Robotics II

July 11, 2018

Exercise 1

Consider a 2R planar robot, with the two links of length l_1 and l_2, having the actuating motors mounted on the axes of the two revolute joints. Each motor delivers its torque to the driven link through an elastic transmission, modeled as a torsional spring of stiffness $k_i > 0$, for $i = 1, 2$. The robot has no motion reduction elements. In the ith motor-link assembly, for $i = 1, 2$, let m_{m_i} and I_{m_i} be, respectively, the balanced mass and the inertia of the rotor of the motor around its spinning axis, and m_i, d_{c_i}, and I_i the link mass, the distance of the center of mass of the link from the preceding joint axis, and the link inertia around its center of mass. Using as generalized coordinates the angle θ_{m_i} of the rotor of motor i w.r.t. the preceding link axis, and the angle θ_i of link i w.r.t. the preceding link axis, for $i = 1, 2$, define the 4×4 inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the robot, where $\boldsymbol{q} = (\boldsymbol{\theta}^T \ \boldsymbol{\theta}_m^T)^T$. State explicitly any simplifying assumption that you may wish to use. Moreover, find a linear parametrization of the inertial term $\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}$ of the robot dynamic model in terms of a minimal set of p suitable dynamic coefficients a_i, $i = 1, \ldots, p$.

Exercise 2

Consider a 4R planar robot with all links of equal length $\ell = 0.2$ [m]. The robot is in the DH configuration $\boldsymbol{q} = (\ 0 \ \ \pi/2 \ \ 0 \ \ \pi/2\ )^T$ [rad] and at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$). In this state, we should assign to the end-effector a desired linear acceleration $\boldsymbol{a} = (\ 5 \ \ 0\ )^T$ [m/s^2]. The joint accelerations are taken as input commands, and are bounded as $|\ddot{q}_i| \le A_i$, $i = 1, \ldots, 4$, with the limits $A_1 = 9$, $A_2 = 6$, $A_3 = 4$, and $A_4 = 2$ [rad/s^2]. Find, if possible, a feasible joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^4$ that executes instantaneously the desired Cartesian task, while satisfying these hard bounds. A solution with a lower norm is preferred, and could be obtained by a straightforward variation of the SNS method moved to the acceleration level.

Exercise 3

In a visual servoing scheme, n point features with coordinates (u_i, v_i), for $i = 1, \ldots, n$, can be extracted from the image. Define the 2×6 interaction matrix $\bar{\boldsymbol{J}}$ between the 6D vector of linear velocity $\boldsymbol{V} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ of the camera and the time derivative of the coordinates $(\bar{u}, \bar{v})$ of the average position of the n point features in the image plane. State all variables that matrix $\bar{\boldsymbol{J}}$ depends upon.

Exercise 4

For a robot with n degrees of freedom, partition the generalized coordinates as $\boldsymbol{q} = (\boldsymbol{q}_a, \boldsymbol{q}_b)$, where $\boldsymbol{q}_a$ has n_a components, $\boldsymbol{q}_b$ has n_b components, and $n_a + n_b = n$. Provide the explicit expressions of the n_a-dimensional reduced robot dynamics and of the constraint-preserving forces $\boldsymbol{\lambda} \in \mathbb{R}^{n_b}$, when the geometric constraint $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{q}_b - \boldsymbol{q}_{b,d} = \boldsymbol{0}$ is imposed at all times, with $\boldsymbol{q}_{b,d}$ being constant.

