Robotics 2

October 21, 2022

Exercise

Consider the PR robot in Fig. 1, moving in a vertical plane.

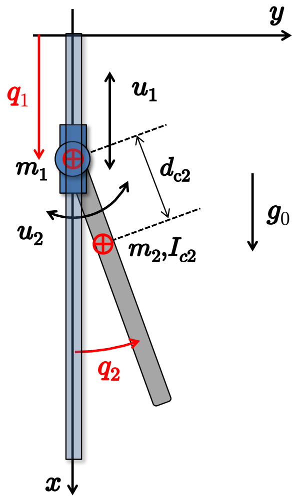

Figure 1: A PR planar robot with the relevant dynamic parameters and variables.

1. Derive the dynamic model of the robot in the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}.$$

2. Find a linear parametrization of the dynamic model in the form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{u},$$

   where $\boldsymbol{a} \in \mathbb{R}^p$ has the minimal possible dimension p (the gravity acceleration g_0 is known).

3. Design a control law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d)$ that globally asymptotically stabilizes the robot to the desired configuration $\boldsymbol{q}_d = (0, \pi)$ [m,rad], when *only* the total mass $m = m_1 + m_2$ of the robot, the acceleration g_0, and the length ℓ_2 of the second link are known.

4. Suppose that the robot is initially in equilibrium at $\boldsymbol{q}_{in} = \boldsymbol{q}(0) = (0, 0)$. Under the action of the control law designed in step 3, determine the sign of the initial acceleration $\ddot{q}_1(0)$ of the first joint, in case this is different from zero.

5. Assume now that all robot dynamic parameters are known. Show how it is possible to design a model-based command $\boldsymbol{u} = \boldsymbol{u}(t)$ that will transfer the robot from the state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{q}_{in}, \boldsymbol{0})$ to the final state $(\boldsymbol{q}(T), \dot{\boldsymbol{q}}(T)) = (\boldsymbol{q}_d, \boldsymbol{0})$ in a given time $T > 0$ and *without* moving the first joint.

6. The robot input commands are now limited as $|u_i(t)| \leq U_i$, $i = 1, 2$. Consider the rest-to-rest task of moving in minimum time T from $\boldsymbol{q}(0) = (q_1(0), \bar{q}_2)$ to $\boldsymbol{q}(T) = (q_1(0) - \Delta, \bar{q}_2)$, with $\Delta > 0$, while keeping the second joint *constantly* at $q_2 = \bar{q}_2 > 0$. Determine the optimal solution in an analytic way and sketch the time-optimal profiles of $\ddot{q}_1(t)$, $\ddot{q}_2(t)$, $u_1(t)$ and $u_2(t)$ for $t \in [0, T]$.

[180 minutes; open books]

Solution

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1. Dynamic model

Kinetic energy

$$T = T_1 + T_2,$$

with

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_{c2}\, \dot{q}_2^2,$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt} \begin{pmatrix} q_1 + d_{c2}c_2 \\ d_{c2}s_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 - d_{c2}s_2\dot{q}_2 \\ d_{c2}c_2\dot{q}_2 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_{c2}^2\dot{q}_2^2 - 2d_{c2}s_2\dot{q}_1\dot{q}_2.$$

Thus

$$T = \frac{1}{2}\left(m_1 + m_2\right)\dot{q}_1^2 + \frac{1}{2}\left(I_{c2} + m_2 d_{c2}^2\right)\dot{q}_2^2 - m_2 d_{c2} s_2 \dot{q}_1 \dot{q}_2.$$

Inertia matrix

$$T = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} s_2 \\ -m_2 d_{c2} s_2 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix}.$$

Coriolis and centrifugal terms

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\,\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})\,\dot{\boldsymbol{q}} \end{pmatrix}, \quad \text{with } \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \quad i = 1, 2.$$

Since

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} c_2 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{O},$$

we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_2 d_{c2} c_2 \dot{q}_2^2 \\ 0 \end{pmatrix}.$$

Potential energy and gravity terms

$$U = U_1 + U_2 = -m_1 g_0 q_1 - m_2 g_0 \left(q_1 + d_{c2} c_2\right),$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} -\left(m_1 + m_2\right) g_0 \\ m_2 d_{c2}\, g_0 s_2 \end{pmatrix}.$$

