x,y) = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \quad JF_{sym}(x,y) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.
\]

Since \( JF_{sym} \) is clearly positive definite and \( F \) is linear, we can easily see that \( F \) is strictly monotone and therefore there is only one solution. Note that actually \( F \) is strongly monotone since \( F \) is linear.

(c) We first look for a solution in the interior of the feasible region. Here, to have a solution we must have \( F(x,y) = 0 \). This is a simple linear system whose unique solution is easily seen to be \((1/4, -3/7)\). Since this point is clearly infeasible we have that there are no solutions in the interior of \( K \). We now start looking at solution on the boundary of \( K \). We begin with points of the type \((x,0)^T\), with \( x \in (-1,1) \). A point of this type is solution only if \( F(x,0) = (0,\beta)^T \) with \( \beta \geq 0 \). We have \( F(x,0) = (3x-1, x+1)^T \) and therefore we have that the first component is zero if and only if \( x = 1/3 \). For this value of \( x \) we actually have \( F(1/3,0) = (0,4/3)^T \) and therefore we can conclude that \((1/3,0)^T\) is a solution of the VI. Since we saw in (b) that this problem has one and only one solution there is no need to continue the exploration of the boundary since we are sure we will not find any other solution.

(d) The KKT system of this VI is

\[
\begin{align*}
3x - y - 1 - \lambda_2 + \lambda_3 &= 0 \\
x + 2y + 1 - \lambda_1 + \lambda_2 + \lambda_3 &= 0 \\
0 &\leq \lambda_1 \perp y \geq 0 \\
0 &\leq \lambda_2 \perp -x + y - 1 \leq 0 \\
0 &\leq \lambda_3 \perp x + y - 1 \leq 0
\end{align*}
\]

(e) Because the constraints are all linear and therefore the KKT conditions are both necessary and sufficient for a point to be a solution.
(f) In $(1/3, 0)^T$ we have $F(1/3, 0) = (0, 4/3)^T$ and only the first constraint is active. Therefore $\lambda_2 = \lambda_3 = 0$ and checking the KKT conditions simply amounts to find a non negative $\lambda_1$ such that

\[
0 = 0 \\
\frac{4}{3} - \lambda_1 = 0
\]

This is surely possible with $\lambda_1 = 4/3$ so that the point $(1/3, 0)^T$ together with $(\lambda_1, \lambda_2, \lambda_3) = (4/3, 0, 0)$ satisfies the KKT conditions.

(g) At the solution $(1/3, 0)^T$ only the first constraint is active, and its gradient is $(0, -1)^T$. Since a non zero vector is always positively linear independent (it is actually linear independent) the MFCQ is satisfied. Alternatively it is easy to see that the vector $d = (0, 1)^T$ is such that $(0, -1)(0, 1)^T = -1 < 0$ which also certifies that the MFCQ is satisfied. It can be shown that this condition is equivalent to the existence of a vector $d$ such that

\[v_1^T d < 0, v_2^T d < 0, \ldots, v_m^T d < 0.\]

7. Consider the same VI described in the previous exercise.

(a) Explain why you can apply the Basic Projection Algorithm.

(b) Compute the maximum value of $\tau$ indicated by the convergence Theorem for the Basic Projection Algorithm.

(c) Apply two iterations of the Basic Projection Algorithm with $\tau = 0.2$ and starting from $(x^0, y^0) = (0, 1)$ (i.e. compute $(x^1, y^1)$ and $(x^2, y^2)$; if necessary compute projections graphically).

8. Consider the VI($K, F$), where

\[
F = \begin{pmatrix}
x^2 y + z - 1 \\
3x + 2y + z - 2 \\
2x - y + z^2 + 1
\end{pmatrix}, \quad K = \{(x, y, z) \in \mathbb{R}^2 : x \geq 0, y \geq 0, z \geq 0, x + y \leq 1\}
\]

Answer the following questions, explaining the reasons of your answers.

(a) Can you be sure that a solution exist?

(b) Will a solution for this KKT surely satisfy the KKT conditions?

(c) By using the KKT conditions, tell which among the following points is a solution

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

\[\text{We recall that a set of vectors } v_1, v_2, \ldots, v_m \text{ is positively linear independent if } \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 \text{ with } \alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_m \geq 0 \quad \implies \quad \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_m = 0\]
9. Consider the NCP($F$) with

$$F = \begin{pmatrix} 2x_1 - x_2 - x_3 - 1 \\ x_1 + x_2^2 + x_2 + x_3 \\ -x_2 + 4x_3 - 4 \end{pmatrix}$$

(a) Is $F$ strongly monotone on $\mathbb{R}_+^3$? (Please, note this function is non-linear and the set is unbounded, so you have to resort to the definition...but it is very easy to answer)

(b) is the point

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

a solution of this NCP?

(c) Write down the KKT system for this NCP and check your answer in the previous point using the KKT conditions.

(d) Reformulate the NCP as a system of equations $\Phi(x) = 0$ by using the Fischer-Burmeister function (please, write the complete system).

10. Consider the VI($K, F$), with

$$F(x, y) = \begin{pmatrix} x + y + xy - \alpha \\ (x - 1)^2 + y^2 - 1 \end{pmatrix}, \quad K = \{(x, y) : x \geq y^2, x \leq 1\}$$

where $\alpha$ is a parameter.

(a) Can you be sure that a solution to this VI surely exists? Why?

(b) Can you be sure that at a solution the KKT conditions are surely satisfied? Why?

(c) Write down the full KKT system for this VI (attention, do now write generic formulas, but the KKT system for this specific VI).

(d) Find out for which values of $\alpha$ the point $(1, 1)$ is a solution of the VI (Suggestion: you can argue geometrically or use the KKT conditions)

(e) Verify that for the values of $\alpha$ for which $(1, 1)$ is a solution the KKT conditions are satisfied in $(1, 1)$ and determine the values of the multipliers (Note, the multipliers may depend on $\alpha$, of course)

(a) Yes because $F$ is continuous and the set $K$ is of course closed and convex (because it is defined by two continuous, convex inequalities) and compact. Indeed, $K$ is closed so to show compactness we only need to verify it is bounded by the first inequality we see that $x$ is non

---

Remember that a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Saying that a set is bounded can be checked, for example and as we do in this exercise, by showing that each variable is bounded. Other common situations where one can easily check boundedness is when the constraints include a constraint of the type $ax^2 + by^2 \leq c$ with $a, b, c$ positive numbers (here and below we consider the case of $\mathbb{R}^2$ but all reasonings can easily be extended to higher dimensions). In this case, it is clear that no variable
negative, while the second tells us that \( x \) is less or equal to 0, in other words \( x \) is between 0 and 1 and therefore bounded. Therefore, by the first inequality we have that \( y^2 \leq 1 \) showing that also \( y \) must be bounded and belong to \([0,1]\). A simple sketch of the feasible region, in Figure (a) below, confirms that the set \( K \) is bounded, and therefore compact. By the fundamental theorem on the existence of solutions of VIs we see that the VI surely has at least a solution.

(b) Yes, because constraints qualifications are satisfied. For example Slater’s CQ holds because, as observed in (a), the constraints are convex inequalities and for the point \((1/2, 0)\) we have \(1/2 > 0\) and \(1/2 < 1\), i.e. all inequalities are satisfied strictly, see the Figure (b) above.

(c) The KKT system is

\[
\begin{align*}
    x + y + xy - \alpha - \lambda_1 + \lambda_2 & = 0 \\
    (x - 1)^2 + y^2 - 1 + 2y\lambda_1 & = 0 \\
    0 & \leq \lambda_1 \perp -x + y^2 & \leq 0 \\
    0 & \leq \lambda_2 \perp x - 1 & \leq 0
\end{align*}
\]

(d) and (e) The point \((1, 1)\) is feasible and in \((1, 1)\) both constraints are active. The KKT system then reduces to

---

\(^9\)Remember that Slater’s CQ guarantees that whatever the solution, the KKT are satisfied. Other CQ must be verified in the specific solution, as they could be satisfied in one point and not in another one. In this example it can be a good exercise to check that the Linear independence of the gradient of the active constraint is satisfied at every feasible point, though.
\[ 3 - \alpha - \lambda_1 + \lambda_2 = 0 \]
\[ 2\lambda_1 = 0 \]
\[ 0 \leq \lambda_1 \]
\[ 0 \leq \lambda_2 \]

For the second equation we get \( \lambda_1 = 0 \) and from the first we therefore get \( \lambda_2 = \alpha - 3 \). By (b) we know that (1,1) is a solution of the VI is and only if it satisfies the KKT system, i.e. the nonnegativity of the multipliers must be satisfied. \( \lambda_1 = 0 \) and therefore it is always nonnegative. Therefore we must only impose \( \lambda_2 = \alpha - 3 \geq 0 \), i.e. \( \alpha \geq 3 \). We conclude that (1,1) is a solution and the multipliers are \( \lambda_1 = 0 \) and \( \lambda_2 = \alpha - 3 \geq 0 \).