[240 minutes; open books, but no computer or smartphone]

Solution

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Exercise 1

We need to compute the kinetic energy of the two motors and the two links, all in planar motion. For the motors, we have

$$T_{m_1} = \frac{1}{2} I_{m_1} \dot{\theta}_{m_1}^2, \qquad T_{m_2} = \frac{1}{2} m_{m_2} l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{m_2} (\dot{\theta}_1 + \dot{\theta}_{m_2})^2,$$

since the first motor is balanced and its center of mass does not move, while the center of mass of the second motor is placed on the second joint axis at a distance equal to the link length l_1.
For the links, we have

$$T_{l_1} = \frac{1}{2} \left(I_1 + m_1 d_{c_1}^2 \right) \dot{\theta}_1^2, \qquad T_{l_2} = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2,$$

with

$$\boldsymbol{p}_{c_2} = \begin{pmatrix} l_1 \cos\theta_1 + d_{c_2} \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + d_{c_2} \sin(\theta_1 + \theta_2) \end{pmatrix}, \quad \boldsymbol{v}_{c_2} = \dot{\boldsymbol{p}}_{c_2} = \begin{pmatrix} -(l_1 \dot{\theta}_1 \sin\theta_1 + d_{c_2} (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)) \\ l_1 \dot{\theta}_1 \cos\theta_1 + d_{c_2} (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{pmatrix},$$

and thus

$$\|\boldsymbol{v}_{c_2}\|^2 = l_1^2 \dot{\theta}_1^2 + d_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 d_{c_2} \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2.$$

Therefore, having set $\boldsymbol{q} = \begin{pmatrix} \boldsymbol{\theta}^T & \boldsymbol{\theta}_m^T \end{pmatrix}^T = \begin{pmatrix} \theta_1 & \theta_2 & \theta_{m_1} & \theta_{m_2} \end{pmatrix}^T$. we can write the total kinetic energy as

$$T = T_{m_1} + T_{l_1} + T_{m_2} + T_{l_2} = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with the 4×4 inertia matrix of the robot given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \begin{matrix} I_1 + m_1 d_{c_1}^2 + I_2 + m_2 d_{c_2}^2 + m_2 l_1^2 \\ + I_{m_2} + m_{m_2} l_1^2 + 2 m_2 l_1 d_{c_2} \cos\theta_2 \end{matrix} & I_2 + m_2 d_{c_2}^2 + m_2 l_1 d_{c_2} \cos\theta_2 & 0 & I_{m_2} \\ I_2 + m_2 d_{c_2}^2 + m_2 l_1 d_{c_2} \cos\theta_2 & I_2 + m_2 d_{c_2}^2 & 0 & 0 \\ 0 & 0 & I_{m1} & 0 \\ I_{m_2} & 0 & 0 & I_{m_2} \end{pmatrix}.$$

Note that, if we assume that in the kinetic energy of the second motor the contribution of the angular velocity due to the previous link carrying the motor can be neglected in comparison with the spinning velocity of the rotor of the motor itself, we would have

$$T_{m_2} = \cdots + \frac{1}{2} I_{m_2} (\dot{\theta}_1 + \dot{\theta}_{m_2})^2 \simeq \cdots + \frac{1}{2} I_{m_2} \dot{\theta}_{m_2}^2, \tag{1}$$

and the off-diagonal terms $M_{14} = M_{41} = I_{m_2}$ of the inertia matrix would disappear. As a result, the matrix would become block diagonal, with two 2×2 blocks (the second being diagonal) that pertain to the link kinetic energy and, respectively, to the motor kinetic energy. This assumption is quite realistic when the motors are connected to the driven links via transmissions with large reduction ratios (which is not, however, the present case), independently from the presence or not of elasticity in the transmissions.

The robot inertia matrix can be rewritten compactly using the following $p = 5$ dynamic coefficients

$$\begin{aligned} a_1 &= I_1 + m_1 d_{c_1}^2 + I_2 + m_2 l_1^2 + I_{m_2} + m_{m_2} l_1^2 \\ a_2 &= m_2 l_1 d_{c_2} \\ a_3 &= I_2 + m_2 d_{c_2}^2 \\ a_4 &= I_{m_1} \\ a_5 &= I_{m_2}, \end{aligned}$$

as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2\cos\theta_2 & a_3 + a_2\cos\theta_2 & 0 & a_5 \\ a_3 + a_2\cos\theta_2 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 \\ a_5 & 0 & 0 & a_5 \end{pmatrix}.$$