Robot equations

$$\begin{gathered} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} \\ \Downarrow \\ \begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} s_2 \\ -m_2 d_{c2} s_2 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} c_2 \dot{q}_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\left(m_1 + m_2\right) g_0 \\ m_2 d_{c2}\, g_0 s_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{gathered} \tag{1}$$

2. Linear parametrization

$$
\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \begin{pmatrix} \ddot{q}_1 - g_0 & -s_2\,\ddot{q}_2 - c_2\,\dot{q}_2^2 & 0 \\ 0 & -s_2\,(\ddot{q}_1 - g_0) & \ddot{q}_2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ m_2 d_{c2} \\ I_{c2} + m_2 d_{c2}^2 \end{pmatrix},
$$

with a minimal number $p = 3$ of dynamic coefficients a_i, $i = 1, 2, 3$.

3. Regulation control

Under the given assumptions, we can design a PD plus constant gravity compensation law as

$$
\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) \tag{2}
$$

with diagonal gain matrices $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and with $K_{Pm} > \alpha$, where $\|\partial\boldsymbol{g}/\partial\boldsymbol{q}\| \leq \alpha$. In fact, for $\boldsymbol{q}_d = (0, \pi)$, the control law (2) becomes simply

$$
\boldsymbol{u} = \begin{pmatrix} -K_{P1}q_1 - K_{D1}\dot{q}_1 - mg_0 \\ K_{P2}(\pi - q_2) - K_{D2}\dot{q}_2 \end{pmatrix}, \tag{3}
$$

where $m\,(= m_1 + m_2)$ and g_0 are known. Moreover,

$$
\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{q}} = \begin{pmatrix} 0 & 0 \\ 0 & m_2 d_{c2}\,g_0 c_2 \end{pmatrix} \quad\Rightarrow\quad \left\|\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{q}}\right\| = \sqrt{\lambda_{\max}\left\{\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{q}}\left(\frac{\partial\boldsymbol{g}}{\partial\boldsymbol{q}}\right)^T\right\}} = m_2 d_{c2}\,g_0|c_2| < m\ell_2 g_0 = \alpha,
$$

being ℓ_2 also known. Thus, to guarantee global asymptotic stabilization to $\boldsymbol{q}_d$ with the control law (3) we choose

$$
K_{Pm} = \min\{K_{P1}, K_{P2}\} \geq m\ell_2 g_0.
$$

4. Initial acceleration

Isolating the acceleration $\ddot{\boldsymbol{q}}$ from the dynamic model (1) and evaluating it at $t = 0$, with initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{0}, \boldsymbol{0})$ and when using the control law (3), gives

$$
\ddot{\boldsymbol{q}}(0) = \boldsymbol{M}^{-1}(\boldsymbol{q}(0))\,(\boldsymbol{u}(0) - \boldsymbol{g}(\boldsymbol{q}(0))) = \frac{1}{\det\boldsymbol{M}(\boldsymbol{q}(0))}\begin{pmatrix} I_{c2} + m_2 d_{c2}^2 & 0 \\ 0 & m_1 + m_2 \end{pmatrix}\begin{pmatrix} 0 \\ K_{P2}\,\pi \end{pmatrix},
$$

since in particular $q_{d1} - q_1(0) = 0$ and $u_1(0) - g_1(0) = -mg_0 + (m_1 + m_2)\,g_0 = 0$. Thus,

$$
\ddot{q}_1(0) = 0.
$$

This should not be unexpected, being the dynamics of the two joints fully decoupled in the initial state and joint 1 still at an equilibrium under the control law (3). Note that a simple PD control law without the gravity compensation term $\boldsymbol{g}(\boldsymbol{q}_d)$ in (3) would result in an initial acceleration $\ddot{q}_1(0) = g_0 = 9.81 > 0$, i.e., the first (prismatic) joint would initially slide downwards.