(d) (again, arguing geometrically this time) We have \( F(1,1) = (3 - \alpha, 0)^T \). Geometrically (see Figure (c) above), this means that \( F(1,1) \) is parallel to the \( x \)-axis and perpendicular to \( x = 1 \), i.e. to the boundary of the constraint \( x \leq 1 \). It is clear, geometrically that (1,1) will be a solution of \( F(1,1) \) points in the negative direction (or is 0), i.e. if \( 3 - \alpha \leq 0 \) or, equivalently, if \( 3 \leq \alpha \) as we found above. Note that this geometric reasoning needs to be complemented by the KKT conditions argument given above to answer (c), but gives an interesting insight on what is the meaning of the fact that \( \lambda_1 = 0 \). Analytically this says that the (gradient of the) first constraint plays no role in the KKT conditions. From a more geometrical point of view, we then expect the first constraint to have no role in (1,1) being a solution of the VI and indeed, see Figure (c) above, if we modify the set \( K \) by eliminating this constraint, we see that (1,1) is still a solution of the resulting VI.

11. Consider MiCP defined by the following \( F \)

\[
F(x, y, z) = \begin{pmatrix}
x^2 + 2x + y + z - 3 \\
x + 2y + z \\
-x - y + 2z + 1
\end{pmatrix}
\]

with the variables \( x \) and \( y \) nonnegative and \( z \) free.

(a) Determine which of the following three points (if any) is a solution

\[
w^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad w^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w^3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad w^4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.
\]

(b) Determine whether this problem is strongly monotone or not.

(c) Are the conditions to apply the basic projection algorithm to this problem satisfied?

(d) Are the conditions to apply the Hyperplane Projection Algorithm satisfied?

(a) The problem is equivalent to the VI \((\mathbb{R}_+ \times \mathbb{R}, F)\) and a feasible point \( w \) is a solution if

\[
0 \leq w_1 \perp F_1(w) \geq 0, \quad 0 \leq w_2 \perp F_2(w) \geq 0, \quad F_3(w) = 0.
\]

Let us then look at the four points. \( w^1 \) is feasible and \( F(w^1) = (-3, 0, 1)^T \) and therefore \( w^1 \) is not a solution, because, for example, \( F_1(w) \not\geq 0 \). \( w^2 \) is feasible and \( F(w^2) = (0, 1, 0)^T \)
and therefore $w^2$ is a solution, because all required conditions are satisfied. $w^3$ is feasible and $F(w^3) = (0, 2, -3)^T$ and therefore $w^3$ is not a solution, because, for example, $F_3(w) \neq 0$. Finally $w^4$ is not feasible, because the first component, $w^4_1$ is negative.

(b) In order to establish whether $F$ is strongly monotone on $K = \mathbb{R}^2_+ \times \mathbb{R}$ we look at the Jacobian and its symmetric part

\[
JF(w) = \begin{bmatrix}
2x + 2 & 1 & 1 \\
1 & 2 & 1 \\
-1 & -1 & 2
\end{bmatrix} \quad JF_{sym}(w) = \begin{bmatrix}
2x + 2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

We look first at the three North-West minors. Recall that on the feasible set $K$ is always non negative. Then, the first minor is $2x + 2 \geq 2 > 0$ for any feasible point. The second minor is $2(2x + 2) - 1 \geq 3 > 0$ for all feasible points. The third minor is the determinant of $JF_{sym}(w)$ and is $4(2x + 2) - 2 \geq 6 > 0$ for all feasible points. We can then see that $F$ is surely strictly monotone on $K$. To establish whether it is actually strongly monotone we must resort to a direct analysis\(^\text{10}\). We know that $F$ is strongly monotone on $K$ if and only if there exists a (at least one) positive $\alpha$ such that $JF(w) - \alpha I$ (or equivalently $JF_{sym}(w) - \alpha I$) is positive semidefinite for all feasible $w$. The analysis we mad to ascertain strict monotonicity easily suggests that this should be true, because all North-West minors are bounded away from zero on $K$. We have

\[
JF_{sym}(w) - \alpha I = \begin{bmatrix}
2x + 2 - \alpha & 1 & 0 \\
1 & 2 - \alpha & 0 \\
0 & 0 & 2 - \alpha
\end{bmatrix}
\]

If we take, for example, $\alpha = 0.5$ we get

\[
JF_{sym}(w) - \alpha I = \begin{bmatrix}
2x + 1.5 & 1 & 0 \\
1 & 1.5 & 0 \\
0 & 0 & 1.5
\end{bmatrix}
\]

and the three North-West minors become (i) $2x + 1.5 \geq 1.5 > 0$ for any feasible point, (ii) $1.5(2x + 1.5) - 1 \geq 1.25 > 0$ for all feasible points, (iii) $2.25(2x + 1.25) - 1.25 \geq 1.875 > 0$ for all feasible points. We then conclude that $F$ is actually also strongly monotone on the set $K^{11}$.

\(^{10}\) In fact, we know, are frequently use the fact that, a strict monotonicity of $F$ on a set $K$ is equivalent to strong monotonicity on the same set if at least one of the following two conditions are met (i) $F$ is linear, (ii) $K$ is compact. However in this exercise $F$ is not linear (because of the term $w^3$ and $K$ is unbounded, and therefore we must use the a more direct approach.

\(^{11}\) It should be clear that every time all North-West minors are all bounded away from zero on the feasible set, strict monotonicity entails strong monotonicity. To make a very simple example of what can happen and make strict and strong monotonicity different, consider the function in dimension $F(x) = -e^{-x}$ and $K = \mathbb{R}_+$. The Jacobian of this function (actually its derivative) is $e^{-x}$ (note that in dimension one all Jacobians are necessarily already symmetric) and is always positive on $K$. This implies that $F$ is actually strictly monotone. However if we look at $JF_{sym}(w) - \alpha I$ as we did before, we get $e^{-x} - \alpha$ and it is clear that however small, but positive, we take $\alpha$ if $x$ is large enough, we have $e^{-x} - \alpha < 0$ and so the function is not strongly monotone. The problem here, is that $e^{-x}$ can become as close to 0 as we like on $K$ and therefore however small $\alpha$ we can find feasible points were the condition for strong monotonicity is violated.
(c) In order to apply the basic Projection Algorithm, $F$ must be strongly monotone and Lipschitz continuous on $K$. The strong monotonicity has already been checked in the previous point, so we only need to look at the Lipschitz property, which, we recall, means $\|F(w) - F(v)\| \leq L\|w - v\|$ for all feasible $w$ and $v$ and for some fixed value $L$. We also recall that the intuitive meaning of this property is that the changes in the function values should be bounded by something proportional to the change in the evaluation point. In one dimension this means that the graph of the function should not become "infinitely steep". Given this it should be rather intuitive that the presence of the term $x^2$, with $x$ that can go to infinity makes this $F$ non Lipschitz continuous on $K$ ($x^2$ is a parabola and when $x$ grows to infinity the graph becomes "infinitely steep"). To see this formally consider the points $w = (\alpha, 0, 0)^T$ and $v = (0, 0, 0)^T$, with $\alpha > 0$; note that these two points belong to $K$. We can write

$$\|F(w) - F(v)\| = \sqrt{(\alpha^2 + \alpha)^2 + \alpha^2 + (-\alpha)^2} = \sqrt{\alpha^4 + 2\alpha^2 + 3\alpha}, \quad \|w - v\| = \alpha.$$  

It is clear that however large we take $L$, when $\alpha$ grows enough (note that it can go to infinity on the feasible set) $\sqrt{\alpha^4 + 2\alpha^2 + 3\alpha} \leq L\alpha$ will always be violated, since the term on the left hand-side grows as $\alpha^2$ while the right hand-side grows as $\alpha$. Therefore $F$ is not Lipschitz continuous on the feasible set and the conditions to apply the basic Projection Algorithm are not satisfied.