Thus, the inertial terms in the robot dynamic model can be given a linearly parametrized form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{\theta}_1 & (2\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 & \ddot{\theta}_2 & 0 & \ddot{\theta}_{m2} \\ 0 & \ddot{\theta}_1\cos\theta_2 & \ddot{\theta}_1 + \ddot{\theta}_2 & 0 & 0 \\ 0 & 0 & 0 & \ddot{\theta}_{m1} & 0 \\ 0 & 0 & 0 & 0 & \ddot{\theta}_1 + \ddot{\theta}_{m2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}.$$

We remark that the simplifying assumption (1) would eliminate from $\boldsymbol{Y}_M$ the presence of $\ddot{\theta}_{m2}$ in element $Y_{M,15}$ and of $\ddot{\theta}_1$ in $Y_{M,45}$, but not reduce the number of dynamic coefficients: as a matter of fact, $p = 5$ is the smallest possible number of such coefficients.

Exercise 2

The task Jacobian of the planar 4R robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \ell \begin{pmatrix} -(s_1 + s_{12} + s_{123} + s_{1234}) & -(s_{12} + s_{123} + s_{1234}) & -(s_{123} + s_{1234}) & -s_{1234} \\ c_1 + c_{12} + c_{123} + c_{1234} & c_{12} + c_{123} + c_{1234} & c_{123} + c_{1234} & c_{1234} \end{pmatrix}, \quad (2)$$

and is used, together with its time derivative $\dot{\boldsymbol{J}}(\boldsymbol{q})$, in the first- and second-order differential mappings

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{a} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{a} \in \mathbb{R}^2, \qquad \ddot{\boldsymbol{q}} \in \mathbb{R}^4.$$

Since a minimum norm solution is being sought at the acceleration level, we solve the second-order differential kinematics in the least squares sense using pseudoinversion as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right). \quad (3)$$

When $\boldsymbol{q} = \bar{\boldsymbol{q}} = \begin{pmatrix} 0 & \pi/2 & 0 & \pi/2 \end{pmatrix}^T$ [rad] and for $\ell = 0.2$ [m], the Jacobian becomes

$$\boldsymbol{J} := \boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.4 & -0.4 & -0.2 & 0 \\ 0 & -0.2 & -0.2 & -0.2 \end{pmatrix}$$

Having $\boldsymbol{J}$ full rank[1], any desired task acceleration $\boldsymbol{a}$ in (3) will be exactly realized in the absence of bounds, or realized at least in direction (possibly in a scaled form) in case the joint accelerations

[1]This property is particularly strong in this case, since *all* 2×2 minors are nonsingular in this configuration.

bounds cannot be satisfied. Moreover, since the robot is at rest ($\dot{\boldsymbol{q}} = 0$) in $\boldsymbol{q} = \bar{\boldsymbol{q}}$, the relation (3) collapses into

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\boldsymbol{a}. \tag{4}$$

In this setting, the SNS (Saturation in the Null Space) method presented originally at the velocity level can be applied without any modification, except for the acceleration limits A_i's replacing the velocity ones.

The pseudoinverse solution in (4) provides the joint acceleration with minimum norm. We have

$$\ddot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{a} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{a} = \begin{pmatrix} -1.6667 & 1.6667 \\ -0.8333 & -0.8333 \\ 0 & -1.6667 \\ 0.8333 & -2.5 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8.3333 \\ -4.1667 \\ 0 \\ 4.1667 \end{pmatrix} \ [\text{rad/s}^2]. \tag{5}$$

The fourth joint acceleration violates the maximum limit, $\ddot{q}_{PS,4} = 4.1667 > 2 = A_4$, so this is not a feasible solution. Thus, we search for a feasible solution by using the SNS method.

