Strategic-form games

Figure 4.11 Three extensive-form games corresponding to the same strategic-form game in Figure 4.10

then those two extensive-form games correspond to the same strategic-form game. He also showed that the other direction obtains: if two games in extensive form yield the same strategic-form game, then they can be transformed into each other by a finite number of these three elementary operations.

4.3 Strategic-form games: solution concepts

We have dealt so far only with the different ways of describing games in extensive and strategic form. We discussed von Neumann’s theorem in the special case of two players and three possible outcomes: victory for White, a draw, or victory for Black. Now we will look at more general games, and consider the central question of game theory: What can we say about what “will happen” in a given game? First of all, note that this question has at least three different possible interpretations:
4.5 Domination

1. An empirical, descriptive interpretation: How do players, in fact, play in a given game?
2. A normative interpretation: How "should" the players play in a given game?
3. A theoretical interpretation: What can we predict will happen in a game given certain assumptions regarding "reasonable" or "rational" behavior on the part of the players?

Descriptive game theory deals with the first interpretation. This field of study involves observations of the actual behavior of players, both in real-life situations and in artificial laboratory conditions where they are asked to play games and their behavior is recorded. This book will not address that area of game theory.

The second interpretation would be appropriate for a judge, legislator, or arbitrator called upon to determine the outcome of a game based on several agreed-upon principles, such as justice, efficiency, nondiscrimination, and fairness. This approach is best suited for the study of cooperative games, in which binding agreements are possible, enabling outcomes to be derived from "norms" or agreed-upon principles, or determined by an arbitrator who bases his decision on those principles. This is indeed the approach used for the study of bargaining games (see Chapter 15) and the Shapley value (see Chapter 18).

In this chapter we will address the third interpretation, the theoretical approach. After we have described a game, in either extensive or strategic form, what can we expect to happen? What outcomes, or set of outcomes, will reasonably ensue, given certain assumptions regarding the behavior of the players?

4.4 Notation

Let \( N = \{1, \ldots, n\} \) be a finite set, and for each \( i \in N \) let \( X_i \) be any set. Denote \( X := \times_{i \in N} X_i \), and for each \( i \in N \) define \( X_{-i} := \times_{j \neq i} X_j \). For each \( i \in N \) we will denote by

\[
X_{-i} = \times_{j \neq i} X_j
\]

the Cartesian product of all the sets \( X_j \) except for the set \( X_i \). In other words,

\[
X_{-i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : x_j \in X_j, \quad \forall j \neq i\}.
\]

An element in \( X_{-i} \) will be denoted \( x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \), which is the \((n - 1)\)-dimensional vector derived from \((x_1, \ldots, x_n) \in X\) by suppressing the \(i\)-th coordinate.

4.5 Domination

Consider the two-player game that appears in Figure 4.12, in which Player I chooses a row and Player II chooses a column.

Comparing Player II's strategies \( M \) and \( R \), we find that:

- If Player I plays \( T \), the payoff to Player II under strategy \( M \) is 2, compared to only 1 under strategy \( R \).
- If Player I plays \( B \), the payoff to Player II under strategy \( M \) is 1, compared to only 0 under strategy \( R \).
Strategic-form games

<table>
<thead>
<tr>
<th></th>
<th>Player II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
</tr>
<tr>
<td>T</td>
<td>1.0</td>
</tr>
<tr>
<td>B</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Figure 4.12 Strategy M dominates strategy R

We see that independently of whichever strategy is played by Player I, strategy $M$ always yields a higher payoff to Player II than strategy $R$. This motivates the following definition:

**Definition 4.6** A strategy $s_i$ of player $i$ is strictly dominated if there exists another strategy $t_i$ of player $i$ such that for each strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i}).$$ (4.7)

*If this is the case, we say that $s_i$ is strictly dominated by $t_i$, or that $t_i$ strictly dominates $s_i$.*

In the example in Figure 4.12 strategy $R$ is strictly dominated by strategy $M$. It is therefore reasonable to assume that if Player II is "rational," he will not choose $R$, because under any scenario in which he might consider selecting $R$, the strategy $M$ would be a better choice. This is the first rationality property that we assume.