(d) The Hyperplane Projection Algorithm only requirement is that $F$ be monotone (and continuous). Since our $F$ is certainly monotone, we already checked that is is actually strongly monotone, we can apply the Hyperplane Projection Algorithm to the given MiCP. Since the problem has one and only one solutions, the sequence produced by this algorithm will converge to the solution.

12. Answer YES or NO, explaining very briefly the reason of your answers (citing a theorem, giving a (counter)example or by making very simple reasonings)

(a) An NCP($F$) with $F$ continuous always has a solution.

NO. Note that existence theorems we studied either require $F$ to be strongly monotone or $K$ to be bounded and neither of these conditions is satisfied in the framework of this statement. Counterexample: In dimension one, take $F(x) = -1$; the corresponding NCP($F$) has no solutions. Indeed, if $x = 0$ we should have a nonnegative function value for $0$ to be a solution, and this is not the case; if $x > 0$ the function should instead be zero, a condition that is clearly not satisfied.

(b) If $\bar{x}$ is a solution of a VI($K$, $F$) with $F$ continuous and $K$ polyhedral, then $\bar{x}$ satisfies the KKT system.

YES. $K$ polyhedral means that the constraints are linear and this is a Constraint Qualification that guarantees that the KKT conditions are satisfied at a solution.

(c) A monotone VI can have no solutions.

YES. The NCP example given in (a) is actually monotone and we already showed it has no solutions.

(d) A strictly monotone system of equations $F(x) = 0$ can have at most one solution.

YES. The system of equations is just the VI($\mathbb{R}^n$, $F$) and we know (see Theorem 1.5.12 (b) in the Lecture Notes) that a strictly monotone VI can have at most one solution.
(e) If a function $F$ is strongly monotone on a convex set $K$, then it will be strongly monotone on any convex subset of $K$.

YES. Indeed, $F$ is strongly monotone on $K$ if $(F(x) - F(y))^T (x - y) \geq \alpha \|x - y\|^2$ for some positive constant $\alpha$ and for all $x$ and $y$ in $K$. It is then clear that if we consider a (convex) subset $K'$ of $K$ the same relation, with the same $\alpha$ holds for all points in this subset $K'$. 
Exercises from the 19/12/2018 and 14/01/2019 Exams
With Solutions

1. Consider a Nash Cournot problem where two firms produce the same good and must decide how much to produce: $x$ is the production of the first firm and $y$ is the production of the second firm. The inverse demand function, giving the unitary price in function of the quantity of good in the market, is $100 - x - y$. The first player, controlling $x$, pays $10x$ to produce $x$, the second $2y^2$ to produce $y$. This setting can be therefore be described by the following game

$$
\min_x - (100 - x - y)x + 10x \quad \min_y - (100 - x - y)y + 2y^2 \quad \text{subject to} \quad x \geq 0, y \geq 0
$$

(1)

A regulator imposes a production tax to avoid congestion. The regulator decides that the total quantity of good on the market should not exceed 44 and imposes a tax $T$ on every unit of good produced.

(a) Check that game (1) can be transformed into a VI and state the VI exactly (i.e. give $F$ and $K$).

(b) Check that the corresponding VI is strongly monotone. How many Nash equilibria does this game have?

(c) What is the equilibrium if the regulator does not impose any tax? I.e. what is the equilibrium of game (1)? (use the information that at equilibrium both firms have a positive production)

(d) What should the tax $T$ be to achieve the regulator goal?

(e) (Optional) How does the game (1) modify after the regulator imposes the tax $T$?

(f) (Optional) Check that in the modified game the total production does not exceed 44 at equilibrium, as desired.

(a) Let’s look at the first player first. The objective function can be written as $x^2 + yx - 90x$ and this is clearly $C^1$, since it is a polynomial. Furthermore, for every given $y$, this function is an upward parabola and therefore convex in $x$ (alternatively, you can compute the second derivative with respect to $x$, which simply gives $2 > 0$ to show convexity). Finally, the feasible set is $x \geq 0$ which is (a closed polyhedron and therefore) closed and convex. We can reason similarly for the second player and conclude that all conditions to transform this game into a VI are satisfied. The corresponding VI($K, F$) is given by

$$
K = X_1 \times X_2 = \{(x, y) : x \geq 0, y \geq 0\}, \quad F(x, y) = \left( \begin{array}{c}
\nabla_x f_1 \\
\nabla_y f_2
\end{array} \right) = \left( \begin{array}{c}
2x + y - 90 \\
6y + x - 100
\end{array} \right).
$$

It may be worth noticing that since the feasible set $K$ is the non negative orthant, this VI is actually the NCP($F$).

(b) Since $F$ is continuously differentiable, we can look at its Jacobian:

$$
\begin{pmatrix}
2 & 1 \\
2 & 6
\end{pmatrix}
$$

which is already symmetric. We can perform the north-west minors test: (a) $2 > 0$ and (b) $2 \cdot 6 - 1 > 0$ and see that $JF$ is positive definite. This is sufficient to conclude that $F$ is strictly monotone, but since $F$ is also linear, strict and strong monotonicity coincide and we can conclude that $F$ is strongly monotone. Because of this we can conclude that the VI has one and only one solution. Furthermore, since the solutions of the VI and of the original NEP coincide, we also conclude that the Nash-Cournot equilibrium problem has one and only one solution.
(c) We are told that at equilibrium both firms have a positive production. This means that no constraints are active and that the equilibrium solution is in the interior of the feasible set. Therefore we can compute the solution by computing the solution of the linear system of two equations in two variables $F(x, y) = (0, 0)$. This easily gives $x = 40$ and $y = 10$.

(d) To achieve the regulator goal the regulator must first find a variational solution of the GNEP

\[
\begin{align*}
\min_x & \quad -(100 - x - y)x + 10x \\
\min_y & \quad -(100 - x - y)y + 2y^2 \\
x \geq 0 & \quad y \geq 0 \\
x + y \leq 44 & \quad x + y \leq 44
\end{align*}
\]

and compute the multiplier of the shared constraint: this will be the tax to be imposed.

We obviously are in the position to write the VI that allows us to compute the variational equilibrium. Indeed, the shared constraint is the same for all players, and it is convex in $x$ and $y$, all the other necessary conditions were already checked in a). The corresponding VI($K, F$) has the same $F$ in a) while the feasible set is

\[ K = X_1 \times X_2 = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 44\}. \]

Note that also this new VI is strongly monotone, since the $F$ did not change: we then see that there is one and only one variational solution of the GNEP (although the GNEP can possibly have other solutions that are not solutions of the VI).

