Robotics 2

July 8, 2025

Exercise 1

Consider a robot with n joints that is redundant with respect to an end-effector velocity task $\dot{\boldsymbol{r}} \in \mathbb{R}^m$, being $m < n$. At a given configuration $\boldsymbol{q}$, the joint velocity command $\dot{\boldsymbol{q}} \in \mathbb{R}^n$ adopts the Projected Gradient (PG) method, using a criterion $H(\boldsymbol{q})$ to be locally maximized, in the form

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_0 + \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\dot{\boldsymbol{r}} - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_0) \qquad \text{with } \dot{\boldsymbol{q}}_0 = \alpha \nabla_{\boldsymbol{q}} H(\boldsymbol{q}), \tag{1}$$

where $\alpha \in [0, \alpha_{\max}]$ is a scalar step to be chosen in the gradient direction of $H(\boldsymbol{q})$. The joint velocity is bounded componentwise by

$$-\dot{\boldsymbol{q}}_{\max} \leq \dot{\boldsymbol{q}} \leq \dot{\boldsymbol{q}}_{\max} \qquad \Longleftrightarrow \qquad -\dot{q}_{\max,i} \leq \dot{q}_i \leq \dot{q}_{\max,i} \qquad i = 1, \ldots, n. \tag{2}$$

Assuming that the minimum norm velocity solution $\dot{\boldsymbol{q}}_r = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{r}}$ is feasible, i.e., satisfies all the bounds (2), determine the largest step α^* in its domain of definition so that the command (1) is still feasible.

Find the analytic expression of the optimal step α^*. Provide then a geometric illustration of the solution for the generic case of a robot with $n = 2$ joints that performs a task of dimension $m = 1$, representing the entire problem in the plane $(\dot{q}_1, \dot{q}_2)$.

Exercise 2

Consider the planar 4-dof robot shown in Fig. 1, moving under gravity. All robot joints are prismatic and thus the joint torques $\boldsymbol{\tau} \in \mathbb{R}^4$ are in fact forces (with units expressed in [N]).

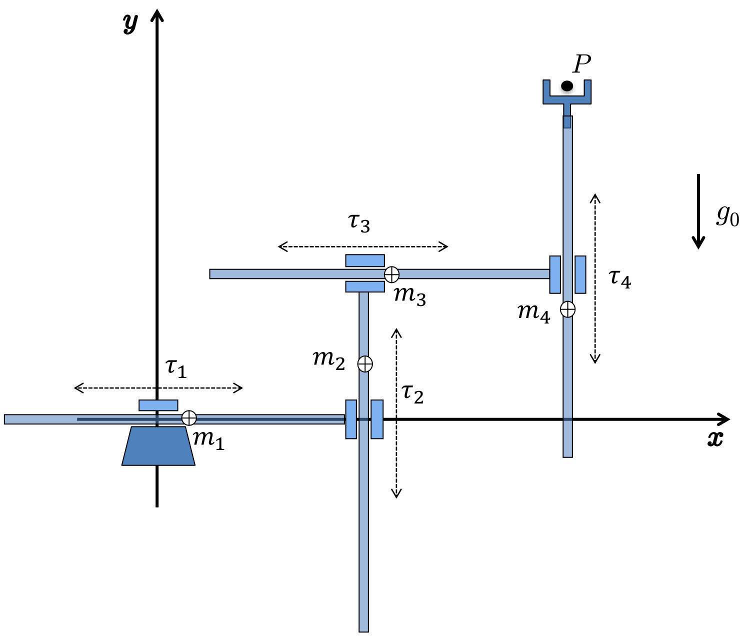

Figure 1: A planar 4P robot in the vertical plane

a) Derive in symbolic form the dynamic model of this robot in the joint space.

b) Determine the 2×2 Cartesian inertia matrix $\boldsymbol{M}_r$ at the end-effector level. Compute then $\boldsymbol{M}_r$ numerically for $\boldsymbol{q} = \boldsymbol{0}$ and equal link masses $m_i = 1$ [kg], $i = 1, \ldots, 4$.

c) Suppose you have no knowledge about the values of the dynamic parameters of this 4P robot (except for the gravity acceleration $g_0 = 9.81$ m/s^2). Design an adaptive control law for $\boldsymbol{\tau}$ that is able to globally asymptotically stabilize the tracking error along a smooth desired trajectory $\boldsymbol{q}_d(t)$. The controller should be of minimal dimension.