5. Inverse dynamics command

The desired motion task is obtained by using inverse dynamics on a suitable rest-to-rest trajectory interpolating the initial and final configuration in a given time T. Consider for instance the cubic polynomial trajectory

$$
\boldsymbol{q}_d(t) = \boldsymbol{q}_{in} + (\boldsymbol{q}_d - \boldsymbol{q}_{in})\left(-2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2\right), \quad t \in [0, T],
$$

or, componentwise,

$$q_{d1}(t) = 0 \quad \Rightarrow \quad \dot{q}_{d1}(t) = \ddot{q}_{d1}(t) = 0,$$

$$q_{d2}(t) = \pi \left(-2 \left(\frac{t}{T} \right)^3 + 3 \left(\frac{t}{T} \right)^2 \right) \Rightarrow \dot{q}_{d2}(t) = \frac{6\pi}{T} \left(- \left(\frac{t}{T} \right)^2 + \frac{t}{T} \right) \Rightarrow \ddot{q}_{d2}(t) = \frac{6\pi}{T^2} \left(1 - 2 \, \frac{t}{T} \right).$$

Accordingly, the required command is computed as

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{u}_d(t) &= \boldsymbol{M}(\boldsymbol{q}_d(t))\ddot{\boldsymbol{q}}_d(t) + \boldsymbol{c}(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{g}(\boldsymbol{q}_d(t)) \\ &= \begin{pmatrix} -m_2 d_{c2} \left(\sin q_{d2}(t) \, \ddot{q}_{d2}(t) - \cos q_{d2}(t) \, \dot{q}_{d2}^2(t) \right) - (m_1 + m_2) \, g_0 \\ \left(I_{c2} + m_2 d_{c2}^2 \right) \ddot{q}_{d2}(t) + m_2 d_{c2} \, g_0 \sin q_{d2}(t) \end{pmatrix}, \quad t \in [0, T]. \end{aligned}$$

6. Minimum time motion

The problem can be formulated as a minimum-time motion on a prescribed path in the joint space, where q_1 needs to move between $q_1(0)$ and $q_1(0) - \Delta$ while q_2 is kept always at the constant value $\bar{q}_2$ (hence, $\dot{q}_2 = \ddot{q}_2 = 0$). Since the velocity term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$ for $\dot{q}_2 = 0$, from eq. (1) the dynamics along this path is then

$$\boldsymbol{M}(\bar{q}_2) \begin{pmatrix} \ddot{q}_1 \\ 0 \end{pmatrix} + \boldsymbol{g}(\bar{q}_2) = \begin{pmatrix} m_1 + m_2 \\ -m_2 d_{c2} \sin \bar{q}_2 \end{pmatrix} (\ddot{q}_1 - g_0) = \boldsymbol{u}, \tag{4}$$

which is parametrized by the single acceleration $\ddot{q}_1(t)$, with $s(t) = q_1(t)$ acting as the (scalar) timing law. Note that the two differential equations in (4) are linear and not independent. This means that one equation should be used in forward dynamics (with the optimal input command) to determine the acceleration $\ddot{q}_1$, while the other will be used in inverse dynamics to find the other input command.

For compactness, define the two constants

$$a = m_1 + m_2 > 0, \qquad b = m_2 d_{c2} \sin \bar{q}_2 > 0.$$

being $\bar{q}_2 \in (0, \pi)$ and thus $\sin \bar{q}_2 > 0$. Then, from (4) and using the bounds on the input commands, we have

$$-U_1 \leq a \, (\ddot{q}_1 - g_0) \leq U_1, \qquad -U_2 \leq -b \, (\ddot{q}_1 - g_0) \leq U_2.$$

Manipulating the inequalities, we obtain that $\ddot{q}_1(t)$, for $t \in [0, T]$, is bounded by

$$\max \left\{ -\frac{U_1}{a} + g_0, -\frac{U_2}{b} + g_0 \right\} = \ddot{q}_1^- \leq \ddot{q}_1(t) \leq \ddot{q}_1^+ = \min \left\{ \frac{U_1}{a} + g_0, \frac{U_2}{b} + g_0 \right\}. \tag{5}$$

While $\ddot{q}_1^+ > 0$ clearly holds, it is necessary to enforce $\ddot{q}_1^- < 0$ in order to be able to perform any desired (rest-to-rest) motion transfer by $\Delta \lesseqgtr 0$ with a suitable sequence of positive and negative acceleration of the first joint. However, this condition is trivially obtained once we realize that the minimum requirement for the actuation torques at each joint is that they should be able to sustain (at least) the robot gravity load in any configuration. As a result, we can safely assume that

$$U_1 > (m_1 + m_2) \, g_0 = a \, g_0, \qquad U_2 > m_2 d_{c2} \, g_0 > m_2 d_{c2} \sin \bar{q}_2 \, g_0 = b \, g_0 \qquad \Rightarrow \qquad \ddot{q}_1^- < 0.$$