**Assumption 4.7** A rational player will not choose a strictly dominated strategy.

We will assume that all the players are rational.

**Assumption 4.8** All players in a game are rational.

Can a strictly dominated strategy (such as strategy $R$ in Figure 4.12) be eliminated, under these two assumptions? The answer is: not necessarily. It is true that if Player II is rational, he will not choose strategy $R$, but if Player I does not know that Player II is rational, he is liable to believe that Player II may choose strategy $R$, in which case it would be in Player I's interest to play strategy $B$. So, in order to eliminate the strictly dominated strategies one needs to postulate that:

- Player II is rational, and
- Player I knows that Player II is rational.

On further reflection, it becomes clear that this, too, is insufficient, and we also need to assume that:

- Player II knows that Player I knows that Player II is rational.

Otherwise, Player II would need to consider the possibility that Player I may play $B$, considering $R$ to be a strategy contemplated by Player II, in which case Player II may be tempted to play $L$. Once again, further scrutiny reveals that this is still insufficient, and we need to assume that:
4.5 Domination

Player II

\[
\begin{array}{c|c|c}
 & L & M \\
\hline
T & 1,0 & 1,2 \\
B & 0,3 & 0,1 \\
\end{array}
\]

Player I

Figure 4.13 The game in Figure 4.12 after the elimination of strategy \( R \)

Player II

L M

Player I T 1,0 1,2

Figure 4.14 The game in Figure 4.12 following the elimination of strategies \( R \) and \( B \)

- Player I knows that Player II knows that Player I knows that Player II is rational.
- Player II knows that Player I knows that Player II knows that Player I knows that Player II is rational.
- And so forth.

If all the infinite conditions implied by the above are satisfied, we say that the fact that Player II is rational is common knowledge among the players. This is an important concept underlying most of our presentation. Here we will give only an informal presentation of the concept of common knowledge. A formal definition appears in Chapter 9, where we extensively study common knowledge.

**Definition 4.9** A fact is common knowledge among the players of a game if for any finite chain of players \( i_1, i_2, \ldots, i_k \) the following holds: player \( i_1 \) knows that player \( i_2 \) knows that player \( i_3 \) knows \ldots that player \( i_k \) knows the fact.

The chain in Definition 4.9 may contain several instances of the same player (as indeed happens in the above example). Definition 4.9 is an informal definition since we have not formally defined what the term "fact" means, nor have we defined the significance of knowing a fact. We will now add a further assumption to the two assumptions listed above:

**Assumption 4.10** The fact that all players are rational (Assumption 4.8) is common knowledge among the players.

Strictly dominated strategies can be eliminated under Assumptions 4.7, 4.8, and 4.10 (we will not provide a formal proof of this claim). In the example in Figure 4.12, this means that, given the assumptions, we should focus on the game obtained by the elimination of strategy \( R \), which appears in Figure 4.13.

In this game strategy \( B \) of Player I is strictly dominated by strategy \( T \). Because the rationality of Player I is common knowledge, as is the fact that \( B \) is a strictly dominated strategy, after the elimination of strategy \( R \), strategy \( B \) can also be eliminated. The players therefore need to consider a game with even fewer strategies, which is given in Figure 4.14.
Strategic-form games

Because in this game strategy $L$ is strictly dominated (for Player II) by strategy $M$, after its elimination only one result remains, $(1, 2)$, which obtains when Player I plays $T$ and Player II plays $M$.

The process we have just described is called iterated elimination of strictly dominated strategies. When this process yields a single strategy vector (one strategy per player), as in the example above, then, under Assumptions 4.7, 4.8, and 4.10, that is the strategy vector that will obtain, and it may be regarded as the solution of the game.

