Robotics 2

Midterm Test – April 30, 2025

Exercise 1

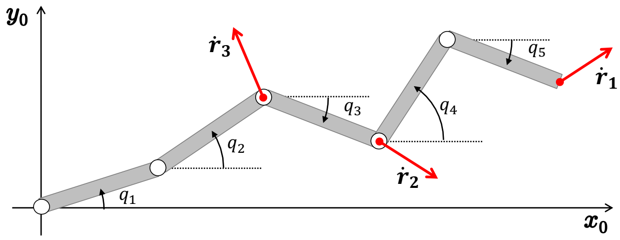


Figure 1: The 5R planar robot with three desired task velocities $\dot{\boldsymbol{r}}_1$, $\dot{\boldsymbol{r}}_2$ and $\dot{\boldsymbol{r}}_3$.

Consider a 5R planar robot with links of unitary length in a generic configuration $\boldsymbol{q}$, where q_i, $i = 1, \dots, 5$, are absolute angles with respect to the $\boldsymbol{x}_0$ axis, as shown in Fig. 1. The robot is controlled by the absolute joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^5$. Three velocity tasks $\dot{\boldsymbol{r}}_i \in \mathbb{R}^2$, $i = 1, 2, 3$, are assigned with priority, respectively, the first for the end-effector, the second for the tip of link 3, and the third for the tip of link 2.

i) Write down the computations to be performed in order to determine the joint velocity command $\dot{\boldsymbol{q}}_{TP}$ according to the Task Priority (TP) method.

ii) Compute the numerical value of $\dot{\boldsymbol{q}}_{TP}$ for the following data:

$$\boldsymbol{q} = (0, \pi/3, -\pi/4, \pi/2, -\pi/3) \ \text{[rad]} \qquad \dot{\boldsymbol{r}}_1 = (2, 3) \qquad \dot{\boldsymbol{r}}_2 = (2, -0.5) \qquad \dot{\boldsymbol{r}}_3 = (-1, 0) \ \text{[m/s]}.$$

Are the tasks exactly realized? If not, compute the norm of the velocity error for each of them.

iii) With the same data as in *ii)*, compute the joint velocity command $\dot{\boldsymbol{q}}_{TA}$ according to the Task Augmentation (TA) method, i.e., without any priority. Determine the norm of the velocity error for each task that is not exactly realized. Discuss the obtained TA results in comparison with the TP method.

iv) Keeping the same task velocities $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{r}}_2$ as in *ii)*, determine, if at all possible, a task velocity $\dot{\boldsymbol{r}}_3 \in \mathbb{R}^2$ such that the joint velocity command $\dot{\boldsymbol{q}}_{TA}$ obtained with the TA method realizes exactly all three tasks.

v) For the solution in *ii)*, find the corresponding Denavit–Hartenberg (relative) joint velocity $\dot{\boldsymbol{\theta}}_{TP}$.

Exercise 2

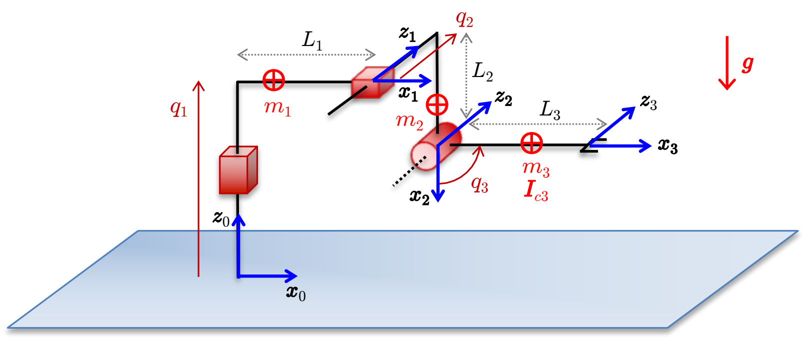


Figure 2: A PPR spatial robot with its D–H frames and relevant kinematic and dynamic parameters.

Figure 2 shows a 3-dof robot moving under gravity in the 3D space, with the associated Denavit–Hartenberg (D–H) frames. The first two joints are prismatic and the third is revolute. The most relevant kinematic and dynamic parameters are also shown. The center of mass of each link is assumed to be on the link kinematic axis ($\boldsymbol{x}_i$ for link i). No special assumption holds for the barycentric inertia matrix $\boldsymbol{I}_{c3}$ of link 3.

i) Derive all terms of the dynamic model in the Lagrangian form

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}.$$

ii) Find two different factorization matrices $\boldsymbol{S}_i(\boldsymbol{q},\dot{\boldsymbol{q}})$, $i = 1, 2$, such that $\boldsymbol{S}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$; show that matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric when $\boldsymbol{S} = \boldsymbol{S}_1$, whereas it is not when $\boldsymbol{S} = \boldsymbol{S}_2$.