Note that the two possible saturation levels for the acceleration of the first joint (at its positive or negative value) correspond to two different physical situations: either because the first actuator pushes/pulls the robot as fast as possible (U_1 saturates), or because the second actuator reaches its limit capability in order to keep the second link at the fixed configuration $\bar{q}_2$ (U_2 saturates). Furthermore, it follows from the expressions of the bounds in (5) that the same actuator will saturate during the acceleration and deceleration phases, while equation (4) shows that the two input commands will always have opposite signs.

With such asymmetric bounds on the feasible acceleration of joint 1, the requested rest-to-rest motion task for a displacement $-\Delta < 0$ (thus, moving against gravity) will be executed in minimum time T by the following acceleration command:

$$\ddot{q}_1(t) = \begin{cases} \ddot{q}_1^-, & t \in [0, T_s), \\ \ddot{q}_1^+, & t \in [T_s, T). \end{cases}$$

The values of T_s and T are obtained from the two relationships:

$$\ddot{q}_1^- \, T_s + \ddot{q}_1^+ \, (T - T_s) = 0 \qquad \text{(rest-to-rest motion enforced)}$$

$$\frac{1}{2}\,\ddot{q}_1^- \, T_s^2 - \frac{1}{2}\,\ddot{q}_1^+ \, (T - T_s)^2 = -\Delta \qquad \text{(net displacement to be achieved).}$$

As a result,

$$T = \sqrt{\frac{2\Delta\left(\ddot{q}_1^+ - \ddot{q}_1^-\right)}{|\ddot{q}_1^-|\,\ddot{q}_1^+}}, \qquad T_s = \frac{\ddot{q}_1^+}{\ddot{q}_1^+ - \ddot{q}_1^-}\,T.$$

Note that in case of opposite values of the positive and negative acceleration ($\ddot{q}_1^+ = -\ddot{q}_1^- = A$), these formulas return the usual symmetric bang-bang profile with $T = \sqrt{4\Delta/A}$ and $T_s = T/2$.

Qualitative optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for $t \in [0, T]$ are sketched in Fig. 2, assuming here that $U_1/a < U_2/b$. Indeed, $\ddot{q}_2(t) = 0$ at any time t (thus, it is not shown).

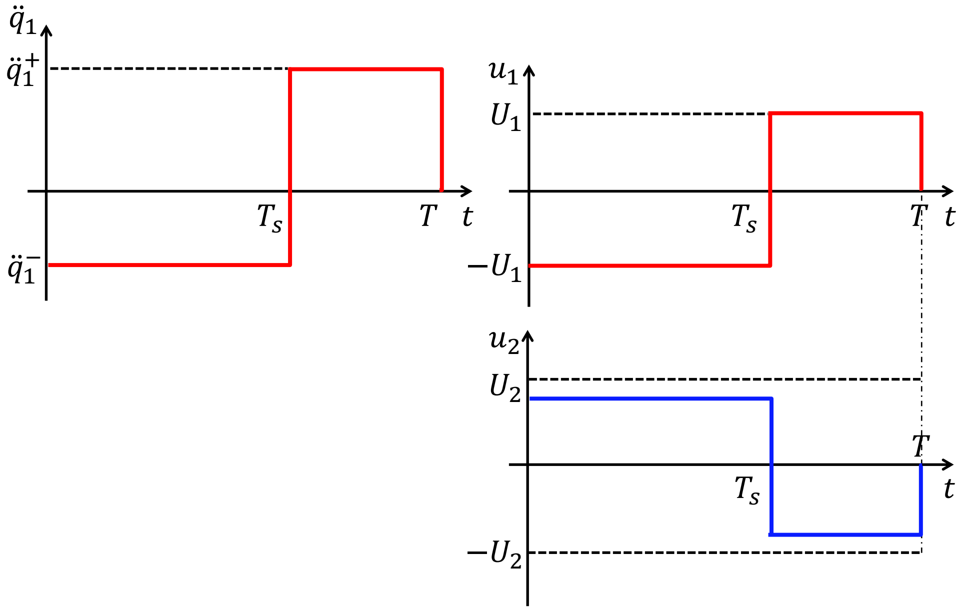

Figure 2: Time-optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for a displacement $-\Delta < 0$ of q_1.

