Robotics II

September 11, 2019

Exercise 1

Consider the 3R robot in Fig. 1, moving on a horizontal plane. The robot has identical links (each of length L, uniformly distributed mass m, and inertia $I_L = mL^2/12$ around the barycentral vertical axis) and is commanded at the joint level by torques $\boldsymbol{\tau}(t) \in \mathbb{R}^3$. Neglect in the following any dissipative/friction effects. With the system at $t = 0$ in a generic initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ with $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$, we want to control the robot so that its kinetic energy $T = T(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in the closed-loop dynamics satisfies the following desired target equation:

$$\frac{dT}{dt} = -\gamma T, \qquad \text{with } \gamma > 0.$$

Determine the expression of the control law $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ that realizes this behavior. For $L = 0.2$ [m], $m = 3$ [kg], $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], $\dot{\boldsymbol{q}}_0 = (0, -\pi, -\pi)$ [rad/s] and $\gamma = 1$, compute the numerical value of such a control torque at $t = 0$, i.e., $\boldsymbol{\tau}(0)$.

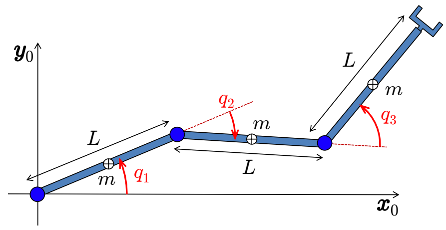

Figure 1: A 3R robot moving on a horizontal plane, and its coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

The RP planar robot shown in Fig. 2 should execute a rest-to-rest motion task in minimum time under torque/force bounds $|\tau_i| \leq \tau_{max,i} > 0$, $i = 1, 2$, with its end-effector moving along a circular path of radius $R > d$ by an angle α from A to B. Determine the analytic expression of the minimum time T^* in terms of the task data and of the robot dynamic parameters. Draw the profile of the two components of the time-optimal command $\boldsymbol{\tau}^*(t)$, for $t \in [0, T^*]$.

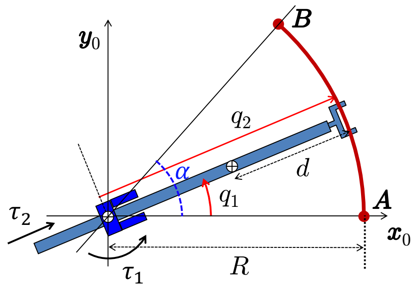

Figure 2: A RP robot moving its end-effector along a circular path on a horizontal plane.

Exercise 3

With reference to Fig. 3, a mass m_1 is moving at constant speed $v_0 > 0$ and collides at some time $t = t_c$ with a mass m_2 which is initially at rest. Assume a purely ideal situation: there is no dissipation due to friction and the collision is perfectly elastic. Therefore, the total kinetic energy T and the total (scalar) momentum P of the two masses will both remain constant over time. Determine the expressions of the velocities $v_1(t_c^+)$ and $v_2(t_c^+)$ of the two masses after the collision. Describe what happens when $m_1 > m_2$, $m_1 = m_2$, or $m_1 < m_2$, and in the limit cases when $m_2 \to 0$ or $m_2 \to \infty$.

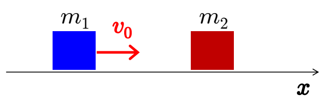

Figure 3: A mass m_1 in motion collides with a second mass m_2 initially at rest.

[open books, 180 minutes]

Solution

September 11, 2019

Exercise 1

The dynamic model of a frictionless robot in the absence of gravity is given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{\tau}, \tag{1}$$

where any factorization matrix $\boldsymbol{S}$ can be used for the (quadratic) Coriolis and centrifugal terms. From the expression of the kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, we obtain

$$\dot{T} = \frac{dT}{dt} = \dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau} - \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\boldsymbol{\tau}, \tag{2}$$

where we have used (1) and the principle of energy conservation (implying $\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} \equiv 0$, $\forall(\boldsymbol{q},\dot{\boldsymbol{q}})$). In order to impose the desired behavior to the Kinetic energy, it follows immediately that

$$\dot{T} = \dot{\boldsymbol{q}}^T\boldsymbol{\tau} = -\gamma\, T = -\frac{\gamma}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Longrightarrow \qquad \boldsymbol{\tau} = -\frac{\gamma}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3}$$

The control law should apply a torque that is the (scaled) negative value of the current generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ of the robot.