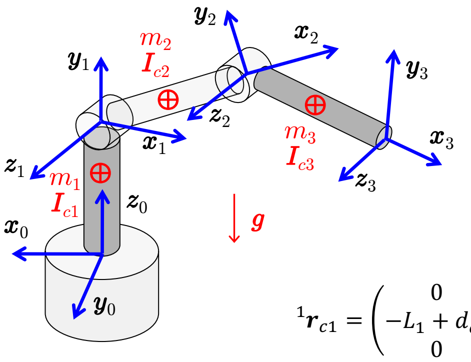
iii) Determine a minimal linear parametrization of the model useful for system identification

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau},$$

providing the symbolic expression of the (unknown) dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ and of the $3 \times p$ regressor matrix $\boldsymbol{Y}$. Assume that the acceleration of gravity $g_0 = 9.81$ m/s^2 is known.

iv) Give the expression of the torque $\boldsymbol{\tau}_d(t) = (\tau_{d1}(t), \tau_{d2}(t)), \tau_{d3}(t))$ needed to execute the desired motion $\boldsymbol{q}_d(t) = (0, \sin(2\omega t), \omega t)$, for all $t \geq 0$. Which should be the initial state $(\boldsymbol{q}(0, \dot{\boldsymbol{q}}(0))$ of the robot so that this joint trajectory is perfectly executed from $t = 0$ on? Discuss the physics of the results.

Exercise 3



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	L_1	q_1
2	0	L_2	0	q_2
3	0	L_3	0	q_3

$${}^i\boldsymbol{I}_{ci} = \begin{pmatrix} I_{xxi} & 0 & 0 \\ 0 & I_{yyi} & 0 \\ 0 & 0 & I_{zzi} \end{pmatrix}, \qquad i = 1,2,3$$

$${}^1\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ -L_1 + d_{c1} \\ 0 \end{pmatrix} \qquad {}^i\boldsymbol{r}_{ci} = \begin{pmatrix} -L_i + d_{ci} \\ 0 \\ 0 \end{pmatrix}, \qquad i = 2,3$$

Figure 3: A 3R robot with D–H frames and table of parameters, and the dynamic assumptions.

For a 3R elbow-type robot, all the kinematic and dynamic information is given in Fig. 3. Derive the robot kinetic energy $T(\boldsymbol{q},\dot{\boldsymbol{q}})$ using the recursive algorithm with moving frames. At a given robot state $(\boldsymbol{q},\dot{\boldsymbol{q}})$, consider then the dynamic optimization problem

$$\min \frac{1}{2}\left(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)^T\left(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right) \qquad \text{s.t.} \quad \begin{cases} \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) \,=\, \boldsymbol{\tau} \\ \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \,=\, \ddot{\boldsymbol{r}}_d, \end{cases}$$

where $\boldsymbol{J}(\boldsymbol{q})$ is the 2×3 Jacobian associated to the task $\boldsymbol{r} = (p_x, p_y) = \boldsymbol{f}(\boldsymbol{q})$, i.e., two of the three coordinates of the robot end-effector position $\boldsymbol{p}$. Provide the symbolic expression of the joint acceleration $\ddot{\boldsymbol{q}}$ that solves this problem. Evaluate numerically this solution when $\boldsymbol{q} = (3\pi/2, \pi/4, 0)$ rad], $\dot{\boldsymbol{q}} = \boldsymbol{0}$, and $\ddot{\boldsymbol{r}}_d = (-1, -1)$ [m/s^2] and with the data

$$L_i = 0.5 \qquad d_{ci} = 0.5\,L_i \quad [\text{m}] \qquad m_i = 2 \text{ kg} \qquad \text{for } i = 1, 2, 3,$$

$$I_{yy1} = \tfrac{1}{40}\,m_1 \qquad I_{xxi} = \tfrac{1}{40}\,m_i \qquad I_{yyi} = I_{zzi} = \tfrac{1}{12}\,m_i\left(0.05 + L_i^2\right) \quad [\text{kgm}^2] \qquad \text{for } i = 2, 3.$$

[240 minutes (4 hours); open books]

Solution

April 30, 2025

Exercise 1

The Jacobian matrices of the three tasks are

$$\begin{aligned}
\boldsymbol{J}_1(\boldsymbol{q}) &= \begin{pmatrix} -s_1 & -s_2 & -s_3 & -s_4 & -s_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix} \\
\boldsymbol{J}_2(\boldsymbol{q}) &= \begin{pmatrix} -s_1 & -s_2 & -s_3 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 \end{pmatrix} \\
\boldsymbol{J}_3(\boldsymbol{q}) &= \begin{pmatrix} -s_1 & -s_2 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

with $s_i = \sin q_i$ and $c_i = \cos q_i$. When evaluated at $\boldsymbol{q} = (0, \pi/3, -\pi/4, \pi/2, -\pi/3)$, the first Jacobian is

$$\boldsymbol{J}_1 = \begin{pmatrix} 0 & -0.8660 & 0.7071 & -1 & 0.8660 \\ 1 & 0.5 & 0.7071 & 0 & 0.5 \end{pmatrix},$$

while $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ follow accordingly. None of the Jacobian matrices loses rank.
The Task Priority (TP) method computes in this case