[continue on the back]

Exercise 3

The dynamics of a robot manipulator with $n = 6$ joints is described by the following equation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F}, \tag{3}$$

where $\boldsymbol{F} \in \mathbb{R}^6$ is an external wrench (force and moment) acting on the robot end effector and $\boldsymbol{J}_g$ is the corresponding geometric Jacobian. The robot should perform a desired motion $\boldsymbol{r}_d(t)$ in a m-dimensional task space, with $m < 6$, characterized by a task function $\boldsymbol{r} = \boldsymbol{t}(\boldsymbol{q})$ with the accociated $m \times 6$ task Jacobian $\boldsymbol{J}_t(\boldsymbol{q}) = \partial \boldsymbol{t}/\partial \boldsymbol{q}$ which is assumed to have full rank.

a) Determine the expression of the control torque $\boldsymbol{\tau} \in \mathbb{R}^6$ that instantaneously executes the task while locally minimizing the inverse inertia weighted torque norm

$$H = \frac{1}{2} \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\|^2_{\boldsymbol{M}^{-1}(\boldsymbol{q})}, \tag{4}$$

where $\boldsymbol{\tau}_0 \in \mathbb{R}^6$ is a preferred joint torque that biases the solution. Assume the full state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ measurable and that a F/T sensor is available to measure $\boldsymbol{F}$.

b) How will the solution torque be modified in the presence of a task error $\boldsymbol{e} = \boldsymbol{r}_d - \boldsymbol{r}$ to be rejected at a desired exponential rate? Will this modified control torque be still optimal in terms of a suitable constrained optimization problem?

c) Consider a planar PR robot, as in the lecture slides `03_LagrangianDynamics_1.pdf`, and apply the result found in item a) for this simpler case, where $n = 2$. The kinematic and dynamic parameters of the robot are

$$l_2 = 0,5 \text{ m} \qquad m_1 = m_2 = 1 \text{ kg} \qquad d_{c2} = 0.25 \text{ m} \qquad I_{c2,zz} = 0.1 \text{ kgm}^2.$$

The robot end effector should perform the one-dimensional motion task $r_d(t) = 1 - \cos 3t$ specified only along the x-direction (thus, $m = 1$). At $t = 0$, the robot is at rest ($\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$) in the configuration $\boldsymbol{q}(0) = (0, \pi/2)$ [m,rad]. The external force $\boldsymbol{F} \in \mathbb{R}^2$ being applied to the robot end effector and the preferred joint torque $\boldsymbol{\tau}_0 \in \mathbb{R}^2$ are

$$\boldsymbol{F} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ [N]} \qquad \text{and} \qquad \boldsymbol{\tau}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{matrix} \text{N} \\ \text{Nm.} \end{matrix}$$

Compute then the numerical value of the optimal torque $\boldsymbol{\tau}(0)$ at the initial instant.

Exercise 4

Consider the same PR robot of Exercise 3c), but now moving freely in a vertical plane, without any interaction with the environment. Design a FDI system that enables to detect and isolate any fault of the robot actuation system, without the need to measure the joint torques, nor compute accelerations, nor invert the inertia matrix. Provide the explicit and computable expression of the so-called residual vector $\boldsymbol{r} \in \mathbb{R}^2$. Sketch the behavior of the two residual components when a total fault of the first actuator occurs at an instant t when the robot is in the state $\boldsymbol{q}(t) = (0, \pi/2)$ [m, rad], $\dot{\boldsymbol{q}}(t) = (1, -1)$ [m/s, rad/s], and a constant torque $\boldsymbol{\tau} = (5, -2)$ [N,Nm] is being requested by the control law.