To realize (3), one needs to derive only the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ for the 3R planar robot at hand. The kinetic energy is given by

$$T = \sum_{I=1}^{3} T_i, \qquad T_i = \frac{1}{2}\,m\,\|\boldsymbol{v}_{ci}\|^2 + \frac{1}{2}\,I_L\,\omega_{z,i}^2, \qquad i = 1,2,3.$$

We compute first

$$T_1 = \frac{1}{2}\,m\left(\frac{L}{2}\,\dot{q}_1\right)^2 + \frac{1}{2}\,I_L\,\dot{q}_1^2 \quad \left(\cdots = \frac{1}{2}\,m\,\frac{L^2}{3}\,\dot{q}_1^2\right).$$

Then, from

$$\boldsymbol{p}_{c2} = \begin{pmatrix} L\cos q_1 + (L/2)\cos(q_1+q_2) \\ L\sin q_1 + (L/2)\sin(q_1+q_2) \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} -L\sin q_1\,\dot{q}_1 - (L/2)\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ L\cos q_1\,\dot{q}_1 + (L/2)\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{pmatrix}$$

and

$$\boldsymbol{p}_{c3} = \begin{pmatrix} L(\cos q_1 + \cos(q_1+q_2)) + (L/2)\cos(q_1+q_2+q_3) \\ L(\sin q_1 + \sin(q_1+q_2)) + (L/2)\sin(q_1+q_2+q_3) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} -L(\sin q_1\,\dot{q}_1 + \sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)) - (L/2)\sin(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3) \\ L(\cos q_1\,\dot{q}_1 + \cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)) + (L/2)\cos(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3) \end{pmatrix},$$

we obtain

$$T_2 = \frac{1}{2}\,m\left(L^2\dot{q}_1^2 + \frac{L^2}{4}(\dot{q}_1+\dot{q}_2)^2 + L^2\cos q_2\,\dot{q}_1(\dot{q}_1+\dot{q}_2)\right) + \frac{1}{2}\,I_L\,(\dot{q}_1+\dot{q}_2)^2$$

and

$$\begin{aligned} T_3 \;=\; & \frac{1}{2}\,m\Big(L^2\dot{q}_1^2 + L^2(\dot{q}_1+\dot{q}_2)^2 + 2L^2\cos q_2\,\dot{q}_1(\dot{q}_1+\dot{q}_2) + \frac{L^2}{4}(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 \\ & + L^2\left(\cos(q_2+q_3)\dot{q}_1 + \cos q_3(\dot{q}_1+\dot{q}_2)\right)(\dot{q}_1+\dot{q}_2+\dot{q}_3)\Big) + \frac{1}{2}\,I_L\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2. \end{aligned}$$

Therefore, using the compact notation for trigonometric functions and substituting for $I_L = mL^2/12$, the inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = mL^2\begin{pmatrix} 4 + 3c_2 + c_3 + c_{23} & \frac{5}{3} + \frac{3}{2}c_2 + c_3 + \frac{1}{2}c_{23} & \frac{1}{3} + \frac{1}{2}\left(c_3 + c_{23}\right) \\ & \frac{5}{3} + c_3 & \frac{1}{3} + \frac{1}{2}c_3 \\ symm & & \frac{1}{3} \end{pmatrix}. \tag{4}$$

Finally, evaluating the control law at $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad] and $\dot{\boldsymbol{q}}_0 = (0, -\pi, -\pi)$ [rad/s] and with the data $L = 0.2$ [m], $m = 3$ [kg] (thus $I_L = 0.01$ [kg·m^2]) and $\gamma = 1$, gives

$$\boldsymbol{\tau}(0) = -\frac{1}{2}\,\boldsymbol{M}(\boldsymbol{q}_0)\,\dot{\boldsymbol{q}}_0 = -\frac{1}{2}\cdot\frac{3}{25}\begin{pmatrix} 3 & \frac{7}{6} & -\frac{1}{6} \\ \frac{7}{6} & \frac{5}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ -\pi \\ -\pi \end{pmatrix} = \frac{\pi}{2}\begin{pmatrix} 0.12 \\ 0.24 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 0.1885 \\ 0.3770 \\ 0.1257 \end{pmatrix} \text{ [Nm]}. \tag{5}$$

Exercise 2

We start by deriving the dynamic model of the RP planar robot in Fig. 2. For the kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

since $\boldsymbol{p}_{c2} = (q_2 - d)\begin{pmatrix} \cos q_1 & \sin q_1 \end{pmatrix}^T$ and $\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2}$, we have

$$T_1 = \frac{1}{2}\, I_{c1}\, \dot{q}_1^2, \qquad T_2 = \frac{1}{2}\, I_{c2}\, \dot{q}_1^2 + \frac{1}{2}\, m_2 \, \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}\left(I_{c2} + m_2(q_2-d)^2\right)\dot{q}_1^2 + \frac{1}{2}\, m_2\, \dot{q}_2^2,$$

with an obvious interpretation of the dynamic parameters. The robot inertia matrix is then