In step 1 of the SNS method, we saturate the overdriven joint by setting $\ddot{q}_4 = A_3 = 2$ [rad/s^2]. Then, the original task is modified by removing the saturated contribution of the fourth joint acceleration (and discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{a}_1 = \boldsymbol{a} - \boldsymbol{J}_4 A_4 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ -0.2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -0.4 & -0.4 & -0.2 \\ 0 & -0.2 & -0.2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{J}_{-4}\,\ddot{\boldsymbol{q}}_{-4},$$

where $\boldsymbol{J}_{-i}$ is the Jacobian obtained by deleting the ith column and, similarly, $\ddot{\boldsymbol{q}}_{-i}$ is the vector of joint accelerations without the ith component. We recompute next the contribution of the remaining active joints, by pseudoinverting the $\boldsymbol{J}_{-4}$ matrix for the modified task. We obtain

$$\ddot{\boldsymbol{q}}_{PS_{-4}} = \boldsymbol{J}_{-4}^{\#}\,\boldsymbol{a}_1 = \begin{pmatrix} -2.2222 & 3.3333 \\ -0.5556 & -1.6667 \\ 0.5556 & -3.3333 \end{pmatrix} \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -9.7778 \\ -3.4444 \\ 1.4444 \end{pmatrix} \ [\text{rad/s}^2],$$

to be completed with the additional choice $\ddot{q}_4 = A_4 = 2$. The first joint acceleration violates now its limit (on the negative side), $\ddot{q}_{PS_{-4},1} = -9.7778 < -9 = -A_1$. So. this is not yet a feasible solution and we proceed with the SNS method.

In step 2, we saturate also the first overdriven joint by setting $\ddot{q}_1 = -A_1 = -9$ [rad/s^2]. The original task is modified by removing both saturated acceleration contributions by the first and fourth joints (discarding the two associated columns of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{a}_2 = \boldsymbol{a} - \boldsymbol{J}_4 A_4 - \boldsymbol{J}_1(-A_1) = \boldsymbol{a}_1 + \boldsymbol{J}_1 A_1 = \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} + 9\begin{pmatrix} -0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix}$$

and

$$\boldsymbol{a}_2 = \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -0.4 & -0.2 \\ -0.2 & -0.2 \end{pmatrix} \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{J}_{-14}\,\ddot{\boldsymbol{q}}_{-14},$$

with obvious notation. We recompute next the contribution of the remaining active joints, by pseudoinverting the (now square and nonsingular) matrix $\boldsymbol{J}_{-23}$ for the modified task. We obtain

$$\ddot{\boldsymbol{q}}_{PS_{-14}} = \boldsymbol{J}_{-14}^{\#}\,\boldsymbol{a}_2 = \boldsymbol{J}_{-14}^{-1}\,\boldsymbol{a}_2 = \begin{pmatrix} -5 & 5 \\ 5 & -10 \end{pmatrix} \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \ [\text{rad/s}^2],$$

with $\ddot{q}_1 = -A_1 = -9$ and $\ddot{q}_4 = A_4 = 2$. All bounds are now satisfied and the obtained joint acceleration is feasible. Recomposing the complete vector, we have the solution

$$\ddot{\boldsymbol{q}}_{SNS} = \begin{pmatrix} -9 \\ -5 \\ 3 \\ 2 \end{pmatrix} \text{ [rad/s}^2\text{]}, \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}_{SNS} = \boldsymbol{a} \text{ and } \|\ddot{\boldsymbol{q}}_{SNS}\| = 10.9087. \tag{6}$$

This feasible solution is the one having the least possible norm.

The solution (6) is not the only feasible one. As a matter of fact, one could have attempted a heuristic procedure to find a (set of) solution(s) in a reasonable but otherwise arbitrary way, e.g., by fixing one component of the input acceleration to one of its (upper or lower) limits, and working out then the rest of the solution. For this, reconsider the original equation to be solved, written explicitly in terms of a linear system in the joint accelerations (scaling the coefficients so as to become all integers):

$$\boldsymbol{J}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{q}} = \boldsymbol{a} \quad \Rightarrow \quad \begin{pmatrix} -0.4 & -0.4 & -0.2 & 0 \\ 0 & -0.2 & -0.2 & -0.2 \end{pmatrix} \ddot{\boldsymbol{q}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{aligned} 2\ddot{q}_1 + 2\ddot{q}_2 + \ddot{q}_3 &= -25 \\ \ddot{q}_2 + \ddot{q}_3 + \ddot{q}_4 &= 0. \end{aligned}$$