$$\begin{aligned}
\boldsymbol{P}_{A,0} &= \boldsymbol{I}_{5\times 5} \quad \text{and} \quad \dot{\boldsymbol{q}}_0 = \boldsymbol{0} \\
\boldsymbol{P}_{10} &= (\boldsymbol{J}_1 \boldsymbol{P}_{A,0})^{\#} \ \left[= \boldsymbol{J}_1^{\#}\right] \\
\dot{\boldsymbol{q}}_1 &= \dot{\boldsymbol{q}}_0 + \boldsymbol{P}_{10} \left(\dot{\boldsymbol{r}}_1 - \boldsymbol{J}_1 \dot{\boldsymbol{q}}_0\right) \ \left[= \boldsymbol{J}_1^{\#} \dot{\boldsymbol{r}}_1\right] \\
\boldsymbol{P}_{A,1} &= \boldsymbol{P}_{A,0} - \boldsymbol{P}_{10} \boldsymbol{J}_1 \boldsymbol{P}_{A,0} \left[= (\boldsymbol{I} - \boldsymbol{J}_1^{\#} \boldsymbol{J}_1)\right] \\
\boldsymbol{P}_{21} &= (\boldsymbol{J}_2 \boldsymbol{P}_{A,1})^{\#} \ \left[= \big(\boldsymbol{J}_2 (\boldsymbol{I} - \boldsymbol{J}_1^{\#} \boldsymbol{J}_1)\big)^{\#}\right] \\
\dot{\boldsymbol{q}}_2 &= \dot{\boldsymbol{q}}_1 + \boldsymbol{P}_{21} \left(\dot{\boldsymbol{r}}_2 - \boldsymbol{J}_2 \dot{\boldsymbol{q}}_1\right) \\
\boldsymbol{P}_{A,2} &= \boldsymbol{P}_{A,1} - \boldsymbol{P}_{21} \boldsymbol{J}_2 \boldsymbol{P}_{A,1} \\
\boldsymbol{P}_{32} &= (\boldsymbol{J}_3 \boldsymbol{P}_{A,2})^{\#} \\
\dot{\boldsymbol{q}}_{TP} &= \dot{\boldsymbol{q}}_2 + \boldsymbol{P}_{32} \left(\dot{\boldsymbol{r}}_3 - \boldsymbol{J}_3 \dot{\boldsymbol{q}}_2\right),
\end{aligned}$$

where the terms in [square brackets] represent some possible simplified/alternative expressions. Note that in this recursive computation there are only *three* operations of pseudoinversion (of 2×5 matrices). Inserting the desired velocities of the three tasks, we obtain

$$\dot{\boldsymbol{q}}_1 = \begin{pmatrix} 1.3913 \\ 0.3191 \\ 1.2912 \\ -0.4348 \\ 1.0722 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_2 = \begin{pmatrix} -0.3477 \\ -1.5756 \\ 0.8987 \\ 6.0622 \\ 7.0000 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{TP} = \begin{pmatrix} -1.3170 \\ -0.8660 \\ 1.7678 \\ 6.0622 \\ 7.0000 \end{pmatrix} \text{[rad/s]}. \tag{1}$$

It is easy to verify that the first two tasks are executed perfectly, whereas the third one has some error:

$$\boldsymbol{e}_1 = \dot{\boldsymbol{r}}_1 - \boldsymbol{J}_1 \dot{\boldsymbol{q}}_{TP} = \boldsymbol{0} \qquad \boldsymbol{e}_2 = \dot{\boldsymbol{r}}_2 - \boldsymbol{J}_2 \dot{\boldsymbol{q}}_{TP} = \boldsymbol{0} \qquad \|\boldsymbol{e}_3\| = \|\dot{\boldsymbol{r}}_3 - \boldsymbol{J}_3 \dot{\boldsymbol{q}}_{TP}\| = 2.4749 \text{ m/s},$$

being

$$\dot{\boldsymbol{r}}_{3,TP} = \boldsymbol{J}_3 \dot{\boldsymbol{q}}_{TP} = \begin{pmatrix} 0.75 \\ -1.75 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_3. \tag{2}$$

This was not unexpected, since the robot has $n = 5$ commands, while the dimension of the augmented task is $m = m_1 + m_2 + m_3 = 2 + 2 + 2 = 6$.

For the Task Augmentation (TA) method, we have

$$\boldsymbol{J}_A = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \\ \boldsymbol{J}_3 \end{pmatrix} \qquad \dot{\boldsymbol{r}}_A = \begin{pmatrix} \dot{\boldsymbol{r}}_1 \\ \dot{\boldsymbol{r}}_2 \\ \dot{\boldsymbol{r}}_3 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_{TA} = \boldsymbol{J}_A^{\#}\,\dot{\boldsymbol{r}}_A = (\boldsymbol{J}_A^T\boldsymbol{J}_A)^{-1}\boldsymbol{J}_A^T\,\dot{\boldsymbol{r}}_A = \begin{pmatrix} -0.9472 \\ 0.1443 \\ 1.7678 \\ 3.6716 \\ 5.2500 \end{pmatrix} \ \text{[rad/s]},$$

where the explicit form of the pseudoinverse can be used since the 6×5 matrix $\boldsymbol{J}_A$ has rank 5. In this case, the error is zero only on the first task:

$$\boldsymbol{e}_1 = \dot{\boldsymbol{r}}_1 - \boldsymbol{J}_1\,\dot{\boldsymbol{q}}_{TA} = \boldsymbol{0} \qquad \|\boldsymbol{e}_2\| = \|\dot{\boldsymbol{r}}_2 - \boldsymbol{J}_2\,\dot{\boldsymbol{q}}_{TA}\| = 1.2374 \qquad \|\boldsymbol{e}_3\| = \|\dot{\boldsymbol{r}}_3 - \boldsymbol{J}_3\,\dot{\boldsymbol{q}}_{TA}\| = 1.2374 \ \text{[m/s]}.$$

