# Robotics 2 - Final Test

**June 1, 2016**

A 2R robot moving in the vertical plane uses the joint coordinates defined in Fig. 1. The inertia matrix of this robot can be written as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}, \tag{1}$$

where the three dynamic coefficients $a_i$, $i = 1, 2, 3$, satisfy $a_1 > a_3 > a_2 > 0$. The robot is initially at rest and should move under the constraint of keeping the *absolute* orientation of the second link at a constant angle $\beta$ w.r.t. the $\boldsymbol{x}_0$ axis, with the value $0 < \beta < \pi/2$ being specified by the initial configuration.

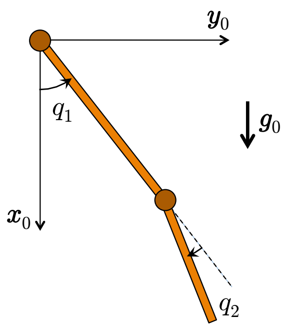


Figure 1: A planar 2R robot

- Derive a one-dimensional *reduced* dynamic model of the robot under the given geometric constraint on its motion. Point out the features of the obtained model. Are there any special (e.g., singular) situations from the dynamic point of view?
- With the robot at rest at $t = 0$ and with $\boldsymbol{q}(0)$ satisfying the geometric constraint, determine *all* feasible torques $\boldsymbol{u}(0)$ that can keep the robot in static equilibrium. Which are the associated values of the scalar constraint force $\lambda(0)$?
- Describe how a simulation with arbitrary joint torque inputs $\boldsymbol{u}(t) \in \mathbb{R}^2$ should be performed when using this model.
- Provide the symbolic expression of the joint torques $u_1$ and $u_2$ that should be applied when the robot is at a given generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ compatible with the constraint, so that the pseudoacceleration is equal to a desired value $\dot{v}_d \in \mathbb{R}$ and the constraint force is $\lambda = 0$.

**[150 minutes; open books]**

# Solution

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We first derive the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}})$ from the expression of $\boldsymbol{B}(\boldsymbol{q})$ in (1), and then compute the potential energy $U(\boldsymbol{q})$ due to gravity and the associated vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model.

From (1) and

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix}, \quad c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right\} \ (i = 1, 2),$$

we have

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{\begin{pmatrix} 0 & -2a_2\sin q_2 \\ 0 & -a_2 \sin q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2\sin q_2 & -a_2\sin q_2 \end{pmatrix} - \boldsymbol{0}\right\} \\ &= \begin{pmatrix} 0 & -a_2\sin q_2 \\ -a_2\sin q_2 & -a_2\sin q_2 \end{pmatrix} \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left\{\begin{pmatrix} 0 & -a_2\sin q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -a_2\sin q_2 & 0 \end{pmatrix} - \begin{pmatrix} -2a_2\sin q_2 & -a_2\sin q_2 \\ -a_2\sin q_2 & 0 \end{pmatrix}\right\} \\ &= \begin{pmatrix} a_2\sin q_2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2\sin q_2\,(2\dot{q}_1 + \dot{q}_2)\,\dot{q}_2 \\ a_2\sin q_2\,\dot{q}_1^2 \end{pmatrix}. \tag{2}$$

The potential energy is given by

$$U = \sum_{i=1}^{2} U_i, \qquad U_i = -m_i \boldsymbol{g}_0^T \boldsymbol{r}_{0,c_i}, \quad i = 1, 2.$$

Since

$$\boldsymbol{g}_0^T = \begin{pmatrix} g_0 & 0 & 0 \end{pmatrix}, \qquad g_0 = 9.81\,[\text{m/s}^2],$$

we need in the computations only the $x$-component of the position vector $\boldsymbol{r}_{0,c_i}$ of the center of mass of the link $i$, for $i = 1, 2$. We have

$$\begin{aligned} U_1 &= -m_1 g_0\, d_1 \cos q_1 \\ U_2 &= -m_2 g_0\,(\ell_1\cos q_1 + d_2\cos(q_1+q_2))\,, \end{aligned}$$

where $m_i$ is the mass of link $i$, $d_i$ is the distance of the CoM of link $i$ from the axis of joint $i$, and $\ell_1$ is the length of link 1. Therefore,

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} a_4\sin q_1 + a_5\sin(q_1+q_2) \\ a_5\sin(q_1+q_2) \end{pmatrix},$$

where we have introduced the dynamic coefficients $a_4 = (m_1 d_1 + m_2\ell_1)g_0$ and $a_5 = m_2 d_2 g_0$.

The scalar geometric constraint ($m = 1$) imposed on the motion of the robot (with a configuration space of dimension $n = 2$) is to keep the second link oriented at a constant angle $\beta$ w.r.t. the axis $\boldsymbol{x}_0$. Thus

$$h(\boldsymbol{q}) = q_1 + q_2 - \beta = 0 \qquad \Rightarrow \qquad \boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0, \quad \text{with } \boldsymbol{A} = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

We note that matrix $\boldsymbol{A}$ is constant since the constraint is linear in $\boldsymbol{q}$. The constrained dynamic model is then

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{A}^T\lambda, \qquad \text{s.t. } h(\boldsymbol{q}) = 0.$$

In order to obtain a reduced dynamic model, we can define a matrix $\boldsymbol{D}$ in the following way:

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{(a nonsingular matrix).} \tag{3}$$

From this

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix},$$

and so we define

$$v = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{q}_2, \quad \dot{v} = \ddot{q}_2 \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} v, \quad \ddot{\boldsymbol{q}} = \boldsymbol{F}\dot{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \dot{v}.$$

Since all these matrices are constant, many simplifications will occur. In particular, the reduced dynamic model (of dimension $n - m = 1$) is expressed by a single differential equation of the form

$$\left(\boldsymbol{F}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}\right)\dot{v} = \boldsymbol{F}^T\left(\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right), \tag{4}$$

while the expression of the scalar constraint force will be given by

$$\lambda = \boldsymbol{E}^T\left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}\dot{v} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{u}\right). \tag{5}$$

These model expressions will hold everywhere, thanks to the global invertibility of the matrix in (3); no singularity will occur with the chosen representation of the reduced dynamics.