By inspection, we find that choosing $\ddot{q}_1 = -A_1 = -9$ will contribute at best to the solution of the first scalar equation, being this variable present only in this equation and having the largest coefficient (and thus the highest sensitivity). In addition, set parametrically $\ddot{q}_4 = \alpha$ in the second scalar equation. We have then

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} -25 + 2A_1 \\ -\ddot{q}_4 \end{pmatrix} = \begin{pmatrix} -7 \\ -\alpha \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} -7 + \alpha \\ 7 - 2\alpha \end{pmatrix},$$

with the components of the parametric solution that need to satisfy the bounds

$$\begin{aligned} |\ddot{q}_2| &\le A_2 = 6 \\ |\ddot{q}_3| &\le A_3 = 4 \\ |\ddot{q}_4| &\le A_4 = 2 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} -6 \le -7 + \alpha &\le 6 \\ -4 \le 7 - 2\alpha &\le 4 \\ -2 \le \alpha &\le 2 \end{aligned} \qquad \Rightarrow \qquad \alpha \in [1.5, 2].$$

The feasible interval for α comes from the simultaneous intersection of the set of inequalities. Therefore, we have a parametrized family of feasible solutions in the form

$$\ddot{\boldsymbol{q}}(\alpha) = \begin{pmatrix} -9 \\ -7 + \alpha \\ 7 - 2\alpha \\ \alpha \end{pmatrix} \text{ [rad/s}^2\text{]}, \qquad \text{with } \|\ddot{\boldsymbol{q}}(\alpha)\| = \sqrt{6\alpha^2 - 42\alpha + 179}, \qquad \alpha \in [1.5, 2]. \tag{7}$$

We see immediately that $\ddot{\boldsymbol{q}}_{SNS} = \ddot{\boldsymbol{q}}(\alpha = 2)$. Moreover, the quadratic polynomial in the norm of $\ddot{\boldsymbol{q}}(\alpha)$ has an unconstrained minimum at $\alpha = 3.5$, which is outside the interval $[1.5, 2]$ of feasibility for α. Therefore, the minimum norm is obtained at the upper limit $\alpha = 2$ of this closed interval, i.e., with $\ddot{\boldsymbol{q}}_{SNS}$. Any other feasible solution will have a larger norm than $\ddot{\boldsymbol{q}}_{SNS}$.

We finally remark that, in order to find a feasible joint acceleration, a different (say, more conventional) solution would have been to use pseudo-inversion with a scaling of the original task acceleration $\boldsymbol{a}$ (in intensity, but without a change in direction). This is done as follows. From (5), we compute the necessary scaling factor $s > 1$ as

$$s = \max\left\{ \frac{|\ddot{q}_{PS,i}|}{A_i}, i = 1, \dots, 4 \right\} = \max\left\{ \frac{8.3333}{9}, \frac{4.1667}{6}, \frac{0}{4}, \frac{4.1667}{2} \right\} = 2.0833.$$

This value is imposed by the exceeding acceleration of the fourth joint. We compute then

$$\boldsymbol{a}_{scaled} = \frac{\boldsymbol{a}}{s} = \begin{pmatrix} 2.4 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}}_{PS,scaled} = \boldsymbol{J}^{\#}\boldsymbol{a}_{scaled} = \begin{pmatrix} -4 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Indeed, the obtained joint acceleration has lower norm than $\ddot{\boldsymbol{q}}_{SNS}$, but realizes in fact only $2.4/5 = 48\%$ of the desired task acceleration of the original problem.