A special case in which such a solution is guaranteed to exist is the family of games in which every player has a strategy that strictly dominates all of his other strategies, that is, a strictly dominant strategy. Clearly, in that case, the elimination of all strictly dominated strategies leaves each player with only one strategy: his strictly dominant strategy. When this occurs we say that the game has a solution in strictly dominant strategies.

Example 4.11. The Prisoner’s Dilemma The “Prisoner’s Dilemma” is a very simple game that is interesting in several respects. It appears in the literature in the form of the following story.

Two individuals who have committed a serious crime are apprehended. Lacking incriminating evidence, the prosecution can obtain an indictment only by persuading one (or both) of the prisoners to confess to the crime. Interrogators give each of the prisoners – both of whom are isolated in separate cells and unable to communicate with each other – the following choices:

1. If you confess and your friend refuses to confess, you will be released from custody and receive immunity as a state’s witness.
2. If you refuse to confess and your friend confesses, you will receive the maximum penalty for your crime (ten years of incarceration).
3. If both of you sit tight and refuse to confess, we will make use of evidence that you have committed tax evasion to ensure that both of you are sentenced to a year in prison.
4. If both of you confess, it will count in your favor and we will reduce each of your prison terms to six years.

This situation defines a two-player strategic-form game in which each player has two strategies: $D$, which stands for Defection, betraying your fellow criminal by confessing, and $C$, which stands for Cooperation, cooperating with your fellow criminal and not confessing the crime. In this notation, the outcome of the game (in prison years) is shown in Figure 4.15.

```

Player II

       D  C

      D | 6, 6 | 0.10

     C | 10, 0 | 1, 1

Player I

```

Figure 4.15 The Prisoner’s Dilemma in prison years

As usual, the left-hand number in each cell of the matrix represents the outcome (in prison years) for Player I, and the right-hand number represents the outcome for Player II.
4.5 Domination

We now present the game in utility units. For example, suppose the utility of both players is given by the following function $u$:

$$u(\text{release}) = 5, \quad u(\text{one year in prison}) = 4, \quad u(\text{6 years in prison}) = 1, \quad u(\text{10 years in prison}) = 0.$$ 

The game in utility terms appears in Figure 4.16.

![Figure 4.16 The Prisoner’s Dilemma in utility units](image)

For both players, strategy $D$ (Defect) strictly dominates strategy $C$ (Cooperate). Elimination of strictly dominated strategies leads to the single solution $(D, D)$ in which both prisoners confess, resulting in the payoff $(1, 1)$.

What makes the Prisoner’s Dilemma interesting is the fact that if both players choose strategy $C$, the payoff they receive is $(4, 4)$, which is preferable for both of them. The solution derived from Assumptions 4.7, 4.8, and 4.10, which appear to be quite reasonable assumptions, is “inefficient”: The pair of strategies $(C, C)$ is unstable, because each individual player can deviate (by defecting) and gain an even better payoff of 5 (instead of 4) for himself (at the expense of the other player, who would receive 0).

In the last example, two strictly dominated strategies were eliminated (one per player), but there was no specification regarding the order in which these strategies were eliminated: was Player I’s strategy $C$ eliminated first, or Player II’s, or were they both eliminated simultaneously? In this case, a direct verification reveals that the order of elimination makes no difference. It turns out that this is a general result: whenever iterated elimination of strictly dominated strategies leads to a single strategy vector, that outcome is independent of the order of elimination. In fact, we can make an even stronger statement: even if iterated elimination of strictly dominated strategies yields a set of strategies (not necessarily a single strategy), that set does not depend on the order of elimination (see Exercise 4.10).

There are games in which iterated elimination of strictly dominated strategies does not yield a single strategy vector. For example, in a game that has no strictly dominated strategies, the process fails to eliminate any strategy. The game in Figure 4.17 provides an example of such a game.