Performing computations, we have

$$\begin{aligned}
&\boldsymbol{F}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F} = a_1 - a_3 > 0 \qquad \text{(constant!)},\\
&\boldsymbol{F}^T\boldsymbol{u} = u_2 - u_1,\\
&-\boldsymbol{F}^T\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) = -a_2\sin q_2\left(\dot{q}_1 + \dot{q}_2\right)^2 + a_4\sin q_1,\\
&\boldsymbol{E}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F} = a_3 - a_1 - a_2\cos q_2,\\
&\boldsymbol{E}^T\boldsymbol{u} = u_1,\\
&\boldsymbol{E}^T\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) = c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + g_1(\boldsymbol{q}) = -a_2\sin q_2\,(2\dot{q}_1 + \dot{q}_2)\dot{q}_2 + a_4\sin q_1 + a_5\sin(q_1 + q_2).
\end{aligned}$$

The reduced model (4) has to be initialized at a state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ that is compatible with the constraint, or

$$q_1(0) + q_2(0) = \beta, \qquad \dot{q}_1(0) + \dot{q}_2(0) = 0. \tag{6}$$

The velocity constraint is indeed satisfied if the robot is initially at rest (although this is not the only case). The constraints (6) will propagate then for all $t > 0$, if we proceed by integrating (4). Thus, $\dot{q}_1 + \dot{q}_2 = 0$ will hold at any time, and the dynamics (4) simplifies finally to

$$(a_1 - a_3)\,\dot{v} = u_2 - u_1 + a_4\sin q_1, \tag{7}$$

while the expression (5) of the constraint force (multiplier) becomes

$$\lambda = (a_3 - a_1 - a_2\cos q_2)\,\dot{v} - a_2\sin q_2\,\dot{q}_1\dot{q}_2 + a_4\sin q_1 + a_5\sin\beta - u_1. \tag{8}$$

The pseudoinertia in (7) is now constant, and the quadratic velocity terms have disappeared as expected. The only residual nonlinearity is due to the gravity.

If the robot starts at rest ($\dot{q}_1(0) = \dot{q}_2(0) = 0$) and satisfies the constraints (6), in order to keep the static equilibrium ($\dot{v} = 0$), from (7) we need to have

$$u_1(0) - u_2(0) = a_4 \sin q_1(0),$$

so that there is (apparently) an infinity of joint torque combinations that will preserve the equilibrium. Associated to each of these, from (8) there is a single constraint force given by

$$\lambda(0) = a_4 \sin q_1(0) + a_5 \sin \beta - u_1(0) = a_5 \sin \beta - u_2(0).$$

Note however that if the constraint $h(\boldsymbol{q}) = 0$ is a *virtual* one, namely it is not imposed by a mechanism but it is enforced only through a control action, then there can be no real constraint forces generated, i.e., $\lambda(0)$ should vanish. In particular, this happens only with the natural choice

$$\boldsymbol{u}(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} a_4 \sin q_1(0) + a_5 \sin \beta \\ a_5 \sin \beta \end{pmatrix} = \boldsymbol{g}(\boldsymbol{q}(0)).$$

More in general, a simulation with the reduced dynamic model (7) and an arbitrary input torque $\boldsymbol{u}(t)$ proceeds, with an integration step $T > 0$, as follows[1]:

1. The initial state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) = (\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ should satisfy (6). Set $k = 0$.
2. At every sampling instant $t_k = kT$, compute $\dot{v}_k = \dot{v}(t_k)$ as

$$\dot{v}_k = \frac{u_{k,2} - u_{k,1} + a_4 \sin q_{k,1}}{a_1 - a_3},$$

evaluate $\ddot{\boldsymbol{q}}_k = \boldsymbol{F} \dot{v}_k$, and use your preferred integration routine to obtain $(\boldsymbol{q}_{k+1}, \dot{\boldsymbol{q}}_{k+1})$.

2'. In alternative to step 2, since $\dot{v} = \ddot{q}_2$, compute

$$\ddot{q}_{k,2} = \frac{u_{k,2} - u_{k,1} + a_4 \sin(\beta - q_{k,2})}{a_1 - a_3},$$

use your preferred integration routine to obtain $(q_{k+1,2}, \dot{q}_{k+1,2})$, and evaluate the remaining components of the state as

$$q_{k+1,1} = \beta - q_{k+1,2}, \qquad \dot{q}_{k+1,1} = -\dot{q}_{k+1,2}.$$

3. Set $k = k + 1$, and cycle over step 2 (or 2').

Finally, to address the last control problem, we evaluate eqs. (7–8) for $\dot{v} = \dot{v}_d$ and $\lambda = 0$ at a generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ obtaining the (unique) needed joint torque

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (a_3 - a_1 - a_2 \cos q_2)\, \dot{v}_d - a_2 \sin q_2\, \dot{q}_1 \dot{q}_2 + a_4 \sin q_1 + a_5 \sin \beta \\ u_1 + (a_1 - a_3)\, \dot{v}_d - a_4 \sin q_1 \end{pmatrix}. \tag{9}$$

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[1] We use the notation $x_{k,i}$ to denote the $i$-th component of a vector $\boldsymbol{x}(t)$ evaluated at time $t_k = kT$.