Exercise 3

The interaction matrix of a generic point feature with image coordinates (u_i, v_i) is known to be

$$\boldsymbol{J}_{p_i}(u_i, v_i, Z_i) = \begin{pmatrix} -\dfrac{\lambda}{Z_i} & 0 & \dfrac{u_i}{Z_i} & \dfrac{u_i v_i}{Z_i} & -\left(\lambda + \dfrac{u_i^2}{\lambda}\right) & v_i \\ 0 & -\dfrac{\lambda}{Z_i} & \dfrac{v_i}{Z_i} & \lambda + \dfrac{v_i^2}{\lambda} & -\dfrac{u_i v_i}{Z_i} & -u_i \end{pmatrix},$$

with

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \boldsymbol{J}_{p_i}(u_i, v_i, Z_i) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

where the parameter $\lambda > 0$ is the constant focal length of the camera and Z_i is the depth of the Cartesian point $\boldsymbol{P}_i \in \mathbb{R}^3$ in the pre-image of (u_i, v_i). The average position of n point features in the image plane has coordinates

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{n} \sum_{i=1}^n u_i \\ \dfrac{1}{n} \sum_{i=1}^n v_i \end{pmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} u_i \\ v_i \end{pmatrix}.$$

Therefore

$$\begin{aligned} \begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{v}} \end{pmatrix} &= \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \left(\frac{1}{n} \sum_{i=1}^n \boldsymbol{J}_{p_i}(u_i, v_i, Z_i) \right) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix} = \bar{\boldsymbol{J}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\lambda}{n} \sum_{i=1}^n \frac{1}{Z_i} & 0 & \frac{1}{n} \sum_{i=1}^n \frac{u_i}{Z_i} & \frac{1}{n} \sum_{i=1}^n \frac{u_i v_i}{Z_i} & -\left(\lambda + \frac{1}{n} \sum_{i=1}^n \frac{u_i^2}{\lambda}\right) & \bar{v} \\ 0 & -\frac{\lambda}{n} \sum_{i=1}^n \frac{1}{Z_i} & \frac{1}{n} \sum_{i=1}^n \frac{v_i}{Z_i} & \lambda + \frac{1}{n} \sum_{i=1}^n \frac{v_i^2}{\lambda} & -\frac{1}{n} \sum_{i=1}^n \frac{u_i v_i}{Z_i} & -\bar{u} \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix}, \end{aligned}$$

with a dependence of the interaction matrix $\bar{\boldsymbol{J}}$ on the components of $\boldsymbol{u} = (u_1 \dots u_n)^T \in \mathbb{R}^n$, $\boldsymbol{v} = (v_1 \dots v_n)^T \in \mathbb{R}^n$, and $\boldsymbol{Z} = (Z_1 \dots Z_n)^T \in \mathbb{R}^n$.

Exercise 4

The n_b-dimensional geometric constraint

$$\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{q}_b - \boldsymbol{q}_{b,d} = \boldsymbol{0} \tag{8}$$

has a simple associated Jacobian

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{h}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \end{pmatrix}.$$

Therefore, considering the decomposition $\boldsymbol{q} = (\boldsymbol{q}_a, \boldsymbol{q}_b)$, the dynamic model of the constrained robot can be partitioned as follows

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}+\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \quad \Rightarrow \quad \begin{pmatrix} \boldsymbol{M}_{aa}(\boldsymbol{q}) & \boldsymbol{M}_{ab}(\boldsymbol{q}) \\ \boldsymbol{M}_{ab}^T(\boldsymbol{q}) & \boldsymbol{M}_{bb}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}}_a \\ \ddot{\boldsymbol{q}}_b \end{pmatrix} + \begin{pmatrix} \boldsymbol{n}_a(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ \boldsymbol{n}_b(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau}_a \\ \boldsymbol{\tau}_b \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\lambda} \end{pmatrix},$$

where $\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})$ collects all non-inertial terms in the model and $\boldsymbol{\lambda} \in \mathbb{R}^{n_b}$ is the vector of Lagrange multipliers associated to the geometric constraints.
To obtain a reduced dynamic model (with only $n - n_b = n_a$ independent coordinates), we proceed in the general way by bordering $\boldsymbol{A}(\boldsymbol{q})$ with the rows of a matrix $\boldsymbol{D}(\boldsymbol{q})$, so as to obtain a square and nonsingular transformation matrix. The situation is particularly simple since $\boldsymbol{A}$ is constant, and so can be chosen $\boldsymbol{D}$. A globally valid choice is then