The KKT conditions for this VI are

\[
\begin{align*}
2x + y - 90 - \lambda_1 + \lambda_3 &= 0 \\
6y + x - 100 - \lambda_2 + \lambda_3 &= 0 \\
0 &\leq \lambda_1 \perp x \geq 0 \\
0 &\leq \lambda_2 \perp y \geq 0 \\
0 &\leq \lambda_3 \perp x + y - 44 \leq 0.
\end{align*}
\]

We must find the unique solution of the VI. We can proceed by inspection; it seems very reasonable to start assuming that both firm have a positive production and that together they produce 44. Remember that without the constraint $x + y \leq 44$ at equilibrium the total production is 50, so assuming that now will be 44, i.e. that the constraint $x + y \leq 44$, seems very reasonable. Please note that this is not crucial, we are just starting our inspection by assuming $x > 0$, $y > 0$, and $x + y = 44$. If we find a solution in this case we are done, and we have found the unique variational equilibrium, if we do not find a solution with these conditions we will simply go ahead examining other cases, until we find the solution. Assuming then $x > 0$, $y > 0$, and $x + y = 44$ the KKT systems reduces to

\[
\begin{align*}
2x + y - 90 + \lambda_3 &= 0 \\
6y + x - 100 + \lambda_3 &= 0 \\
x + y - 44 &= 0 \\
\lambda_3 &\geq 0,
\end{align*}
\]

with $\lambda_1 = \lambda_2 = 0$. The first three equations form a (nonsingular) system of three equations in three variables. We can compute its solution, which turns out to be $x = 35$, $y = 9$, and $\lambda_3 = 11$. Since $\lambda_3$ is non negative we see that $x = 35$, $y = 9$, $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = 11$ solves the KKT system and therefore $x = 35$, $y = 9$ is the variational solution of the GNEP. The taxation is $T = \lambda_3 = 11$. 
(e) After the regulator imposes a taxation the player's problem becomes

\[
\begin{align*}
\min_x & - (100 - x - y)x + 10x + 11(x + y - 44) \\
\text{subject to } & x \geq 0
\end{align*}
\]

\[
\begin{align*}
\min_y & - (100 - x - y)y + 2y^2 + 11(x + y - 44) \\
\text{subject to } & y \geq 0
\end{align*}
\]

or, recalling that the first player minimizes with respect to \( x \) and the second with respect to \( y \) and quantities not depending on \( x \) or \( y \) respectively can be considered as constants that therefore do not influence the minimization process, as

\[
\begin{align*}
\min_x & -(100 - x - y)x + 10x + 11 \\
\text{subject to } & x \geq 0
\end{align*}
\]

\[
\begin{align*}
\min_y & -(100 - x - y)y + 2y^2 + 11y \\
\text{subject to } & y \geq 0
\end{align*}
\]

(f) We know already from the theory that the solution of this new game with taxation should be \( x = 35, y = 9 \), which give a total production of 44, so let's just check whether this is true. We can do so in several ways, we just see two methods to check the result.

(a) The new game with taxation can obviously be transformed into an equivalent VI (by a) we know all the necessary conditions are satisfied, since we only added a linear term into each objective function), therefore the NEP with taxation is equivalent to the VI \((K, F)\), where

\[
\begin{align*}
K &= X_1 \times X_2 = \{(x, y) : x \geq 0, y \geq 0\}, \\
F(x, y) &= \left( \begin{array}{c}
\nabla_x f_1 \\
\nabla_y f_2
\end{array} \right) = \left( \begin{array}{c}
2x + y - 79 \\
6y + x - 89
\end{array} \right).
\end{align*}
\]

It is easy to see now that \( F(35, 9) = (0, 0) \) and therefore that \( x = 35, y = 9 \) is the unique solution of this VI and therefore the unique solution of the NEP with taxation.

(b) Alternatively we could also check that \( x = 35, y = 9 \) is solution of the NEP with taxation by using the definition of Nash equilibrium. So, for the first player, set \( y = 9 \); her optimization problem becomes

\[
\begin{align*}
\min_x & x^2 - 70x \\
\text{subject to } & x \geq 0
\end{align*}
\]

The objective function is an upward parabola whose vertex is \( x = 35 \); since this is feasible for the problem, this is also its optimal solution. Analogously, for the second player, set \( x = 35 \); her problem becomes

\[
\begin{align*}
\min_y & 3y^2 - 54y \\
\text{subject to } & x \geq 0
\end{align*}
\]

Again, the objective function is an upward parabola whose vertex is \( y = 9 \); since this is feasible for the problem, this is also its optimal solution. We have therefore checked, by using the definition of Nash equilibrium, that \( x = 35, y = 9 \) is actually a Nash equilibrium. Note that without the conversion to VI it is harder to ascertain that this is actually the unique Nash equilibrium of the game, a fact that we could derive in a) by observing that the VI is strongly monotone.
2. Consider a finite game with two players where the functions of the two players are defined by
the following two matrices (the players minimize and the first player control the rows while
the second the columns)

\[
A = \begin{pmatrix}
1 & -2 & -1 \\
0 & -1 & 2
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 0 & -2 \\
2 & -1 & 1
\end{pmatrix}
\]

(a) Find, by inspection, the unique Nash equilibrium of this game
(b) Formulate the mixed strategy extension of this game
(c) Check that the Nash equilibrium of the original game corresponds to an equilibrium in
the extended game
(d) Does any of the players have a weakly dominated strategy?

(a) There are 6 possibilities (1,1), (1,2), (1,3), (2,1), (2,2), and (2,3). Let’s check them one
by one.

(1,1) is not an equilibrium because, if the second player plays 1, for the first player is
better to switch to 2.

(1,2) is not an equilibrium because if the first player plays 1, for the second player will
be better off by switching to 3.

(1,3) instead is a Nash equilibrium because both players cannot improve their outcome
by unilaterally deviating from 1 and 3 respectively. Indeed, -1 is the minimum over the
third column of A and -2 is the minimum on the first row of B.

Since in the text it is written that this game has a unique Nash equilibrium we can stop
here and conclude that the unique Nash equilibrium is (1,3).

(b) The mixed strategies extension of this game is

\[
\min_x \quad x^T A y \\
x_1 + x_2 = 1 \\
x_1, x_2 \geq 0
\]

\[
\min_y \quad x^T B y \\
y_1 + y_2 + y_3 = 1 \\
y_1, y_2, y_3 \geq 0
\]

Substituting to A and B the values in the text, we obtain a more explicit version of the
game

\[
\min_x \quad x_1 y_1 - 2x_1 y_2 - x_1 y_3 - x_2 y_2 + 2x_2 y_3 \\
x_1 + x_2 = 1 \\
x_1, x_2 \geq 0
\]

\[
\min_y \quad x_1 y_1 - 2x_1 y_2 + 2x_2 y_1 - x_2 y_2 + x_2 y_3 \\
y_1 + y_2 + y_3 = 1 \\
y_1, y_2, y_3 \geq 0
\]

(c) The original Nash equilibrium, (1,3) corresponds, in mixed strategies, to the strategy
(1,0),(0,0,1). We check, using the definition, that this corresponds to a Nash equi-
librium for the extended game. Substituting the strategy (0,0,1) in the first player
problem, we obtain

\[
\min_x \quad -x_1 + 2x_2 \\
x_1 + x_2 = 1 \\
x_1, x_2 \geq 0
\]

and it is clear that (1,0) is actually the optimal solution of this problem, because “all
the weight” is given to x_1 that has the smallest coefficient on the objective function.
Analogously, if we substitute (1,0) in the second players problem, we get

\[
\min_y \quad y_1 - 2y_3 \\
y_1 + y_2 + y_3 = 1 \\
y_1, y_2, y_3 \geq 0
\]
and again, it is clear that \((0, 0, 1)\) is the optimal solution because "all the weight" is given to \(y_3\) that has the smallest coefficient on the objective function. It may be useful to point out that in this case \(y_2\) has coefficient 0.

(d) A strategy \(a\) is weakly dominated by another strategy \(b\) if, whatever strategy the other player chooses, the cost incurred by playing \(a\) is always greater or equal than the cost incurred by playing \(b\) and there is at least one other player’s choice for which the cost of playing \(a\) is actually worst than that of playing \(b\). The first player has not dominated strategy because if the second player plays 1 than for the first player it is better to play 2, while if the second player plays 2, for the first player it is better to play 1. on the other hand, the first strategy of the second player is dominated by both the second and third strategy of that player. Indeed, whatever the choice of the first player, the costs incurred by playing 1 are always greater than those incurred by playing 2 or 3. Please note for terming strategy 1 “dominated” it is not necessary that it is dominated by both 2 and 3, it is enough that it dominated by 1 other strategy.

3. Consider an antagonistic game (i.e. a finite, zero-sum, two-players game) described by the following matrix

\[
A = \begin{pmatrix}
-1 & 2 & 1 \\
0 & 1 & 3
\end{pmatrix}
\]

(a) Does this game have a Nash equilibrium?
(b) Formulate the two LPs that allow you to compute the mixed strategy solution
(c) Check any of the following pairs is a solution in mixed strategies: (a) \((1,0)\) and \((0,0,1)\); (b) \((2/3, 1/3)\) and \((0, 2/3, 1/3)\); (c) \((1/3, 1/3)\) and \((1/3, 1/3, 1/3)\).