Although there are no strictly dominated strategies in this game, strategy $B$ does have a special attribute: although it does not always guarantee a higher payoff to Player I relative to strategy $T$, in all cases it does grant him a payoff at least as high, and in the special case in which Player II chooses strategy $L$, $B$ is a strictly better choice than $T$. In this case we say that strategy $B$ weakly dominates strategy $T$ (and strategy $T$ is weakly dominated by strategy $B$).
Strategic-form games

![Payoff matrix](image)

**Figure 4.17** A game with no strictly dominated strategies

**Definition 4.12** Strategy $s_i$ of player $i$ is termed weakly dominated if there exists another strategy $t_i$ of player $i$ satisfying the following two conditions:

(a) For every strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \leq u_i(t_i, s_{-i}).$$

(b) There exists a strategy vector $t_{-i} \in S_{-i}$ of the other players such that

$$u_i(s_i, t_{-i}) < u_i(t_i, t_{-i}).$$

In this case we say that strategy $s_i$ is weakly dominated by strategy $t_i$, and that strategy $t_i$ weakly dominates strategy $s_i$.

If strategy $t_i$ dominates (weakly or strictly) strategy $s_i$, then $s_i$ does not (weakly or strictly) dominate $t_i$. Clearly, strict domination implies weak domination. Because we will refer henceforth almost exclusively to weak domination, we use the term “domination” to mean “weak domination,” unless the term “strict domination” is explicitly used. The following rationality assumption is stronger than Assumption 4.7.

**Assumption 4.13** A rational player does not use a dominated strategy.

Under Assumptions 4.8, 4.11, and 4.13 we may eliminate strategy $T$ in the game in Figure 4.17 (as it is weakly dominated), and then proceed to eliminate strategy $R$ (which is strictly dominated after the elimination of $T$). The only remaining strategy vector is $(B, L)$, with a payoff of $(2, 2)$. Such a strategy vector is called rational, and the process of iterative elimination of weakly dominated strategies is called rationalizability. The meaning of “rationalizability” is that a player who expects a certain strategy vector to obtain can explain to himself why that strategy vector will be reached, based on the assumption of rationality.

**Definition 4.14** A strategy vector $s \in S$ is termed rational if it is the unique result of a process of iterative elimination of weakly dominated strategies.

Whereas Assumption 4.7 looks reasonable, Assumption 4.13 is quite strong. Reinhard Selten, in trying to justify Assumption 4.13, suggested a concept he termed the trembling hand principle. The basic postulate of this principle is that every single strategy available to a player may be used with positive probability, which may well be extremely small. This may happen simply by mistake (the player’s hand might tremble as he reaches to press the button setting in motion his chosen strategy, so that by mistake the button associated
4.6 Second-price auctions

With a different strategy is activated instead), by irrationality on the part of the player, or because the player chose a wrong strategy due to miscalculations. This topic will be explored in greater depth in Section 7.3 (page 262).

To illustrate the trembling hand principle, suppose that Player II in the example of Figure 4.17 chooses strategies $L$ and $R$ with respective probabilities $x$ and $1-x$, where $0 < x < 1$. The expected payoff to Player I in that case is $x + 2(1-x) = 2 - x$ if he chooses strategy $T$, as opposed to a payoff of 2 if he chooses strategy $B$. It follows that strategy $B$ grants him a strictly higher expected payoff than $T$, so that a rational Player I facing Player II who has a trembling hand will choose $B$ and not $T$; i.e., he will not choose the weakly dominated strategy.

The fact that strategy $s_j$ of player $i$ (weakly or strictly) dominates his strategy $t_i$ depends only on player $i$'s payoff function, and is independent of the payoff functions of the other players. Therefore, a player can eliminate his dominated strategies even when he does not know the payoff functions of the other players. This property will be useful in Section 4.6. In the process of rationalizability we eliminate dominated strategies one after the other. Eliminating strategy $s_i$ of player $i$ after strategy $s_j$ of player $j$ means that we assume that player $i$ believes that player $j$ will not implement $s_j$. This assumption is reasonable only if player $i$ knows player $j$'s payoff function. Therefore, the process of iterative elimination of dominated strategies can be justified only if the payoff functions of the players are common knowledge among them; if this condition does not hold, the process is harder to justify.