[4 hours, open books]

Solution

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Exercise 1

Reorganize the PG method (1) as

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_r + \alpha\,\dot{\boldsymbol{q}}_n \qquad \text{with } \dot{\boldsymbol{q}}_n = \Big(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\Big)\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \boldsymbol{P}(\boldsymbol{q})\nabla_{\boldsymbol{q}} H(\boldsymbol{q}),$$

in terms of the minimum norm solution $\dot{\boldsymbol{q}}_r = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{r}}$ for the task and a null-space joint velocity $\dot{\boldsymbol{q}}_n$, with the gradient of the criterion $H(\boldsymbol{q})$ premultiplied by the projection matrix $\boldsymbol{P}(\boldsymbol{q})$ in the null space of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$. From (2), we have

$$-\left(\dot{\boldsymbol{q}}_{\max} + \dot{\boldsymbol{q}}_r\right) \leq \alpha\,\dot{\boldsymbol{q}}_n \leq \dot{\boldsymbol{q}}_{\max} - \dot{\boldsymbol{q}}_r,$$

where all components of the vectors on the left side and on the right side of this chain of inequalities are non-positive and, respectively, non-negative, being $\dot{\boldsymbol{q}}_r$ by assumption a feasible velocity. Thus, both inequalities are certainly satisfied for $\alpha = 0$. By working componentwise, in order to find the largest feasible value for α we can distinguish three situations, depending on the sign of the i-th component of vector $\dot{\boldsymbol{q}}_n$.

- For all $i \in I^+ \subseteq \{1, \ldots, n\}$ such that $\dot{q}_{n,i} > 0$, we have

$$-\frac{\dot{q}_{\max,i} + \dot{q}_{r,i}}{\dot{q}_{n,i}} \leq 0 \leq \alpha \leq A^+_{\max,i} = \frac{\dot{q}_{\max,i} - \dot{q}_{r,i}}{\dot{q}_{n,i}} \geq 0.$$

- For all $i \in I^- \subseteq \{1, \ldots, n\}$ such that $\dot{q}_{n,i} < 0$, we have

$$-\frac{\dot{q}_{\max,i} - \dot{q}_{r,i}}{|\dot{q}_{n,i}|} \leq 0 \leq \alpha \leq A^-_{\max,i} = \frac{\dot{q}_{\max,i} + \dot{q}_{r,i}}{|\dot{q}_{n,i}|} \geq 0.$$

- For all $i \in I^0 = \{1, \ldots, n\} - I^+ - I^-$ such that $\dot{q}_{n,i} = 0$, no bounds are set on α.

If the sets I^+ and I^- are both empty, then all components of the projected gradient vector $\dot{\boldsymbol{q}}_n$ are zero, and the robot is at a configuration that already satisfies the local necessary condition for a constrained maximum[1] of the criterion $H(\boldsymbol{q})$. Therefore, the obvious (and useless) choice is to set $\alpha^* = 0$. In all other cases, define

$$\alpha^+ = \min_{i \in I^+} A^+_{\max,i} \qquad \alpha^- = \min_{i \in I^-} A^-_{\max,i},$$

with the understanding that one of these quantities is not computed when the associated set is empty (at this stage, I^+ and I^- cannot be both empty). The largest feasible step in the gradient direction is then chosen as

$$\alpha^* = \min\{\alpha^+, \alpha^-, \alpha_{\max}\} \geq 0.$$

If the result is $\alpha^* = 0$, it means that either there exist (at least) a component $\dot{q}_{r,i} = \dot{q}_{\max,i}$ with $i \in I^+$ or a component $\dot{q}_{r,i} = -\dot{q}_{\max,i}$ with $i \in I^-$. Then, any additional null-space term used beside the minimum norm solution $\dot{\boldsymbol{q}}_r$ would violate feasibility.