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1} + I_{c2} + m_2(q_2-d)^2 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{6}$$

From this, we compute the Coriolis/centrifugal terms using the matrices of Christoffel symbols

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right)\right], \qquad i = 1, 2.$$

We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2(q_2-d) \\ m_2(q_2-d) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2(q_2-d) & 0 \\ 0 & 0 \end{pmatrix},$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})\dot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} 2m_2\,(q_2-d)\,\dot{q}_1\dot{q}_2 \\ -m_2\,(q_2-d)\,\dot{q}_1^2 \end{pmatrix}. \tag{7}$$

From (6) and (7), we write the (unconstrained) dynamic equations in their scalar form as

$$\left(I_{c1} + I_{c2} + m_2(q_2-d)^2)\right)\ddot{q}_1 + 2m_2\,(q_2-d)\,\dot{q}_1\dot{q}_2 \;=\; \tau_1, \tag{8}$$

$$m_2\ddot{q}_2 - m_2\,(q_2-d)\,\dot{q}_1^2 \;=\; \tau_2. \tag{9}$$

In order to execute the task, the second joint variable should remain constant at all times, namely $q_2 = R$, $\dot{q}_2 = \ddot{q}_2 = 0$. Therefore, from (8) with $q_2 = R$ and $\dot{q}_2 = 0$, the robot dynamics along the path can be described by

$$I_0\,\ddot{q}_1 = \tau_1, \qquad \text{with } I_0 = I_{c1} + I_{c2} + m_2(R-d)^2 > 0, \tag{10}$$

whereas, from (9) used as inverse dynamics with $q_2 = R$ and $\ddot{q}_2 = 0$, the second motor should apply the force

$$\tau_2(t) = -m_2\,(R-d)\,\dot{q}_1^2(t) \tag{11}$$

in order to have the end-effector remaining perfectly on the path. Equations (10–11) are the core of the solution. Based on the linear dynamics (10), to perform the desired rest-to-rest motion task in minimum time, the first motor should apply a bang-bang torque profile $\tau_1(t)$ (with maximum positive and negative torque $\pm\tau_{max,1}$, each applied for half of the motion interval). The total motion time should be sufficient to complete the rotation $\Delta q_1 = \alpha > 0$. Again from (10), this corresponds to using a maximum (absolute) acceleration bound in the definition of the time-optimal motion of joint 1, i.e,

$$|\ddot{q}_1| \le A_{max,1} = \frac{\tau_{max,1}}{I_0}. \tag{12}$$

While doing so, however, the velocity $\dot{q}_1(t)$ of the first joint will increase linearly and, according to (11), the force that the second motor needs to apply in order to keep the robot end-effector on the path will increase quadratically. As a result, the second actuator may exceed its dynamic capabilities. Therefore, the bound $|\tau_2| \le \tau_{max,2}$ will impose also a bound $V_{max,1}$ on the (absolute) velocity that the first joint can reach. We have[1]

$$|\tau_2| = m_2\,(R-d)\,\dot{q}_1^2 \le \tau_{max,2} \qquad \Longrightarrow \qquad |\dot{q}_1| \le V_{max,1} = \sqrt{\frac{\tau_{max,2}}{m_2\,(R-d)}}. \tag{13}$$

Under the combined velocity/torque (viz. velocity/acceleration) bounds for the motion of joint 1, the minimum time solution will have in general a bang-coast-bang profile for the first torque (and its acceleration as well). The motion time T^* is computed then from known formulas.

[1]Note that $R - d > 0$ by assumption, so the argument of the square root is positive.

If $\alpha > V_{max,1}^2/A_{max,1}$, a coast phase will exist. Then

$$T_s = \frac{V_{max,1}}{A_{max,1}} \qquad \Longrightarrow \qquad (T^* - T_s)V_{max,1} = \alpha \qquad \Longrightarrow \qquad T^* = \frac{\alpha}{V_{max,1}} + \frac{V_{max,1}}{A_{max,1}}, \tag{14}$$

where one should replace the definitions of bounds in (12) and (13). The (qualitative) plots of the resulting torque/force vector $\boldsymbol{\tau}^*(t)$ are reported in Fig. 4. The second joint force $\tau_2^*(t)$ follows from (11), with a quadratic time profile where the velocity of the first joint is linear in time and a constant value where $\dot{q}_1$ is constant. The other special cases (with pure bang-bang commands) are treated similarly.