On the other hand, Figure 3 shows the numerical results obtained with the data

$$m_1 = 8, \;\; m_2 = 5 \text{ [kg]}, \quad d_{c2} = 1 \text{ [m]}, \quad \bar{q}_2 = \frac{\pi}{4} \text{ [rad]}, \quad U_1 = 260 \text{ [N]}, \;\; U_2 = 100 \text{ [Nm]}, \qquad (6)$$

yielding for a displacement of $-\Delta = -1$ [m] of q_1:

$$\ddot{q}_1^- = -10.19, \quad \ddot{q}_1^+ = 29.81 \text{ [rad/s}^2\text{]}, \qquad T = 0.5132, \quad T_s = 0.3825 \text{ [s]}.$$

The first joint saturates its command ($|u_1| = U_1 = 260$ [N]), whereas the command to the second joint is set to a maximum (absolute) value of $|u_2| = -b\left(\ddot{q}_1^- - g_0\right) = 70.7107 < 100 = U_2$ [Nm].

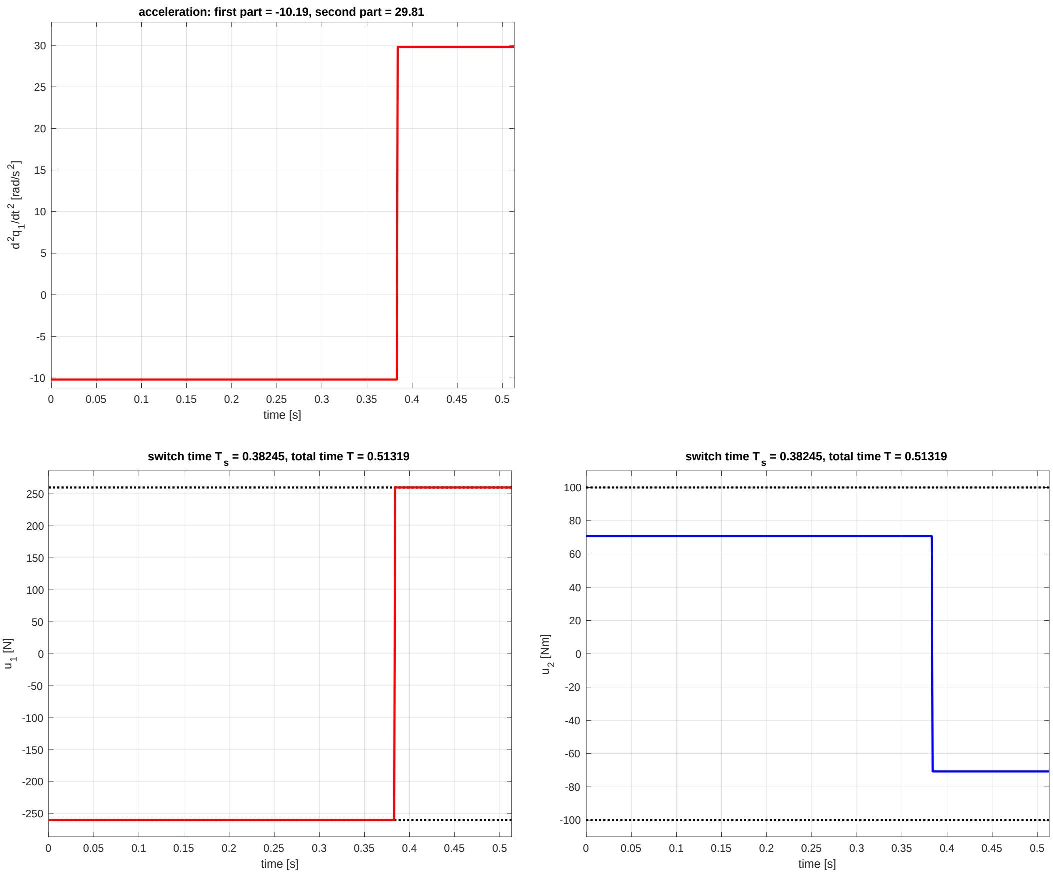

Figure 3: Time-optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for the data in (6) and $-\Delta = -1$.

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