This algorithm is illustrated geometrically in Fig. 2, for a generic case with $n = 2$ and $m = 1$. At a given $\boldsymbol{q}$, the shaded green box represents the feasible region (2) in the plane $(\dot{q}_1, \dot{q}_2)$. The blue line is the linear equality $J\dot{\boldsymbol{q}} = \dot{r}$, being J a 1×2 matrix. The gradient $\nabla_q H(q)$ and its projection $\dot{\boldsymbol{q}}_n$ in the null space of J with a 2×2 matrix $\boldsymbol{P}$ are shown in red. The final result $\dot{\boldsymbol{q}}^*_{PG} = \dot{\boldsymbol{q}}_r + \alpha^*\dot{\boldsymbol{q}}_n$ is shown in orange.

[1] The equation $\dot{\boldsymbol{q}}_n = \boldsymbol{P}(\boldsymbol{q})\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \boldsymbol{0}$ is actually a stationarity condition, i.e., the robot could also be at a constrained minimum, or even at a saddle point. In these cases, standard gradient techniques have not enough information on where to move to improve the criterion.

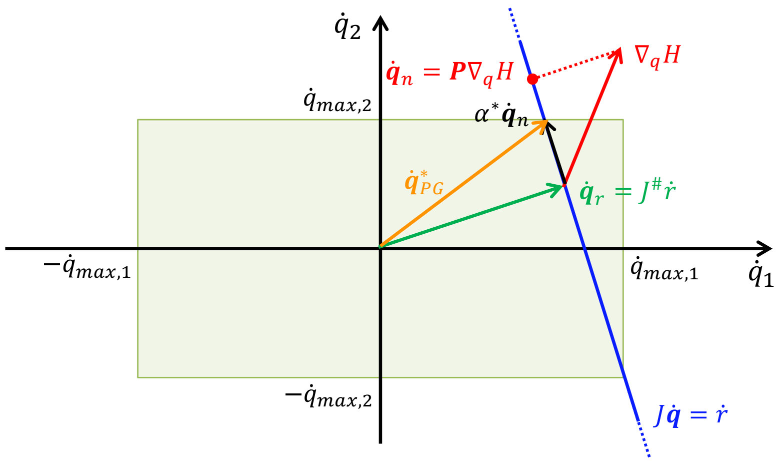

Figure 2: Geometric illustration of the selection of α^* in the PG method (1) for $n = 2$ and $m = 1$

Exercise 2

We compute the kinetic energy of the links as follows:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{q}_1^2 \\ T_2 &= \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2\right) \\ T_3 &= \frac{1}{2} m_3 \left((\dot{q}_1 + \dot{q}_3)^2 + \dot{q}_2^2\right) \\ T_4 &= \frac{1}{2} m_4 \left((\dot{q}_1 + \dot{q}_3)^2 + (\dot{q}_2 + \dot{q}_4)^2\right). \end{aligned}$$

Thus,

$$T = \sum_1^4 T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}}$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 + m_3 + m_4 & 0 & m_3 + m_4 & 0 \\ 0 & m_2 + m_3 + m_4 & 0 & m_4 \\ m_3 + m_4 & 0 & m_3 + m_4 & 0 \\ 0 & m_4 & 0 & m_4 \end{pmatrix}.$$

Being the inertia matrix constant, then $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$. As for the potential energy of the links, one has

$$\begin{aligned} U_1 &= U_{01} \\ U_2 &= U_{02} + m_2 g_0 q_2 \\ U_3 &= U_{03} + m_3 g_0 q_2 \\ U_4 &= U_{04} + m_4 g_0 (q_2 + q_4), \end{aligned}$$

and so

$$U = \sum_1^4 U_i \qquad \Rightarrow \qquad \boldsymbol{g} = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ (m_2 + m_3 + m_4)\, g_0 \\ 0 \\ m_4 g_0 \end{pmatrix}.$$

As a result, the dynamic model of the robot has only configuration-independent terms