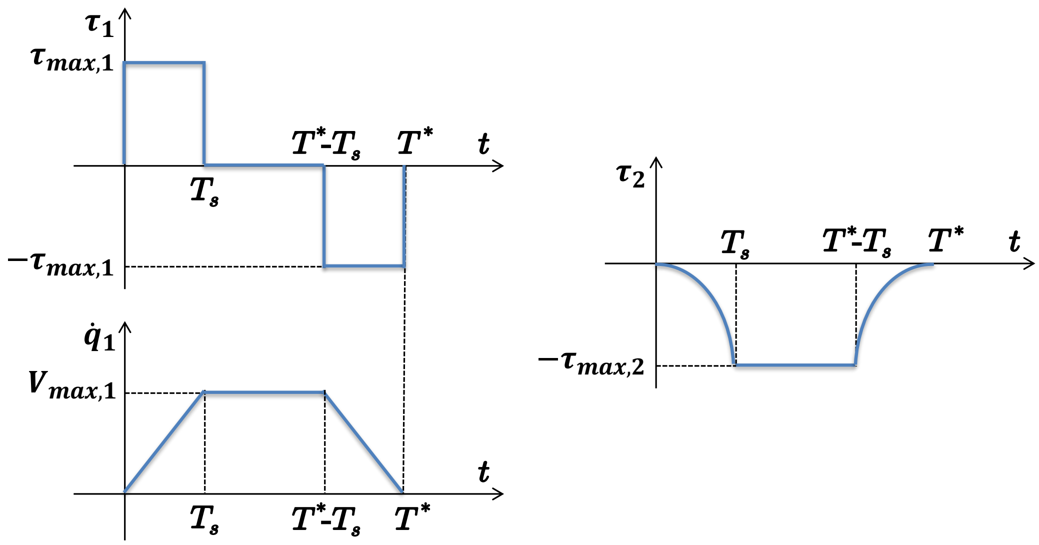

Figure 4: Optimal profiles of the torque τ_1^*, of the related velocity $\dot{q}_1^*$, and of the force τ_2^* for the requested rest-to-rest minimum time motion of the RP robot in Fig. 2.

Exercise 3

This is a simple application of conservation principles of the total kinetic energy T and total momentum P (along the direction x) for the system with the two masses m_1 and m_2. In formulas,

$$T(t) = \frac{1}{2}m_1 v_1^2(t) + \frac{1}{2}m_2 v_2^2(t) = \text{constant}, \qquad P(t) = m_1 v_1(t) + m_2 v_2(t) = \text{constant}, \qquad \forall t.$$

We apply these identities around the collision time $t = t_c$, just before $(t = t_c^-)$ and just after $(t = t_c^+)$. Let

$$v_1 = v_1(t_c^+), \qquad v_1(t_c^-) = v_0 > 0, \qquad v_2 = v_2(t_c^+), \qquad v_2(t_c^-) = 0,$$

where v_1 and v_2 are the unknowns of our problem. Thus,

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_0^2 \tag{15}$$

and

$$m_1 v_1 + m_2 v_2 = m_1 v_0. \tag{16}$$

Equations (15) and (16) are rewritten respectively as

$$m_1(v_1^2 - v_0^2) + m_2 v_2^2 = m_1(v_1 - v_0)(v_1 + v_0) + m_2 v_2^2 = 0 \tag{17}$$

and

$$m_1(v_1 - v_0) = -m_2 v_2. \tag{18}$$

Substituting (18) in (17) and simplifying yields

$$v_2 = v_1 + v_0. \tag{19}$$

Plugging (19) back into (16) leads to

$$m_1 v_1 + m_2(v_1 + v_0) = m_1 v_0 \qquad \Longrightarrow \qquad v_1 = \frac{m_1 - m_2}{m_1 + m_2}\, v_0. \tag{20}$$

Finally, substituting v_1 in (19) gives

$$v_2 = \frac{2m_1}{m_1 + m_2}\, v_0. \tag{21}$$

From (20–21), we conclude that:

$$\begin{cases} m_2 \to 0 & \Longrightarrow & v_1 = v_0 > 0, & v_2 = 2v_0 > 0, \\ m_2 < m_1 & \Longrightarrow & v_0 > v_1 > 0, & v_2 > v_0 > 0, \\ m_2 = m_1 & \Longrightarrow & v_1 = 0, & v_2 = v_0 > 0, \\ m_2 > m_1 & \Longrightarrow & -v_0 < v_1 < 0, & 0 < v_2 < v_0, \\ m_2 \to \infty & \Longrightarrow & v_1 = -v_0 < 0, & v_2 = 0. \end{cases}$$

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