(a) By taking the maximum along the rows and the minimum along the columns we get

\[
\begin{pmatrix}
-1 & 2 & 1 \\
0 & 1 & 3
\end{pmatrix}
\begin{pmatrix}
2 \\
3 \\
1
\end{pmatrix}
\]

Since \(\min\{2,3\} \neq \max\{-1,1,1\}\) we know that no (pure strategy) Nash equilibrium exists.

(b) The two LP are

\[
\begin{align*}
\min_{x, \lambda} & \quad \lambda \\
\text{s.t.} & \quad \lambda \geq -x_1 \\
& \quad \lambda \geq 2x_1 + x_2 \\
& \quad \lambda \geq x_1 + 3x_3 \\
& \quad x_1 + x_2 = 1 \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y, \mu} & \quad \mu \\
\text{s.t.} & \quad \mu \leq -y_1 + 2y_2 + y_3 \\
& \quad \mu \leq y_2 + 3y_3 \\
& \quad y_1 + y_2 + y_3 = 1 \\
& \quad y \geq 0
\end{align*}
\]
(c) The couple (a) corresponds to the (1,3) solution in pure strategies. Since we already found out that there are no pure strategies solution, this (1,0), (0,0,1) cannot be a solution in mixed strategies. The couple (c) also cannot be an equilibrium, because the strategy proposed for the first players, (1/3, 1/3) is not feasible for the first LP above, in that 1/3 + 1/3 ≠ 1. So we are only left with (b). If we substitute the indicated values in the two LPs we get

\[
\begin{align*}
\min_{x, \lambda} & \quad \lambda \\
\lambda & \geq -2/3 \\
\lambda & \geq 5/3 \\
\lambda & \geq 5/3 \\
2/3 + 1/3 & = 1 \\
2/3, 1/3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{\mu, \nu} & \quad \mu \\
\mu & \leq 5/3 \\
\mu & \leq 5/3 \\
y_1 + y_2 + y_3 & = 1 \\
y & \geq 0
\end{align*}
\]

We see then that the points \((\lambda, x_1, x_2) = (5/3, 2/3, 1/3)\) and \((\mu, y_1, y_2, y_3) = (5/3, 0, 2/3, 1/3)\) are feasible for the two LPs. Since these two LPs are dual one of the other and the objective function value is the same, 5/3, we know that two given points are optimal solution of the two LPs and, therefore we can conclude that the solution in (b) is a Nash equilibrium for the extended game.

4. Consider the VI\((K, F)\), where

\[
F = \begin{pmatrix}
x^2y + z - 1 \\
3x + 2y + z - 2 \\
2x - y + z^2 + 1
\end{pmatrix}, \quad K = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}
\]

Answer the following questions, explaining the reasons of your answers.

(a) Can you be sure that a solution exist?

(b) A solution for this KKT will surely satisfy the KKT conditions?

(c) By using the KKT conditions, tell which among the following points is a solution

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

(d) Is \(F\) monotone on \(K\)?

(a) Yes, because \(F\) is continuous, being defined by polynomials, and the feasible set is closed convex, because it is a polyhedron, and bounded, because all variables can only take values in \([0, 1]\). Therefore, for the fundamental theorem about the existence of a solution of a VI, we have that a solution must exists (remember that closed + bounded = compact).

(b) Yes, because the constraints are linear, which is a CQ, and therefore every solution will satisfy the KKT conditions by a theorem we proved.

(c) (a) \((0,0,0)\) is feasible and \(F(0,0,0) = (-1, -2, 1)\). It should be clear geometrically that this is not a solution, but we check that using the KKT conditions. The KKT system for this VI is
\[ x^2y + z - 1 - \lambda_1 + \lambda_4 = 0 \\
3x + 2y + z - 2 - \lambda_2 + \lambda_4 = 0 \\
2x - y + x^2 + 1 - \lambda_3 + \lambda_4 = 0 \\
0 \leq \lambda_1 \perp x \geq 0 \\
0 \leq \lambda_2 \perp y \geq 0 \\
0 \leq \lambda_3 \perp z \geq 0 \\
0 \leq \lambda_4 \perp x + y + z - 1 \leq 0. \]

In \((0, 0, 0)\) the first three constraint are active, while the last one is not, thus implying \(\lambda_4 = 0\). Taking into account this, the KKT system is \((0,0,0)\) reduces to

\[-1 - \lambda_1 = 0 \\
-2 - \lambda_2 = 0 \\
1 - \lambda_3 = 0 \\
0 \leq \lambda_1 \\
0 \leq \lambda_2 \\
0 \leq \lambda_3 \]

from which we get, for example \(\lambda_1 = -1 < 0\), showing that \((0,0,0)\) does not satisfy the KKT conditions and therefore, see (b), this point is not a solution of the VI.

Consider now the point \((1,0,0)\). The point is feasible, and in this point the second, third and fourth constraints are active, implying \(\lambda_1 = 0\). Since \(F(1,0,0) = (-1, 1, 3)\), the KKT system reduces to

\[-1 + \lambda_4 = 0 \\
1 - \lambda_2 + \lambda_4 = 0 \\
3 - \lambda_3 + \lambda_4 = 0 \\
0 \leq \lambda_2 \\
0 \leq \lambda_3 \\
0 \leq \lambda_4. \]

It is easily seen them that it must be \(\lambda_2 = 2\), \(\lambda_3 = 4\), and \(\lambda_4 = 1\) and that the point \((1,0,0)\), together with the multipliers \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 2, 4, 1)\) satisfies the KKT system and is therefore a solution of the VI.

Finally, the point \((1,1,1)\) is not feasible, since \(1 + 1 + 1 \neq 1\), and therefore this point cannot be a solution of the VI.
5. Consider the VI\((K,F)\), where

\[
F = \begin{pmatrix} x^2 y + z - 1 \\ 3x + 2y + z - 2 \\ 2x - y + z^2 + 1 \end{pmatrix}, \quad K = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y \leq 1\}
\]

Answer the following questions, explaining the reasons of your answers.

(a) Can you be sure that a solution exist?
(b) Will a solution for this KKT surely satisfy the KKT conditions?
(c) By using the KKT conditions, tell which among the following points is a solution

\[
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

This exercise is practically identical to the previous one except that the feasible set is different, in that the last linear constraint is \(x + y \leq 1\) instead of \(x + y + z \leq 1\). Given the similarities we only discuss point (a). In this course, we only have two tools to establish existence of solution to a VI: (i) the basic existence theorem requiring continuity of \(F\) and compactness of \(K\) and (ii) the result that says that a strongly monotone VI has one and only one solution. In this case we cannot apply (i), because the set \(K\) is not bounded (and therefore it is not compact; remember, compactness = closedness + boundedness). In fact, it is clear that the onyl constraint on \(z\) is \(z \geq 0\) and \(z\) can therefore take arbitrarily large values without violating any constraint. For example, the point \((0,0,\alpha)\) is feasible for any non negative \(\alpha\) and \(\alpha\) can be as large as we want. Therefore we can try (ii). Since the \(F\) is continuously differentiable we can look at its Jacobian. We have

\[
JF(x, y, z) = \begin{pmatrix} 2xy & x^2 & 1 \\ 3 & 2 & 1 \\ 2 & -1 & 2z \end{pmatrix}, \quad JF_z(x, y, z) = \begin{pmatrix} 2xy \\ (x^2 + 3)/2 \\ 3/2 \end{pmatrix}, \quad JF_x = \begin{pmatrix} (x^2 + 3)/2 \\ 2 \\ 0 \end{pmatrix}, \quad JF_y = \begin{pmatrix} 3/2 \\ 2 - \alpha \\ 0 \end{pmatrix}
\]

We recall that \(F\) is strongly monotone if a positive constant \(\alpha\) exists such that the symmetric part of the Jacobian, \(JF(x, y, z)_s\), is such that \(JF(x, y, z)_s - \alpha I\) is positive semidefinite for all points \((x, y, z) \in K\). It is then clear that \(F\) is not strongly monotone, in fact \((0,0,0)\) is feasible and

\[
JF_z(0,0,0) - \alpha I = \begin{pmatrix} -\alpha & 3/2 & 3/2 \\ 3/2 & 2 - \alpha & 0 \\ 3/2 & 0 & -\alpha \end{pmatrix}
\]

But we know that a monotone matrix cannot have any diagonal element which is negative and since \(\alpha\) is fixed but positive, \(JF(0,0,0)_s - \alpha I\) cannot be semidefinite.