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{\tau}. \tag{5}$$

To evaluate the Cartesian inertia matrix $\boldsymbol{M}_r$, we need the Jacobian of the end effector velocity. From

$$\boldsymbol{p} = \begin{pmatrix} q_1 + q_3 \\ q_2 + q_4 \end{pmatrix}$$

one has

$$\boldsymbol{J} = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Thus, after some elementary symbolic computations,

$$\boldsymbol{M}_r = \left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1} = \begin{pmatrix} m_3 + m_4 & 0 \\ 0 & m_4 \end{pmatrix}.$$

Using the given link mass data, one has

$$\boldsymbol{M}_r = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The left-hand side of (5) can be linearly parametrized in (at least) two alternative and minimal ways as

$$\boldsymbol{Y}_{\boldsymbol{m}}(\ddot{\boldsymbol{q}})\,\boldsymbol{m} = \boldsymbol{Y}(\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau},$$

either directly with the unknown mass parameters

$$\boldsymbol{Y}_{\boldsymbol{m}}(\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & \ddot{q}_1 & \ddot{q}_1 + \ddot{q}_3 & \ddot{q}_1 + \ddot{q}_3 \\ 0 & \ddot{q}_2 + g_0 & \ddot{q}_2 + g_0 & \ddot{q}_2 + \ddot{q}_4 + g_0 \\ 0 & 0 & \ddot{q}_1 + \ddot{q}_3 & \ddot{q}_1 + \ddot{q}_3 \\ 0 & 0 & 0 & \ddot{q}_2 + \ddot{q}_4 + g_0 \end{pmatrix} \qquad \boldsymbol{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix},$$

or with the dynamic coefficients appearing in the inertia matrix $\boldsymbol{M}$

$$\boldsymbol{Y}(\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & 0 & \ddot{q}_3 & 0 \\ 0 & \ddot{q}_2 + g_0 & 0 & \ddot{q}_4 \\ 0 & 0 & \ddot{q}_1 + \ddot{q}_3 & 0 \\ 0 & 0 & 0 & \ddot{q}_2 + \ddot{q}_4 + g_0 \end{pmatrix} \qquad \boldsymbol{a} = \begin{pmatrix} m_1 + m_2 + m_3 + m_4 \\ m_2 + m_3 + m_4 \\ m_3 + m_4 \\ m_4 \end{pmatrix}.$$

The adaptive control law is particularly simple, since there are no velocity terms and the dynamics is linear. For example, in the second case we have

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{Y}(\ddot{\boldsymbol{q}}_r)\,\hat{\boldsymbol{a}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma}\boldsymbol{Y}^T(\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}) \qquad \dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q}) \end{aligned}$$

with diagonal gains $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$, $\boldsymbol{\Gamma} > 0$, and $\boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P$.

Exercise 3

The problem is solved using the general results on Linear Quadratic (LQ) optimization problems, as applied to dynamic redundancy resolution. The task has to be considered at the acceleration level, thus the linear equation for the problem is

$$\boldsymbol{J}_t(\boldsymbol{q})\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Using the input torque from (3)

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} \tag{6}$$

in (4), the objective function H is rewritten as

$$\begin{aligned} H &= \frac{1}{2}\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{\tau}_0\right)^T \boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{\tau}_0\right), \\ &= \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \left(\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{\tau}_0\right)^T\ddot{\boldsymbol{q}} + k(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\tau}_0, \boldsymbol{F}), \end{aligned}$$

where k is an irrelevant constant at the current robot and interaction state. This is a complete quadratic objective function in the unknown $\ddot{\boldsymbol{q}}$, with a weighting matrix $\boldsymbol{M}(\boldsymbol{q})$ and a vector with three contributions in the linear term multiplying the acceleration. When compared with the general LQ formulation in 02_KinematicRedundancy_1.pdf