Moreover, for the augmented task one has

$$\|\boldsymbol{e}_A\| = \|\dot{\boldsymbol{r}}_A - \boldsymbol{J}_A\,\dot{\boldsymbol{q}}_{TA}\| = 1.7500 \text{ m/s}.$$

The norm of the error $\boldsymbol{e}_A$ obtained with the TA method is smaller than the norm of the error $\boldsymbol{e}_3$ of the third task alone, as obtained with the TP method. This is because the pseudoinverse solution in the TA method minimizes the error norm of the entire augmented task, at the price of distributing it over multiple subtasks; on the other hand, the TP method is designed for zeroing the error on higher priority tasks, at the price of having larger errors (in norm) on tasks with lower priority. For the same reason, the norms of $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are equal in the TA method, since pseudoinversion tends to equally distribute the error, when present, among the components.

Before proceeding, it is instructive to check also the situation in which the third task is completely discarded from the TA method. In such case, we would have

$$\boldsymbol{J}_{A,2} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} \quad \dot{\boldsymbol{r}}_{A,2} = \begin{pmatrix} \dot{\boldsymbol{r}}_1 \\ \dot{\boldsymbol{r}}_2 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_{TA,2} = \boldsymbol{J}_{A,2}^{\#}\,\dot{\boldsymbol{r}}_{A,2} = \boldsymbol{J}_{A,2}^T(\boldsymbol{J}_{A,2}\boldsymbol{J}_{A,2}^T)^{-1}\,\dot{\boldsymbol{r}}_{A,2} = \begin{pmatrix} -0.3477 \\ -1.5756 \\ 0.8987 \\ 6.0622 \\ 7.0000 \end{pmatrix} \ \text{[rad/s]},$$

obtaining perfect execution of the two considered tasks

$$\boldsymbol{e}_1 = \dot{\boldsymbol{r}}_1 - \boldsymbol{J}_1\,\dot{\boldsymbol{q}}_{TA,2} = \boldsymbol{0} \qquad \boldsymbol{e}_2 = \dot{\boldsymbol{r}}_2 - \boldsymbol{J}_2\,\dot{\boldsymbol{q}}_{TA,2} = \boldsymbol{0},$$

while a larger error norm results for the discarded task,

$$\|\boldsymbol{e}_3\| = \left\|\dot{\boldsymbol{r}}_3 - \boldsymbol{J}_3\,\dot{\boldsymbol{q}}_{TA,2}\right\| = 2.6230 \ \text{[m/s]},$$

being

$$\dot{\boldsymbol{r}}_{3,TA,2} = \boldsymbol{J}_3\,\dot{\boldsymbol{q}}_{TA,2} = \begin{pmatrix} 1.3645 \\ -1.1355 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_3. \tag{3}$$

The correct execution of the first two tasks is a consequence of the fact that the 4×5 Jacobian $\boldsymbol{J}_{A,2}$ has (full) rank 4. Note that the joint velocity solution $\dot{\boldsymbol{q}}_{TA,2}$ coincides with the intermediate solution $\dot{\boldsymbol{q}}_2$ of the TP method, which considers at that stage only the first two tasks. In fact, when a combined task is realizable, the priority order has no relevance. However, the error norm on the third task is larger with $\dot{\boldsymbol{q}}_{TA,2}$ than with the final $\dot{\boldsymbol{q}}_{TP}$, simply because this task is here completely excluded from the picture, as opposed to the TP case.

Having shown that the TA method is capable of realizing exactly the first two tasks, when only these two are being considered, the solution to item *iv)* is easily found using the computations already done either with the complete TP method or with the TA method applied to two tasks only.

In fact, if the desired velocity of the third task were $\dot{\boldsymbol{r}}_{3,TP}$ in (2) rather than $\dot{\boldsymbol{r}}_3$, then all three tasks would be realizable, For such a modified augmented task, it is

$$\dot{\boldsymbol{r}}_{A,m_1} = \begin{pmatrix} \dot{\boldsymbol{r}}_1 \\ \dot{\boldsymbol{r}}_2 \\ \dot{\boldsymbol{r}}_{3,TP} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ -0.5 \\ 0.75 \\ -1.75 \end{pmatrix} \in \mathcal{R}(\boldsymbol{J}_A),$$

and thus

$$\dot{\boldsymbol{q}}_{TA,m_1} = \boldsymbol{J}_A^{\#}\, \dot{\boldsymbol{r}}_{A,m_1} = \begin{pmatrix} -1.3170 \\ -0.8660 \\ 1.7678 \\ 6.0622 \\ 7.0000 \end{pmatrix} \text{[rad/s]} \qquad \Rightarrow \qquad \boldsymbol{e}_{A,m_1} = \dot{\boldsymbol{r}}_{A,m_1} - \boldsymbol{J}_A\, \dot{\boldsymbol{q}}_{TA,m_1} = \boldsymbol{0}.$$

The same solution is obtained when using the TP method for this modified problem ($\dot{\boldsymbol{q}}_{TP,m_1} = \dot{\boldsymbol{q}}_{TA,m_1}$, and $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \boldsymbol{e}_{3,m_1} = \boldsymbol{0}$); again, when all considered tasks are compatible, their priority is irrelevant.