The process of rationalizability – iterated elimination of dominated strategies – is an efficient tool that leads, sometimes surprisingly, to significant results. The following example, taken from the theory of auctions, provides an illustration.

4.6 Second-price auctions

A detailed study of auction theory is presented in Chapter 12. In this section we will concentrate on the relevance of the concept of dominance to auctions known as sealed-bid second-price auctions, which are conducted as follows:

- An indivisible object is offered for sale.
- The set of buyers in the auction is denoted by $N$. Each buyer $i$ attaches a value $v_i$ to the object; that is, he is willing to pay at most $v_i$ for the object (and is indifferent between walking away without the object and obtaining it at price $v_i$). The value $v_i$ is buyer $i$'s private value, which may arise from entirely subjective considerations, such as his preference for certain types of artistic objects or styles, or from potential profits (for example, the auctioned object might be a license to operate a television channel). This state of affairs motivates our additional assumption that each buyer $i$ knows his own private value $v_i$ but not the values that the other buyers attach to the object. This does not, however, prevent him from assessing the private values of the other buyers, or from believing that he knows their private values with some level of certainty.
- Each buyer $i$ bids a price $b_i$ (presented to the auctioneer in a sealed envelope).
Strategic-form games

- The winner of the object is the buyer who makes the highest bid. That may not be surprising, but in contrast to the auctions most of us usually see, the winner does not proceed to pay the bid he submitted. Instead he pays the second-highest price offered (hence the name second-price auction). If several buyers bid the same maximal price, a fair lottery is conducted between them to determine who will receive the object in exchange for paying that amount (which in this case is also the second-highest price offered.)

Let us take a concrete example. Suppose there are four buyers respectively bidding 5, 15, 2, and 21. The buyer bidding 21 is the winner, paying 15 in exchange for the object. In general, the winner of the auction is a buyer $i$ for which

$$b_i = \max_{j \in N} b_j, \quad (4.10)$$

If buyer $i$ is the winner, the amount he pays is $\max_{j \neq i} b_j$. We now proceed to describe a sealed-bid second-price auction as a strategic-form game.\(^3\)

1. The set of players is the set $N$ of buyers in the auction.
2. The set of strategies available to buyer $i$ is the set of possible bids $S_i = [0, \infty)$.
3. The payoff to buyer $i$, when the strategy vector is $b = (b_1, \ldots, b_n)$, is

$$u_i(b) = \begin{cases} 0 & \text{if } b_i < \max_{j \in N} b_j, \\ \frac{b_i - \max_{j \neq i} b_j}{\|k: b_k = \max_{j \in N} b_j\|} & \text{if } b_i = \max_{j \in N} b_j. \end{cases} \quad (4.11)$$

How should we expect a rational buyer to act in this auction? At first glance, this appears to be a very difficult problem to solve, because no buyer knows the private values of his competitors, let alone the prices they will bid. He may not even know how many other buyers are participating in the auction. So what price $b_i$ will buyer $i$ bid? Will he bid a price lower than $v_i$, in order to ensure that he does not lose money in the auction, or higher than $v_i$, in order to increase his probability of winning, all the while hoping that the second-highest bid will be lower than $v_i$? The process of rationalizability leads to the following result:

**Theorem 4.15** In a second-price sealed-bid auction, the strategy $b_i = v_i$ weakly dominates all other strategies.

In other words, under Assumptions 4.8, 4.10, and 4.13, the auction will proceed as follows:

- Every buyer will bid $b_i = v_i$.
- The winner will be the buyer whose private valuation of the object is the highest.\(^4\) The price paid by the winning buyer (i.e., the object’s sale price) is the second-highest private value. If several buyers share the same maximal bid, one of them, selected randomly by a fair lottery, will get the object, and will pay his private value (which in this special case is also the second-highest bid, and his profit will therefore be 0).