$$\min_{\boldsymbol{x}} H(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{x}_0\right)^T \boldsymbol{W}\left(\boldsymbol{x}-\boldsymbol{x}_0\right) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{W}\boldsymbol{x} - \boldsymbol{x}_0^T\boldsymbol{W}\boldsymbol{x} + k(\boldsymbol{x}_0)$$
$$\text{subject to } \boldsymbol{J}\boldsymbol{x} = \boldsymbol{y},$$

we find the following correspondences:

$$\begin{aligned}
\boldsymbol{x} &= \ddot{\boldsymbol{q}} \\
\boldsymbol{y} &= \ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}} \\
\boldsymbol{J} &= \boldsymbol{J}_t(\boldsymbol{q}) \\
\boldsymbol{W} &= \boldsymbol{M}(\boldsymbol{q}) \\
-\boldsymbol{W}\boldsymbol{x}_0 &= \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{\tau}_0. \\
&\Rightarrow \boldsymbol{x}_0 = \boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0 + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})\right).
\end{aligned}$$

Therefore, the general solution

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{J}_W^{\#}\left(\boldsymbol{y} - \boldsymbol{J}\boldsymbol{x}_0\right) = \boldsymbol{J}_W^{\#}\boldsymbol{y} + \left(\boldsymbol{I} - \boldsymbol{J}_W^{\#}\boldsymbol{J}\right)\boldsymbol{x}_0,$$

where $\boldsymbol{J}_W^{\#} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right)^{-1}$ is the weighted pseudoinverse of $\boldsymbol{J}$ (in the full rank case), translates here to the acceleration solution

$$\begin{aligned}
\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0 + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})\right) + \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}_t^T(\boldsymbol{q})\left(\boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}_t^T(\boldsymbol{q})\right)^{-1} \\
\left(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0 + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\right).
\end{aligned}$$

Plugging this back into (6) yields the torque solution (in equivalent forms):

$$\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{\tau}_0 + \boldsymbol{M}(\boldsymbol{q})(\boldsymbol{J}_t(\boldsymbol{q}))^{\#}_{\boldsymbol{M}(\boldsymbol{q})}(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0 + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)) \\
&= \boldsymbol{\tau}_0 + \boldsymbol{J}_t^T(\boldsymbol{q})\left(\boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_0 + \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)) \\
&= \boldsymbol{J}_t^T(\boldsymbol{q})\left(\boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}(\ddot{\boldsymbol{r}}_d - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{J}_g^T(\boldsymbol{q})\boldsymbol{F}\right)) \\
&\quad + \left(\boldsymbol{I} - \boldsymbol{J}_t^T(\boldsymbol{q})\left(\boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\boldsymbol{J}_t(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\right)\boldsymbol{\tau}_0.
\end{aligned} \tag{7}$$

Note that when $\boldsymbol{\tau}_0 = \boldsymbol{0}$ and $\boldsymbol{F} = \boldsymbol{0}$, the last expression coincides with the one provided in the lecture slides.

In the presence of a task error $\boldsymbol{e} = \boldsymbol{r}_d - \boldsymbol{r}$, the only needed modification in the control torque (7) is to replace

$$\ddot{\boldsymbol{r}}_d \quad \longrightarrow \quad \ddot{\boldsymbol{r}}_d + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} = \ddot{\boldsymbol{r}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}) + \boldsymbol{K}_P(\boldsymbol{r}_d - \boldsymbol{r})$$

with positive definite, possibly diagonal gain matrices $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$. The task error will then be rejected at an exponential rate, which can be tuned by a suitable choice of the two gain matrices. The modified control torque will remain optimal in terms of the same objective function, with the linear constraint replaced by

$$\boldsymbol{J}_t(\boldsymbol{q})\ddot{\boldsymbol{q}} = (\ddot{\boldsymbol{r}}_d + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}) - \dot{\boldsymbol{J}}_t(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