We then conclude that we cannot say \textit{a priori} whether this VI has a solution. We remark that this does not mean the VI has no solutions, only that we are not able to predict whether it surely has one. Indeed, we saw in the previous exercise, and this can be checked also for this one, that \((1,0,0)\) is indeed a solution.

6. Consider the NCP\((F)\) with

\[
F = \begin{pmatrix} e^{\alpha(x+1)} + y - 1 \\ 3xy + 1 - \alpha \end{pmatrix}
\]

For which values of the parameter \(\alpha\) the origin is a solution of NCP\((F)\)? Take \(\alpha = 0\) and find by inspection all solutions of the corresponding NCP\((F)\).
(a) For a \((0,0)\) to be a solution of and NCP\((F)\), we must have \(F(0,0) \geq (0,0)\). We have \(F(0,0) = (e^\alpha - 1, 1 - \alpha)\). These two components of \(F\) are easily seen to be non negative for \(\alpha \in [0,1]\) and it is exactly for those \(\alpha\) that \((0,0)\) is a solution of the NCP\((F)\).

(b) For \(\alpha = 0\) we get \(F(x,y) = (y, 3xy + 1)\). We already know that the origin is a solution, we are therefore left with the inspection of the three regions \(R_1 = \{x = 0, y > 0\}\), \(R_2 = \{y = 0, x > 0\}\), and \(R_3 = \{x > 0, y > 0\}\).

\(R_1\). This is the positive \(y\)-semiaxis. In this region, a point is a solution if the first component of \(F\) is non negative and the second is 0. Since the second component in this case is always 1, no point in \(R_1\) can be a solution.

\(R_2\). This is the positive \(x\)-semiaxis. In this region, a point is a solution if the first component of \(F\) is 0 and the second is non negative. In this region, we have \(F(x,0) = (0,1)\). Therefore, all points in \(R_2\) are solutions.

\(R_3\). In this region, the interior of the feasible region, a point is a solution if \(F(x,y) = (0,0)\). But it is easy to see that if both \(x\) and \(y\) are positive, we have that both components of \(F\) are positive and so we conclude that there are no solution in \(R_3\).

In conclusion, we see that the solution of the NCP for \(\alpha = 0\) are all the point on the non negative \(x\)-semiaxis, that is all points in \(\{(x,y): x \geq 0, y = 0\}\).

7. Answer YES or NO, explaining very briefly the reason of your answers (citing a theorem, giving a (counter)example or by making very simple reasonings)

- A VI always has a solution. NO, for example the VI\((\mathbb{R}, e^x)\), equivalent to \(e^x = 0\), has no solution.

- If a point satisfies the KKT system of a VI, the corresponding \(x\) is a solution of the VI. YES, there is a theorem stating so.

- A monotone VI can have exactly 2 solutions. NO, the solution set of a monotone VI is convex and therefore if it contains two distinct points it must contain the whole segment joining those two points.

- A strictly monotone system of equations \(F(x) = 0\) can have at most one solution. YES, there is a theorem stating so.

- If a function \(F\) is monotone on a convex set \(K\), then it will be monotone on any convex subset of \(K\). YES, the definition of monotonicity on a set requires that a certain inequality be satisfied for any pair of points in the set. Therefore if this inequality is satisfied for any two points in \(K\), it will be satisfied also for any two points in a (convex) subset of \(K\).

- A contraction on a closed set can have exactly 2 fixed points. NO, there is a theorem stating that a contraction has one and only one fixed point.

- The FB function is differentiable. NO, it is not differentiable in the origin (its square is differentiable everywhere, though).

- A zero-sum game always admits a Nash equilibrium. NO, we studied that a certain condition must be satisfied for a finite zero-sum game to have an equilibrium so in general the answer is no. Furthermore, the exercise 3 above gives an example of a finite zero-sum game with no solution (in pure strategies).

- The mixed extension of an antagonistic game can have more than one solution. YES.

- In a zero-sum finite game conservative strategies always exist and are unique. NO, they always exist but need not be unique. Again, in exercise 3 above, the second player has two conservative strategies: 2 and 3.
8. Consider the following GNEP

\[
\begin{align*}
\min_x & \quad \frac{1}{2}x^4 - yx - x \\
\text{subject to} & \quad x \geq 1 \\
& \quad x + y \leq 3\sqrt{2}
\end{align*}
\]

\[
\begin{align*}
\min_y & \quad \frac{1}{2}(y - 2x)^2 - y \\
\text{subject to} & \quad y \geq -1 \\
& \quad x + y \leq 3\sqrt{2}
\end{align*}
\]

(a) Show (with some, minimal, detail, not just stating that the assumptions are satisfied!) that this GNEP can be associated to a VI and write down this VI.

(b) Check whether this VI strongly monotone (on its feasible set).

(c) How many variational solutions does the GNEP have?

(d) Write down the KKT system for this VI.

(e) Use the KKT system to check which one is a solution

\[
\begin{pmatrix}
1 \\
-1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
2\sqrt{2}
\end{pmatrix}, \quad \begin{pmatrix}
\sqrt{2} \\
2\sqrt{2}
\end{pmatrix}
\]

(f) Suppose that the game is the one described above without the constraint \(x + y \leq 3\sqrt{2}\) (that is, suppose the game is a NEP and not a GNEP). What taxation should a regulator impose in order for the solution of this NEP to be the variational solution found before? Write down the NEP the players solve after the regulator imposed the taxation.

(a) This is a GNEP with shared constraints. The private constraints \((x \geq 1 \text{ and } y \geq -1)\) are linear so that the private feasible sets are polyhedrons and therefore closed and convex. The shared constraint, \(x + y \leq 3\sqrt{2}\), is linear and therefore convex in both constraints. The objective functions are polynomials and therefore continuously differentiable. Let's check the convexity requirement. The objective function of the first player is \(\frac{1}{2}x^4 - yx - x\) and, for every fixed \(y\), \(-yx - x\) is linear in \(x\), while the term \(\frac{1}{2}x^4\) is also convex. In fact, the second derivative of this term is \(3x^2\) which is always non-negative and therefore \(\frac{1}{2}x^4\) is convex. Therefore, for every fixed \(y\) the objective function of the first player is the sum of convex functions in \(x\) and is therefore convex in \(x\). Consider now the second objective function, \(\frac{1}{2}(y - 2x)^2 - y\). For a fixed value of \(x\) this is an upward parabola and therefore convex in \(y\). Therefore we can associate to the GNEP the VI\((K, F)\), with

\[
K = \{(x, y) : x \geq 1, y \geq -1, x + y \leq 3\sqrt{2}\}, \quad F(x, y) = \begin{pmatrix}
x^2 - y - 1 \\
x - 2x - 1
\end{pmatrix}.
\]

We recall that for a GNEP we know that every solution of the associated VI is a solution of the GNEP but not all solutions of the GNEP are necessarily solutions of the VI.