Similarly, if the desired velocity of the third task were $\dot{\boldsymbol{r}}_{3,TA,2}$ in (3) rather than $\dot{\boldsymbol{r}}_3$, then again all three tasks would be realizable,

$$\dot{\boldsymbol{r}}_{A,m_2} = \begin{pmatrix} \dot{\boldsymbol{r}}_1 \\ \dot{\boldsymbol{r}}_2 \\ \dot{\boldsymbol{r}}_{3,TA,2} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ -0.5 \\ 1.3645 \\ -1.1355 \end{pmatrix} \in \mathcal{R}(\boldsymbol{J}_A),$$

and thus

$$\dot{\boldsymbol{q}}_{TA,m_2} = \boldsymbol{J}_A^{\#}\, \dot{\boldsymbol{r}}_{A,m_2} = \begin{pmatrix} -0.3477 \\ -1.5756 \\ 0.8987 \\ 6.0622 \\ 7.0000 \end{pmatrix} \text{[rad/s]} \qquad \Rightarrow \qquad \boldsymbol{e}_{A,m_2} = \dot{\boldsymbol{r}}_{A,m_2} - \boldsymbol{J}_A\, \dot{\boldsymbol{q}}_{TA,m_2} = \boldsymbol{0}.$$

Indeed, and as before, the same solution is obtained when using the TP method for this other modified problem ($\dot{\boldsymbol{q}}_{TP,m_2} = \dot{\boldsymbol{q}}_{TA,m_2}$, and $\boldsymbol{e}_1 = \boldsymbol{e}_2 = \boldsymbol{e}_{3,m_2} = \boldsymbol{0}$).

As for item *v)*, for the planar case the coordinate transformation from relative (D–H) angles to absolute angles (w.r.t. the $\boldsymbol{x}_0$ axis) and viceversa is

$$q_i = \sum_{j=1}^{i} \theta_j \quad \text{for } i = 1, \ldots, n \qquad \Longleftrightarrow \qquad \theta_1 = q_1 \qquad \theta_i = q_i - q_{i-1} \quad \text{for } i = 2, \ldots, n.$$

Being this mapping linear, it can be represented in matrix form as $\boldsymbol{q} = \boldsymbol{T}\boldsymbol{\theta}$ and $\boldsymbol{\theta} = \boldsymbol{T}^{-1}\boldsymbol{q}$. The same holds also for the velocity transformation. In the case of $n = 5$ joints, we have

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Thus, from (1) we obtain

$$\dot{\boldsymbol{\theta}}_{TP} = \boldsymbol{T}^{-1}\dot{\boldsymbol{q}}_{TP} = \begin{pmatrix} -1.3170 \\ 0.4510 \\ 2.6338 \\ 4.2944 \\ 0.9378 \end{pmatrix} \text{ [rad/s]}.$$

Exercise 2

This exercise can be completed without using any symbolic toolbox or recursive computation method, taking advantage of the simple kinematic structure of the PPR robot. The kinetic energies of the first two links are given by

$$T_1 = \frac{1}{2}\, m_1 \dot{q}_1^2 \qquad T_2 = \frac{1}{2}\, m_2 \left(\dot{q}_1^2 + \dot{q}_2^2\right).$$

For the third link, denoting by $d_{c3} > 0$ the distance of its CoM from the joint axis 3, we have

$${}^0\boldsymbol{p}_{c3} = \begin{pmatrix} L_1 - d_{c3}\sin q_3 \\ q_2 \\ q_1 - L_2 - d_{c3}\cos q_3 \end{pmatrix} \quad \Rightarrow \quad {}^0\boldsymbol{v}_{c3} = {}^0\dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -d_{c3}\cos q_3\, \dot{q}_3 \\ \dot{q}_2 \\ \dot{q}_1 + d_{c3}\sin q_3\, \dot{q}_3 \end{pmatrix} \qquad {}^3\boldsymbol{\omega}_3 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_3 \end{pmatrix},$$

and then

$$T_3 = \frac{1}{2}\, m_3 \left\|{}^0\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\, {}^3\boldsymbol{\omega}_3^T\, {}^3\boldsymbol{I}_{c3}\, {}^3\boldsymbol{\omega}_3 = \frac{1}{2}\, m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\, \dot{q}_3^2 + 2\, d_{c3}\sin q_3\, \dot{q}_1\dot{q}_3\right) + \frac{1}{2}\, I_{zz3}\, \dot{q}_3^2.$$

Therefore,

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\, \dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & m_3 d_{c3}\sin q_3 \\ 0 & m_2 + m_3 & 0 \\ m_3 d_{c3}\sin q_3 & 0 & I_{zz3} + m_3 d_{c3}^2 \end{pmatrix}.$$

The velocity terms are computed from the Christoffel matrices as

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}}{\partial q_i}\right) \qquad i = 1,\ldots,n,$$

where $\boldsymbol{m}_i(\boldsymbol{q})$ is the i-th column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_3 d_{c3}\cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = m_3 d_{c3}\cos q_3\, \dot{q}_3^2,$$

while $\boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{C}_3(\boldsymbol{q}) = \boldsymbol{O}$, which imply also $c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{0}$. Thus,