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix}.$$

Thus, the following bidirectional mappings are established between the generalized velocity $\dot{\boldsymbol{q}}$ (and acceleration $\ddot{\boldsymbol{q}}$) and the pseudo-velocity $\boldsymbol{v}$ (and pseudo-acceleration $\dot{\boldsymbol{v}}$):

$$\boldsymbol{v} = \boldsymbol{D}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_a, \qquad \dot{\boldsymbol{v}} = \ddot{\boldsymbol{q}}_a \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}\boldsymbol{v} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{v}, \qquad \ddot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} \dot{\boldsymbol{v}}.$$

Dropping dependencies, the reduced inertia matrix and the reduced non-inertial dynamic terms are computed as

$$\boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \boldsymbol{M} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} = \boldsymbol{M}_{aa},$$

$$\boldsymbol{F}^T(\boldsymbol{\tau} - \boldsymbol{n}) = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_a - \boldsymbol{n}_a \\ \boldsymbol{\tau}_b - \boldsymbol{n}_b \end{pmatrix} = \boldsymbol{\tau}_a - \boldsymbol{n}_a.$$

Therefore, taking into account that $\boldsymbol{q}_b = \boldsymbol{q}_{b,d}$ and $\dot{\boldsymbol{q}}_b = \ddot{\boldsymbol{q}}_b = \boldsymbol{0}$ from (8), the reduced dynamic model becomes

$$\boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) = \boldsymbol{\tau}_a,$$

while the Lagrange multipliers (i.e., the forces that will preserve the geometric constraints when attempting their violation) takes the expression

$$\boldsymbol{\lambda} = \boldsymbol{E}^T(\boldsymbol{M}\boldsymbol{F}\dot{\boldsymbol{v}} + \boldsymbol{n} - \boldsymbol{\tau}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \left(\begin{pmatrix} * \\ \boldsymbol{M}_{ab}^T \end{pmatrix} \ddot{\boldsymbol{q}}_a + \begin{pmatrix} * \\ \boldsymbol{n}_b \end{pmatrix} - \begin{pmatrix} * \\ \boldsymbol{\tau}_b \end{pmatrix} \right),$$

or, by expliciting the dependencies,

$$\boldsymbol{\lambda} = \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) - \boldsymbol{\tau}_b.$$

We conclude with two extra comments. The following torque command, expressed as a function of the constrained robot state and of the arbitrary input torque $\boldsymbol{\tau}_a$,

$$\begin{aligned} \boldsymbol{\tau}_b &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) \\ &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\boldsymbol{M}_{aa}^{-1}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\left(\boldsymbol{\tau}_a - \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0})\right) + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}), \end{aligned}$$

will guarantee $\boldsymbol{\lambda} \equiv \boldsymbol{0}$ at all times, resulting in a feasible motion with minimal internal effort. On the other hand, the feedback linearizing control law that achieves (in a decoupled way) a desired value $\boldsymbol{a}$ for the acceleration $\ddot{\boldsymbol{q}}_a$ of the free variables and a desired constraint force $\boldsymbol{\lambda} = \boldsymbol{\lambda}_d$ is given by

$$\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}) \\ \boldsymbol{M}_{ab}^T \boldsymbol{q}_a, \boldsymbol{q}_{b,d}) \end{pmatrix} \boldsymbol{a} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{\lambda}_d + \boldsymbol{n}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}),$$

or

$$\begin{aligned} \boldsymbol{\tau}_a &= \boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\, \boldsymbol{a} + \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}), \\ \boldsymbol{\tau}_b &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\, \boldsymbol{a} + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) + \boldsymbol{\lambda}_d. \end{aligned}$$

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