(b) Let's check whether the VI is strictly monotone. We have

\[
JF(x, y) = \begin{pmatrix}
3x^2 & -1 \\
-2 & 1
\end{pmatrix}, \quad JF_s(x, y) = \begin{pmatrix}
3x^2 & -3/2 \\
-3/2 & 1
\end{pmatrix}.
\]

We check the north-west minors, they are (i) \(3x^2\) and (ii) \(3x^2 - 9/4\). Since on the feasible set we always have \(x \geq 1\), we see easily that for every feasible point (i) \(3x^2 \geq 3 > 0\) and (ii) \(3x^2 - 9/4 \geq 3 - 9/4 = 3/4 > 0\). Therefore we know that the VI has at most one solution. Since the feasible region is compact we also know that this VI has at least one solution. We can conclude that the VI has exactly one solution. Since the variational equilibria of the GNEP are by definition the solutions of the associated VI, we can conclude that this VI has exactly one variational equilibrium. Alternatively, in order to see that the VI has exactly one solution we could reason as follows: \(F\) is strictly monotone on \(K\), but since \(K\) is compact, this implies that \(F\) is also strongly monotone on \(K\) and therefore we can conclude the VI has one and only one solution.
(c) The KKT system for this VI is

\[
\begin{align*}
\quad x^3 - y - 1 - \lambda_1 + \lambda_3 &= 0 \\
\quad y - 2x - 1 - \lambda_2 + \lambda_3 &= 0 \\
\quad 0 \leq \lambda_1 &\perp 1 - x \leq 0 \\
\quad 0 \leq \lambda_2 &\perp -1 - y \leq 0 \\
\quad 0 \leq \lambda_3 &\perp x + y - 3\sqrt{2} \leq 0
\end{align*}
\]

(d) Let’s examine the three points

The point \((1, -1)\) is feasible and we have \(F(1, -1) = (1, -4)\). The active constraints are the first two, thus implying \(\lambda_3 = 0\). The KKT system therefore reduces to

\[
\begin{align*}
1 - \lambda_1 &= 0 \\
-4 - \lambda_2 &= 0 \\
0 &\leq \lambda_1 \\
0 &\leq \lambda_2
\end{align*}
\]

The second equality implies \(\lambda_2 = -4\) contradicting \(\lambda_2 \geq 0\). Therefore \((1, -1)\) does not satisfy the KKT conditions.

The point \((1, 2\sqrt{2})\) is feasible and we have \(F(1, 2\sqrt{2}) = (-2\sqrt{2}, 2\sqrt{2} - 3)\). The only active constraint is the first one thus implying \(\lambda_2 = 0 = \lambda_3\). The KKT system therefore reduces to

\[
\begin{align*}
-2\sqrt{2} - \lambda_1 &= 0 \\
2\sqrt{2} - 3 &= 0 \\
0 &\leq \lambda_1
\end{align*}
\]

and the second equality is impossible thus showing that also \((1, 2\sqrt{2})\) does not satisfy the KKT system (we could also observe that the first equality gives \(\lambda_1 < 0\) to conclude that the KKT system is not satisfied).

The point \((\sqrt{2}, 2\sqrt{2})\) is feasible and we have \(F(\sqrt{2}, 2\sqrt{2}) = (-1, -1)\). The only active constraint is the third one thus implying \(\lambda_3 = 0 = \lambda_2\). The KKT system therefore reduces to

\[
\begin{align*}
-1 + \lambda_3 &= 0 \\
-1 + \lambda_3 &= 0 \\
0 &\leq \lambda_3
\end{align*}
\]

which gives \(\lambda_3 = 1\). We therefore see that \((\sqrt{2}, 2\sqrt{2})\) with the multipliers \((\lambda_1, \lambda_2, \lambda_3) = (0, 0, 1)\) satisfies the KKT system and is therefore a variational equilibrium.

We can conclude that of the three proposed point only \((\sqrt{2}, 2\sqrt{2})\) is a variational equilibrium of the GNEP.

9. Consider a finite game with two players where the functions of the two players are defined by the following two matrices (player 1 controls the rows, player 2 the columns)

\[
A = \begin{pmatrix}
1 & -1 & \alpha \\
0 & -2 & 2
\end{pmatrix} \quad B = \begin{pmatrix}
-1 & 0 & -2 \\
2 & 1 & 1
\end{pmatrix}
\]

(a) For which values of \(\alpha\) is \((2,3)\) a Nash equilibrium of this game?

(b) For which values of \(\alpha\) does the first player have a weakly dominated strategy? Which one?

(c) Does the second player have a weakly dominant strategy?
(d) Compute the solution of this game (if any) for $\alpha = 0$

(e) Compute the conservative strategies for the two players (of course, the first player conservative strategy depends on $\alpha$, detail its expression).

(a) For $(2,3)$ to be a Nash equilibrium we must have, according to the definition of Nash equilibrium and recalling that the first player controls the rows and the second the columns,

$$A(2,3) \leq A(1,3)$$
$$B(2,3) \leq B(2,1) \text{ and } B(2,3) \leq B(2,2).$$

The inequalities on $B$ do not depend on $\alpha$ and are satisfied, while the one on $A$ require $2 \leq \alpha$. So $(2,3)$ is a Nash equilibrium for every $\alpha \geq 2$.

(b) Since $A(2,1) < A(1,1)$ and $A(2,2) < A(1,2)$ it is clear that $1$ is never dominant for the first player. For $2$ to be dominant we must have $A(2,3) \leq A(1,3)$, i.e. $2 \leq \alpha$ (Note the use of $<$ and $\leq$; since for the first player we have the first two elements in the second row strictly less than the corresponding elements in the first row, in order to have a weakly dominant strategy, it is enough to have $A(2,3) \leq A(1,3)$; had the exercise asked for a dominant strategy we should have required $A(2,3) < A(1,3)$, i.e. $2 < \alpha$).

(c) The function of the second player, described by $B$, does not depend on $\alpha$ and it is easy to see that the third strategy is weakly dominant because

$$B(1,3) < B(1,1), \quad B(2,3) < B(2,1), \quad B(1,3) < B(1,2), \quad B(2,3) \leq B(2,2).$$

(Again note that since we actually have $B(2,3) = B(2,2)$, $3$ is a weakly dominant strategy, but not a dominant strategy, with strategy 1 dominated by strategy 3 and strategy 2 weakly dominated by strategy 3).

(d) For $\alpha = 0$ we get

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 2 \end{pmatrix} \quad \quad B = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

and it is easy to check by inspection that there are two Nash equilibria: $(1, 3)$ and $(2, 2)$.

(e) The worst case function for player two is $\tilde{C}_2(y) = (2,1,1)$ and it is clear that the conservative strategies, i.e. the strategies minimizing the worst case scenario, are 2 and 3. For the first player we have $\tilde{C}_1(x) = \max\{1, \alpha\}, 2$. Therefore it is clear that (i) if $\alpha < 2$ the conservative strategy is 1, (ii) if $\alpha = 2$ both 1 and 2 are conservative strategies and if $\alpha > 2$ then the conservative strategy is 2.

10. Consider an antagonistic game (i.e. a finite, zero-sum, two-players game) described by the following matrix

$$A = \begin{pmatrix} 0 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

(a) Does this game have a Nash equilibrium?

(b) Formulate the two LPs that allow you to compute the mixed strategy solution

(c) Check whether any of the following pairs is a solution in mixed strategies: (a) $(0,1)$ and $(0,1,0)$; (b) $(2/3, 1/3)$ and $(0, 2/3, 1/3)$; (c) $(1/2, 1/2)$ and $(0, 1/2, 1/2)$. 

(a) By taking the maximum along the rows and the minimum along the columns we get

\[
\begin{pmatrix}
0 & 3 & 1 \\
-1 & 1 & 3 \\
-1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
3 \\
3
\end{pmatrix}
\]

Since \(\min\{3,3\} \neq \max\{-1,1,1\}\) we know that no (pure strategy) Nash equilibrium exists.

(b) The two LP are

\[
\begin{align*}
\min_{x_1, x_2} & \quad \lambda \\
\lambda & \geq -x_2 \\
\lambda & \geq 3x_1 + x_2 \\
\lambda & \geq x_1 + 3x_3 \\
x_1 + x_2 & = 1 \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y_1, y_2} & \quad \mu \\
\mu & \leq 3y_2 + y_3 \\
\mu & \leq -y_1 + y_2 + 3y_3 \\
y_1 + y_2 + y_3 & = 1 \\
y & \geq 0
\end{align*}
\]

(c) The pair (a) corresponds to the (2,2) solution in pure strategies. Since we already found out that there are no pure strategies solution, this (0,1), (0,1,0) cannot be a solution in mixed strategies.