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} m_3 d_{c3}\cos q_3\, \dot{q}_3^2 \\ 0 \\ 0 \end{pmatrix}.$$

Two possible factorizations $\boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}$ of $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the skew-symmetric property of $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ are

$$S_1'(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & 0 & m_3 d_{c3}\cos q_3\, \dot{q}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_1''(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 + m_3 d_{c3}\cos q_3\, \dot{q}_3 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & 0 \end{pmatrix}.$$

The first matrix is obtained directly from the Christoffel matrices, with the first row being $\dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})$ and the remaining two rows being zero (since $\boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{C}_3(\boldsymbol{q}) = \boldsymbol{O}$). The second matrix is obtained by adding to the first one a skew-symmetric matrix $\boldsymbol{S}(\dot{\boldsymbol{q}})$ built from $\dot{\boldsymbol{q}} \times \dot{\boldsymbol{q}} = \boldsymbol{S}(\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{0}$. Being

$$\dot{\boldsymbol{M}}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ 0 & 0 & 0 \\ m_3 d_{c3} \cos q_3\, \dot{q}_3 & 0 & 0 \end{pmatrix},$$

the two factorizations lead to the skew-symmetric matrices

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1' = \begin{pmatrix} 0 & 0 & -m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ 0 & 0 & 0 \\ m_3 d_{c3} \cos q_3\, \dot{q}_3 & 0 & 0 \end{pmatrix}$$

and

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1'' = \begin{pmatrix} 0 & 2\dot{q}_3 & -2\dot{q}_2 - m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ -2\dot{q}_3 & 0 & 2\dot{q}_1 \\ 2\dot{q}_2 + m_3 d_{c3} \cos q_3\, \dot{q}_3 & -2\dot{q}_1 & 0 \end{pmatrix}.$$

On the other hand, two possible factorizations $\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}$ of vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ such that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2$ is not skew-symmetric are

$$S_2'(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & 0 & m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ -\dot{q}_2 & \dot{q}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_2''(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 + m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

leading respectively to

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2' = \begin{pmatrix} 0 & 0 & -m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ 2\dot{q}_2 & -2\dot{q}_1 & 0 \\ m_3 d_{c3} \cos q_3\, \dot{q}_3 & 0 & 0 \end{pmatrix}$$

and

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2'' = \begin{pmatrix} 0 & 2\dot{q}_3 & -2\dot{q}_2 - m_3 d_{c3} \cos q_3\, \dot{q}_3 \\ 0 & 0 & 0 \\ m_3 d_{c3} \cos q_3\, \dot{q}_3 & 0 & 0 \end{pmatrix}.$$

The potential energies of the links are given by

$$\begin{aligned} U_1 &= m_1 g_0 q_1 + U_{10} \\ U_2 &= m_2 g_0 q_1 + U_{20} \\ U_3 &= m_3 g_0 \left(q_1 - d_{c3} \cos q_3\right) + U_{30}. \end{aligned}$$

Therefore,

$$U = U_1 + U_2 + U_3 \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} (m_1 + m_2 + m_3)\, g_0 \\ 0 \\ m_3 d_{c3} g_0 \sin q_3 \end{pmatrix}.$$

As a result, defining the following $p = 4$ dynamic coefficients

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= m_2 + m_3 \\ a_3 &= m_3 d_{c3} \\ a_4 &= I_{zz3} + m_3 d_{c3}^2, \end{aligned}$$

the dynamic model can be written as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau}$$

where

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 + g_0 & 0 & \ddot{q}_3 \sin q_3 + \dot{q}_3^2 \cos q_3 & 0 \\ 0 & \ddot{q}_2 & 0 & 0 \\ 0 & 0 & (\ddot{q}_1 + g_0) \sin q_3 & \ddot{q}_3 \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Finally, given the desired joint trajectory

$$\boldsymbol{q}_d(t) = \begin{pmatrix} 0 \\ \sin(2\omega t) \\ \omega t \end{pmatrix} \;\Rightarrow\; \dot{\boldsymbol{q}}_d(t) = \begin{pmatrix} 0 \\ 2\omega \cos(2\omega t) \\ \omega \end{pmatrix} \;\Rightarrow\; \ddot{\boldsymbol{q}}_d(t) = \begin{pmatrix} 0 \\ -4\omega^2 \sin(2\omega t) \\ 0 \end{pmatrix} \qquad \text{for } t \geq 0,$$

the expression of the inverse dynamics torque is

$$\boldsymbol{\tau}_d(t) = \begin{pmatrix} \tau_{d1}(t) \\ \tau_{d2}(t) \\ \tau_{d3}(t) \end{pmatrix} = \begin{pmatrix} (m_1 + m_2 + m_3)\, g_0 + m_3 d_{c3}\, \omega^2 \cos(\omega t) \\ -4\,(m_2 + m_3)\, \omega^2 \sin(2\omega t) \\ m_3 d_{c3} g_0 \sin(\omega t) \end{pmatrix} \qquad \text{for } t \geq 0. \tag{4}$$

In order to be matched with the desired trajectory at time $t = 0$, the initial robot state should be

$$\boldsymbol{q}(0) = \boldsymbol{q}_d(0) = \boldsymbol{0} \qquad \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_d(0) = \begin{pmatrix} 0 \\ 2\omega \\ \omega \end{pmatrix}.$$

Starting from this state and using the torque command (4), the desired joint trajectory $\boldsymbol{q}_d(t)$ is perfectly executed for all $t \geq 0$ (at least, in nominal conditions).