For the planar PR robot in the lecture slides 03_LagrangianDynamics_1.pdf, the relevant dynamic model terms are

$$\boldsymbol{M}(q_2) = \begin{pmatrix} m_1 + m_2 & -m_2 d_{c2}\sin q_2 \\ -m_2 d_{c2}\sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \qquad \boldsymbol{c}(q_2,\dot{q}_2) = \begin{pmatrix} -m_2 d_{c2}\cos q_2\,\dot{q}_2^2 \\ 0 \end{pmatrix}, \tag{8}$$

with the Jacobian transpose matrix that relates end-effector forces to joint torques

$$\boldsymbol{J}_g^T(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 \\ -l_2\sin q_2 & l_2\cos q_2 \end{pmatrix},$$

i.e., the transpose of the first two rows of the geometric Jacobian. The one-dimensional task and its 1×2 Jacobian are defined by

$$r = t(\boldsymbol{q}) = q_1 + l_2 \cos q_2 \qquad \Rightarrow \qquad \boldsymbol{J}_t(q_2) = \begin{pmatrix} 1 & -l_2 \sin q_2 \end{pmatrix},$$

whereas the desired task acceleration is

$$\ddot{r}_d(t) = 9 \cos 3t.$$

At the initial time $t = 0$ the robot is at rest, so the torque (7) simplifies to

$$\boldsymbol{\tau}(0) = \boldsymbol{\tau}_0 + \boldsymbol{M}(\boldsymbol{q}(0))(\boldsymbol{J}_t(\boldsymbol{q}(0)))^{\#}_{\boldsymbol{M}(\boldsymbol{q}(0))}(\ddot{r}_d(0) - \boldsymbol{J}_t(\boldsymbol{q}(0))\boldsymbol{M}^{-1}(\boldsymbol{q}(0))\,(\boldsymbol{\tau}_0(0) + \boldsymbol{J}_g^T(\boldsymbol{q}(0))\boldsymbol{F}))$$

Moreover, in the initial configuration $\boldsymbol{q}(0) = (0, \pi/2)$ [m,rad] and with the given dynamic parameters, we have

$$\boldsymbol{M}(\boldsymbol{q}(0)) = \begin{pmatrix} 2 & -0.25 \\ -0.25 & 0.1625 \end{pmatrix} \qquad \boldsymbol{J}_g^T(\boldsymbol{q}(0)) = \begin{pmatrix} 1 & 0 \\ -0.5 & 0 \end{pmatrix} \qquad \boldsymbol{J}_t(q_2(0)) = \begin{pmatrix} 1 & -0.5 \end{pmatrix}.$$

Performing computations and using the given values for $\boldsymbol{\tau}_0$ and $\boldsymbol{F}$, the final result is

$$\boldsymbol{\tau}(0) = \begin{pmatrix} -2 \\ 2.5 \end{pmatrix} \begin{matrix} \text{N} \\ \text{Nm.} \end{matrix}$$

Exercise 4

For the PR robot moving in a vertical plane, beside the dynamic model terms in (8), we need also the gravity vector

$$U = U_0 + m_2 g_0 d_{c2} \sin q_2 \qquad \Rightarrow \qquad \boldsymbol{g}(q_2) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ m_2 d_{c2} g_0 \cos q_2 \end{pmatrix}.$$

Further, the generalized momentum of this robot is

$$\boldsymbol{p} = \boldsymbol{M}(q_2)\,\dot{\boldsymbol{q}} = \begin{pmatrix} (m_1 + m_2)\,\dot{q}_1 - m_2 d_{c2} \sin q_2\,\dot{q}_2 \\ -m_2 d_{c2} \sin q_2\,\dot{q}_1 + (I_{c2,zz} + m_2 d_{c2}^2)\,\dot{q}_2 \end{pmatrix},$$

In the absence of dissipative effects, the vector $\boldsymbol{\alpha} \in \mathbb{R}^2$ in the expression of the residual is

$$\alpha_i = -\frac{1}{2}\,\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\,\dot{\boldsymbol{q}} + g_i(\boldsymbol{q}) \qquad i = 1, 2 \qquad \Rightarrow \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 \\ m_2 d_{c2} \cos q_2\,(\dot{q}_1 \dot{q}_2 + g_0) \end{pmatrix}.$$