Consider now the pair (b). If we substitute the indicated values in the two LPs we get

\[
\begin{align*}
\min_{x_1, x_2} & \quad \lambda \\
\lambda & \geq -1/3 \\
\lambda & \geq 7/3 \\
\lambda & \geq 5/3 \\
2/3 + 1/3 & = 1 \\
2/3, 1/3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y_1, y_2, y_3} & \quad \mu \\
\mu & \leq 7/3 \\
\mu & \leq 5/3 \\
0 + 2/3 + 1/3 & = 1 \\
0, 2/3, 1/3 & \geq 0
\end{align*}
\]

We see then that the points \((\lambda, x_1, x_2) = (7/3, 2/3, 1/3)\) and \((\mu, y_1, y_2, y_3) = (5/3, 0, 2/3, 1/3)\) are feasible for the two LPs. Since these two LPs are dual one of the other and the objective function values are different, we know that two given points are not optimal solution of the two LPs and, therefore, we can conclude that the solution in (b) is not a Nash equilibrium for the extended game.

Consider now the pair in (c). If we substitute the indicated values in the two LPs we get

\[
\begin{align*}
\min_{x_1, x_2} & \quad \lambda \\
\lambda & \geq -1/2 \\
\lambda & \geq 2 \\
\lambda & \geq 2 \\
2/3 + 1/3 & = 1 \\
2/3, 1/3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y_1, y_2, y_3} & \quad \mu \\
\mu & \leq 2 \\
\mu & \leq 2 \\
0 + 2/3 + 1/3 & = 1 \\
0, 2/3, 1/3 & \geq 0
\end{align*}
\]

We see then that the points \((\lambda, x_1, x_2) = (2, 1/2, 1/2)\) and \((\mu, y_1, y_2, y_3) = (2, 0, 1/2, 1/2)\) are feasible for the two LPs. Since these two LPs are dual one of the other and the objective function values are equal these are optimal solutions of the two LPs and the indicated pair is a Nash equilibrium for the extended game.

11. Consider the NCP(F) with

\[
F = \begin{pmatrix}
2x_1 - x_2 - x_3 - 1 \\
x_1 + x_2^2 + x_2 + x_3 \\
-x_2 + 4x_3 - 4
\end{pmatrix}
\]
(a) Is \( F \) strongly monotone on \( \mathbb{R}^3_+ \)? (Please, note this function is non linear and the set is unbounded, so you have to resort to the definition...but it is very easy to answer)

(b) is the point
\[
\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]
a solution of this NCP?

(c) Write down the KKT system for this NCP and check your answer in the previous point using the KKT conditions.

(d) Reformulate the NCP as a system of equations \( \Phi(x) = 0 \) by using the Fischer-Burmeister function (please, write the complete system).

(a) Since the \( F \) is continuously differentiable we can look at its Jacobian:

\[
JF(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 2x_2 + 1 & 1 \\ 0 & -1 & 4 \end{pmatrix}, \quad JF_a(x_1, x_2, x_3) = \begin{pmatrix} 2 & 0 & -1/2 \\ 0 & 2x_2 + 1 & 0 \\ -1/2 & 0 & 4 \end{pmatrix}.
\]

Since, as indicated in the text, this function is not linear and the set \( K \) is unbounded, we use the condition on the symmetric part of the Jacobian and simply check whether, for some positive and fixed value of \( \alpha \) the matrix \( JF_a(x_1, x_2, x_3) - \alpha I \) is positive semidefinite.

\[
JF_a(x_1, x_2, x_3) - \alpha I = \begin{pmatrix} 2 - \alpha & 0 & -1/2 \\ 0 & 2x_2 + 1 - \alpha & 0 \\ -1/2 & 0 & 4 - \alpha \end{pmatrix}
\]

and it is easy to see that if \( \alpha \) is small this matrix is positive semidefinite for any \( x \geq 0 \). Indeed, tak for example \( \alpha = 1/2 \), we get

\[
JF_a(x_1, x_2, x_3) - \frac{1}{2} I = \begin{pmatrix} 3/2 & 0 & -1/2 \\ 0 & 2x_2 + 1/2 & 0 \\ -1/2 & 0 & 7/2 \end{pmatrix}.
\]

The north-west minors are (i) \( 3/2 \), (ii) \( 3x_2 + 3/4 \), and (iii) \( \frac{3}{2}x_2 + \frac{21}{8} - \frac{1}{2}x_2 - \frac{1}{8} = 10x_2 + \frac{3}{2} \) and are all clearly positive on the non negative orthant. Therefore, \( F \) is strongly monotone on \( \mathbb{R}^3_+ \).

(b) \( F(1, 0, 1) = (0, 2, 0); \) since we have

\[
0 \leq \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \perp \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \geq 0
\]

\((1, 0, 1) \) is a solution of NCP(\( F \)).

(c) NCP(\( F \)) is just the VI(\( \mathbb{R}^3_+, F \)) and noting that \( \mathbb{R}^3_+ = \{(x_1, x_2, x_3): x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\} \), we can easily write the KKT system as

\[
\begin{align*}
2x_1 - x_2 - x_3 - 1 - \lambda_1 &= 0 \\
x_1 + x_2^2 + x_2 + x_3 - \lambda_2 &= 0 \\
-x_2 + 4x_3 - 4 - \lambda_3 &= 0 \\
0 \leq \lambda_1 &\perp -x_1 \leq 0 \\
0 \leq \lambda_2 &\perp -x_2 \leq 0 \\
0 \leq \lambda_3 &\perp -x_3 \leq 0
\end{align*}
\]
The point $(1,0,1)$ is feasible and $F(1,0,1) = (0,2,0)$. In this point only the second constraint is active and therefore $\lambda_1 = 0 = \lambda_3$. The KKT system therefore reduces to

$$
\begin{align*}
0 &= 0 \\
2 - \lambda_2 &= 0 \\
0 &= 0 \\
0 &\leq \lambda_2
\end{align*}
$$

from which we get $\lambda_2 = 2$. It is then clear that $(1,0,1)$, together with $(\lambda_1, \lambda_2, \lambda_3) = (0,2,0)$ satisfies the KKT system thus confirming that $(1,0,1)$ is a solution.

(d) It is enough to recall that $\phi_{FB}(a,b) = \sqrt{a^2 + b^2} - (a + b)$ to get the FB-equation reformulation of the NCP($F$)

$$
\begin{pmatrix}
\phi_{FB}(x_1, F_1(x)) \\
\phi_{FB}(x_2, F_2(x)) \\
\phi_{FB}(x_3, F_3(x))
\end{pmatrix} =
\begin{pmatrix}
\sqrt{x_1^2 + (2x_1 - x_2 - x_3 - 1)^2} - (x_1 + 2x_1 - x_2 - x_3 - 1) \\
\sqrt{x_2^2 + (x_1 + x_2 + 2x_2 + x_3)^2} - (x_2 + x_1 + x_2 + x_2 + x_3) \\
\sqrt{x_3^2 + (-x_2 + 4x_3 - 4)^2} - (x_3 - x_2 + 4x_3 - 4)
\end{pmatrix} = 0.
$$

12. Consider the two players game

$$
\begin{align*}
\min_x (x - 2)^2 \\
s.t. & \quad x + y \leq \frac{3}{2} \\
& \quad 0 \leq x \leq 1
\end{align*} \quad \quad \begin{align*}
\min_y \frac{1}{2} y^2 + 2xy - 2y \\
s.t. & \quad x + y \leq \frac{3}{2} \\
& \quad 0 \leq y \leq 1
\end{align*}
$$

Check, by using the definition of Nash equilibrium, that $(1,0)$ is a Nash equilibrium.

Let us set $y = 0$ in the first player’s problem, we get

$$
\begin{align*}
\min_x (x - 2)^2 \\
& \quad x \leq \frac{3}{2} \\
& \quad 0 \leq x \leq 1
\end{align*}
$$

We must check that 1 is the solution of this first player’s problem. And indeed, the objective function is an upward parabola with vertex in 2. Since the feasible region is $[0,1]$, it is clear that the minimum is 1 (draw a picture!).

Consider now the second player’s problem, where we set $x = 1$, we get

$$
\begin{align*}
\min_y \frac{1}{2} y^2 \\
& \quad y \leq \frac{1}{2} \\
& \quad 0 \leq y \leq 1
\end{align*}
$$

We must check that 0 is the solution of this second player’s problem. And indeed, the objective function is an upward parabola with vertex in 0. Since 0 is feasible for the second player’s problem (the feasible region is $[0,1/2]$), it is clear that the minimum is 0.