Note that the initial torque is different from zero only for the first component $\tau_{d1}(0)$; this is composed by the constant term $(m_1 + m_2 + m_3)\, g_0$ needed to balance gravity in any position, and by a term $m_3 d_{c3}\, \omega^2$ contrasting the centrifugal force due to the rotation of link 3 at the constant speed $\dot{q}_3 = \omega$ that would otherwise move joint 1. The periodic motion of joint 2 is fully decoupled from that of the other two degrees of freedom of the robot; in fact, the second dynamic equation, $(m_2 + m_3)\, \ddot{q}_2 = \tau_2$, is independent of the other two (and $q_2(t)$ does not enter in the first and third equations of motion). Finally, the inverse dynamics torque on joint 3 is only due to the need of compensating the time-varying (periodic) gravity load, in order to keep the uniform rotation of link 3; because dissipative effects are neglected, if gravity were not present, no torque would be needed at this joint to sustain its constant angular speed $\dot{q}_3 = \omega$.

Exercise 3

From the D–H table of parameters and from the location of the link CoMs in Fig. 3, we extract the information needed by the recursive algorithm for computing the robot kinetic energy in moving frames:

$$\begin{aligned} {}^0\boldsymbol{R}_1(q_1) &= \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix} & {}^0\boldsymbol{r}_{0,1} &= \begin{pmatrix} 0 \\ 0 \\ L_1 \end{pmatrix} \\ {}^1\boldsymbol{R}_2(q_2) &= \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & {}^1\boldsymbol{r}_{1,2} &= \begin{pmatrix} L_2 \cos q_2 \\ L_2 \sin q_2 \\ 0 \end{pmatrix} \\ {}^2\boldsymbol{R}_3(q_3) &= \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} & {}^2\boldsymbol{r}_{2,3} &= \begin{pmatrix} L_3 \cos q_3 \\ L_3 \sin q_3 \\ 0 \end{pmatrix}, \end{aligned}$$

The steps of the recursive algorithm for revolute joints only (like for our 3R robot) are as follows.

step 0 (initialization): ${}^0\boldsymbol{\omega}_0 = \mathbf{0}$, ${}^0\boldsymbol{v}_0 = \mathbf{0}$; $\boldsymbol{z}_0 = (0, 0, 1)$.

step 1 (link 1):

$$
{}^1\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{\omega}_0 + \dot{q}_1\boldsymbol{z}_0\right) = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix}
$$

$$
{}^1\boldsymbol{v}_1 = {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{v}_0 + {}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{r}_{0,1}\right) = {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_1 \\ 0 \end{pmatrix} = \mathbf{0}
$$

$$
{}^1\boldsymbol{v}_{c1} = {}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -L_1 + d_{c1} \\ 0 \end{pmatrix} = \mathbf{0}
$$

$$
T_1 = \frac{1}{2}\, m_1 \left\|{}^1\boldsymbol{v}_{c1}\right\|^2 + \frac{1}{2}\, {}^1\boldsymbol{\omega}_1^T\, {}^1\boldsymbol{I}_{c1}\, {}^1\boldsymbol{\omega}_1 = \frac{1}{2}\, {}^1\boldsymbol{\omega}_1^T\, {}^1\boldsymbol{I}_{c1}\, {}^1\boldsymbol{\omega}_1 = \frac{1}{2}\, I_{yy1}\, \dot{q}_1^2.
$$

step 2 (link 2);

$$
{}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{\omega}_1 + \dot{q}_2\boldsymbol{z}_0\right) = {}^1\boldsymbol{R}_2^T(q_2)\begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
$$

$$
{}^2\boldsymbol{v}_2 = {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{r}_{1,2}\right) = {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{1,2} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix}
$$

$$
{}^2\boldsymbol{v}_{c2} = {}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{c2} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -L_2 + d_{c2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d_{c2}\,\dot{q}_2 \\ -d_{c2} c_2\,\dot{q}_1 \end{pmatrix}
$$

$$
T_2 = \frac{1}{2}\, m_2 \left\|{}^2\boldsymbol{v}_{c2}\right\|^2 + \frac{1}{2}\, {}^2\boldsymbol{\omega}_2^T\, {}^2\boldsymbol{I}_{c2}\, {}^2\boldsymbol{\omega}_2 = \frac{1}{2}\, m_2 d_{c2}^2\left(\dot{q}_2^2 + c_2^2\,\dot{q}_1^2\right) + \frac{1}{2}\left(\left(s_2^2 I_{xx2} + c_2^2 I_{yy2}\right)\dot{q}_1^2 + I_{zz2}\,\dot{q}_2^2\right)
$$

step 3 (link 3);

$$
{}^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{\omega}_2 + \dot{q}_3\boldsymbol{z}_0\right) = {}^2\boldsymbol{R}_3^T(q_3)\begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} = \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix}
$$