Note that an equivalent expression of $\boldsymbol{\alpha}$ in vector form is obtained using a factorization of $\boldsymbol{c}(q_2, \dot{q}_2)$ that satisfies the skew-symmetry of matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$, namely

$$\boldsymbol{c}(q_2, \dot{q}_2) = \boldsymbol{S}(q_2, \dot{q}_2)\,\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2\,\dot{q}_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}.$$

We have in fact

$$\boldsymbol{\alpha} = -\boldsymbol{S}^T(q_2, \dot{q}_2)\,\dot{\boldsymbol{q}} + \boldsymbol{g}(q_2).$$

As a result, the residual vector for detection and isolation of actuator faults $\boldsymbol{\tau}_f$ in

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{\tau}_f$$

is

$$\boldsymbol{r} = \boldsymbol{K}\left[\int (\boldsymbol{\tau} - \boldsymbol{\alpha} - \boldsymbol{r})\,dt - \boldsymbol{p}\right] \qquad \text{with } \boldsymbol{K} > 0 \text{ and diagonal.} \tag{9}$$

In particular, the explicit form of the terms in (9) is

$$\begin{aligned} r_1 &= K_1\left[\int (\tau_1 - r_1)\,dt - ((m_1 + m_2)\,\dot{q}_1 - m_2 d_{c2} \sin q_2\,\dot{q}_2)\right] \\ r_2 &= K_2\left[\int (\tau_2 - m_2 d_{c2} \cos q_2 (\dot{q}_1 \dot{q}_2 + g_0) - r_2)\,dt - \left((I_{c2,zz} + m_2 d_{c2}^2)\,\dot{q}_2 - m_2 d_{c2} \sin q_2\,\dot{q}_1\right)\right]. \end{aligned}$$

It is easy to see that the dynamics of $\boldsymbol{r}$ satisfies

$$\dot{\boldsymbol{r}} = \boldsymbol{K}(\boldsymbol{\tau}_f - \boldsymbol{r}) \qquad \Longleftrightarrow \qquad \begin{aligned} \dot{r}_1 &= K_1\,(\tau_{f,1} - r_1) \\ \dot{r}_2 &= K_2\,(\tau_{f,2} - r_2)\,. \end{aligned}$$

Therefore, the time profiles of the residuals in response to actuator faults $\tau_{f,i}$, for $i = 1, 2$, are decoupled for each joint and completely independent of the robot state. When a total fault occurs at $t = 0$ on joint 1, and this actuator is being commanded with a constant force of 5 N, the behavior of $r_1(t)$ is shown in Fig. 3 for two possible values of the first gain, namely for $K_1 = 10$ (in red) and for $K_1 = 3$ (in blue). In both cases, the exponential response converges, faster or slower, to the missing value of the commanded torque ($\tau_1 = 5$ N) —exactly for this reason, we can conclude that a total fault occured ($\tau_f = \tau_1$) —a fact that was not known a priori! On the other hand, the second residual remains always unaffected ($r_2(t) \equiv 0$) and therefore is not reported.

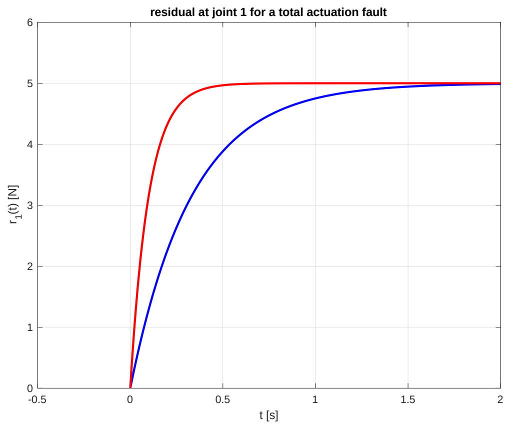

Figure 3: Responses of $r_1(t)$ for a total fault of the first actuator when $\tau_1 = 5$ N is commanded

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