---

\(^3\) The relation between this auction method and other, more familiar, auction methods is discussed in Chapter 12.

\(^4\) This property is termed **efficiency** in the game theory and economics literature.
4.6 Second-price auctions

Figure 4.18 The payoff function for strategy \( b_i = v_i \)

Each buyer knows his private value \( v_i \) and therefore he also knows his payoff function. Since buyers do not necessarily know each other’s private value, they do not necessarily know each other’s payoff functions. Nevertheless, as we mentioned on page 91, the concept of domination is defined also when a player does not know the other players’ payoff functions.

**Proof:** Consider a buyer \( i \) whose private value is \( v_i \). Divide the set of strategies available to him, \( S_i = [0, \infty) \), into three subsets:

- The strategies in which his bid is less than \( v_i \): \( [0, v_i) \).
- The strategy in which his bid is equal to \( v_i \): \( \{ v_i \} \).
- The strategies in which his bid is higher than \( v_i \): \( (v_i, \infty) \).

We now show that strategy \( b_i = v_i \) dominates all the strategies in the other two subsets. Given the procedure of the auction, the payment eventually made by buyer \( i \) depends on the strategies selected by the other buyers, through their highest bid, and the number of buyers bidding that highest bid. Denote the maximal bid put forward by the other buyers by

\[
B_{-i} = \max_{j \neq i} b_j,
\]

and the number of buyers who offered this bid by

\[
N_{-i} = \left\lfloor \frac{k}{b_k = \max_{j \neq i} b_j} \right\rfloor.
\]

The payoff function of buyer \( i \), as a function of the strategy vector \( b \) (i.e., the vector of all the bids made by the buyers) is

\[
u_i(b) = \begin{cases} 0 & \text{if } b_i < B_{-i}, \\
\frac{v_i - B_{-i}}{N_{-i}+1} & \text{if } b_i = B_{-i}, \\
v_i - B_{-i} & \text{if } b_i > B_{-i}.
\end{cases}
\]

Since the only dependence that the payoff function \( u_i(b) \) has on the bids \( b_{-i} \) of the other buyers is via the highest bid, \( B_{-i} \), we sometimes denote this function by \( u_i(b_i, B_{-i}) \). If buyer \( i \) chooses strategy \( b_i = v_i \), his payoff as a function of \( B_{-i} \) is given in Figure 4.18.

If buyer \( i \) chooses strategy \( b_i \) satisfying \( b_i < v_i \), his payoff function is given by Figure 4.19.
The height of the dot in Figure 4.19, when $b_i = B_{-i}$, depends on the number of buyers who bid $B_{-i}$.

The payoff function in Figure 4.18 (which corresponds to the strategy $b_i = v_i$) is (weakly) greater than the one in Figure 4.19 (corresponding to a strategy $b_i$ with $b_i < v_i$). The former is strictly greater than the latter when $b_i \leq B_{-i} < v_i$. It follows that strategy $b_i = v_i$ dominates all strategies in which the bid is lower than buyer i’s private value.

The payoff function for a strategy $b_i$ satisfying $b_i > v_i$ is displayed in Figure 4.20. Again, we see that the payoff function in Figure 4.18 is (weakly) greater than the payoff function in Figure 4.20. The former is strictly greater than the latter when $v_i < B_{-i} \leq b_i$. It follows that the strategy in which the bid is equal to the private value weakly dominates all other strategies, as claimed. □

Theorem 4.15 holds also when some buyers do not know the number of buyers participating in the auction, their private values, their beliefs (about the number of buyers and the private values of the other buyers), and their utility functions (for example, information on whether the other players are risk seekers, risk averse, or risk neutral; see Section 2.7). The only condition needed for Theorem 4.15 to hold is that each buyer know the rules of the auction.