$$
\begin{aligned}
{}^3\boldsymbol{v}_3 &= {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_3 \times {}^2\boldsymbol{r}_{2,3}\right) = {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{v}_2 + {}^3\boldsymbol{\omega}_3 \times {}^3\boldsymbol{r}_{2,3} \\
&= \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 + L_3\left(\dot{q}_2 + \dot{q}_3\right) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\dot{q}_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^3\boldsymbol{v}_{c3} = {}^3\boldsymbol{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\boldsymbol{r}_{c3} &= \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 + L_3\left(\dot{q}_2 + \dot{q}_3\right) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot{q}_1 \\ c_{23}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} \times \begin{pmatrix} -L_3 + d_{c3} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} L_2 s_3\,\dot{q}_2 \\ L_2 c_3\,\dot{q}_2 + d_{c3}\left(\dot{q}_2 + \dot{q}_3\right) \\ -\left(L_2 c_2 + d_{c3} c_{23}\right)\dot{q}_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T_3 &= \frac{1}{2}\, m_3 \left\|{}^3\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\, {}^3\boldsymbol{\omega}_3^T\, {}^3\boldsymbol{I}_{c3}\, {}^3\boldsymbol{\omega}_3 \\
&= \frac{1}{2}\, m_3\left(\left(d_{c3} c_{23} + L_2 c_2\right)^2 \dot{q}_1^2 + L_2^2 s_3^2 \dot{q}_2^2 + \left(d_{c3}(\dot{q}_2 + \dot{q}_3) + L_2 c_3 \dot{q}_2\right)^2\right) \\
&\quad + \frac{1}{2}\left(\left(I_{xx3} s_{23}^2 + I_{yy3} c_{23}^2\right)\dot{q}_1^2 + I_{zz3}\left(\dot{q}_2 + \dot{q}_3\right)^2\right)
\end{aligned}
$$

As a result, from

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

we extract the inertia matrix of the 3R robot as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 c_2^2 + a_3 s_2^2 + a_4 c_{23}^2 + a_5 s_{23}^2 + 2a_6 c_2 c_{23} & 0 & 0 \\ 0 & a_7 + 2a_6 c_3 & a_8 + a_6 c_3 \\ 0 & a_8 + a_6 c_3 & a_8 \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= I_{yy1} \\ a_2 &= I_{yy2} + m_2 d_{c2}^2 + m_3 L_2^2 \\ a_3 &= I_{xx2} \\ a_4 &= I_{yy3} + m_3 d_{c3}^2 \\ a_5 &= I_{xx3} \\ a_6 &= m_3 d_{c3} L_2 \\ a_7 &= I_{zz2} + m_2 d_{c2}^2 + I_{zz3} + m_3 d_{c3}^2 + m_3 L_2^2 \\ a_8 &= I_{zz3} + m_3 d_{c3}^2 \end{aligned}$$

is a set (not necessarily minimal — this was by no means requested!) of dynamic coefficients used here just for compactness.

At this stage, no further dynamic computation is needed for solving the second part of the problem. The requested task is described by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} c_1\,(L_2 c_2 + L_3 c_{23}) \\ s_1\,(L_2 c_2 + L_3 c_{23}) \end{pmatrix}$$

with associated 2×3 Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1\,(L_2 c_2 + L_3 c_{23}) & -c_1\,(L_2 s_2 + L_3 s_{23}) & -c_1 L_3 s_{23} \\ c_1\,(L_2 c_2 + L_3 c_{23}) & -s_1\,(L_2 s_2 + L_3 s_{23}) & -s_1 L_3 s_{23} \end{pmatrix}.$$

Since from the dynamic model one has $\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$, the proposed optimization problem can be reformulated at a given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in terms of the unknown $\ddot{\boldsymbol{q}} \in \mathbb{R}^3$ as

$$\min_{\ddot{\boldsymbol{q}}} \frac{1}{2}\,\ddot{\boldsymbol{q}}^T \boldsymbol{M}^2(\boldsymbol{q})\ddot{\boldsymbol{q}} \qquad \text{s.t.} \qquad \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

which is in the standard form of an LQ problem (without null space term). Its explicit solution is given by

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{M}^2}(\boldsymbol{q})\left(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) = \boldsymbol{M}^{-2}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^{-2}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\left(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \tag{5}$$

under the assumption that the Jacobian has full rank (= 2). The locally optimal acceleration is thus given by the *squared inertia-weighted pseudoinverse* of the task Jacobian.

Replacing now the numerical data of the problem, we obtain for the two relevant matrices in the solution (5) at $\boldsymbol{q} = (3\pi/2, \pi/4, 0)$ [rad]:

$$\boldsymbol{M} = \begin{pmatrix} 0.7750 & 0 & 0 \\ 0 & 1.3500 & 0.4250 \\ 0 & 0.4250 & 0.1750 \end{pmatrix} \qquad \boldsymbol{J} = \begin{pmatrix} 0.7071 & 0 & 0 \\ 0 & 0.7071 & 0.3536 \end{pmatrix}.$$

The Jacobian is of rank 2. Since the current joint velocity is $\dot{\boldsymbol{q}} = \boldsymbol{0}$, we do not need the time derivative of the Jacobian (the term $\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is zero). The solution for $\ddot{\boldsymbol{r}}_d = (-1, -1)$ [m/s^2] is then

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} -1.4142 \\ 2.4965 \\ -7.8214 \end{pmatrix} \ [\text{rad/s}^2].$$

* * * * *