Robotics II

June 6, 2016

Exercise 1

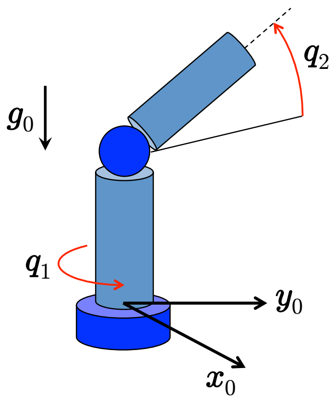

Figure 1: A 2R polar robot

Derive the dynamic model of a 2R polar robot moving in the presence of gravity, using the generalized coordinates $\boldsymbol{q} = (q_1, q_2)$ defined in Fig. 1. Assume that the links have cylindric form (as in the picture) and uniformly distributed mass.

Provide for this robot the explicit expression of the terms of an adaptive control law that guarantees asymptotic tracking of a desired smooth joint trajectory $\boldsymbol{q}_d(t)$, without any a priori knowledge about the robot dynamic parameters. Which is the minimum dimension of such an adaptive controller?

Exercise 2

For the robot in Fig. 1, write down all different symbolic expressions of control laws that you are aware of, which guarantee regulation to a desired (generic) constant configuration $\boldsymbol{q}_d$. Specify for each law the design conditions for success and the type of convergence/stability achieved.

[180 minutes; open books]

Solution

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Exercise 1

The definition of the joint variables q_1 and q_2 follows the Denavit-Hartenberg convention. We show in Fig. 2 the two DH frames attached to the two moving links (and indexed with 1 and 2), which will be used for defining conveniently their inertial parameters. The rotation matrices between frames 0 and 1 and between frames 1 and 2 are found easily by inspection (without the need of explicitly defining a DH table of parameters) as

$$^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad ^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the cylindric links have uniform mass, their center of mass will lie along the $\boldsymbol{y}_1$-axis for link 1 and along the $\boldsymbol{x}_2$-axis for link 2. We denote with $d_2 > 0$ the distance of the center of mass of link 2 from the axis of joint 1 (slightly more than half of the link length). The inertia matrix of each link is diagonal when referred to the kinematic reference frame attached to the link, as well as when referred to the frame with origin in the center of mass and having the same orientation. We denote the link inertia matrices in the latter case as

$$^i\boldsymbol{I}_i = \begin{pmatrix} I_{ix} & & \\ & I_{iy} & \\ & & I_{iz} \end{pmatrix}, \qquad i = 1, 2.$$

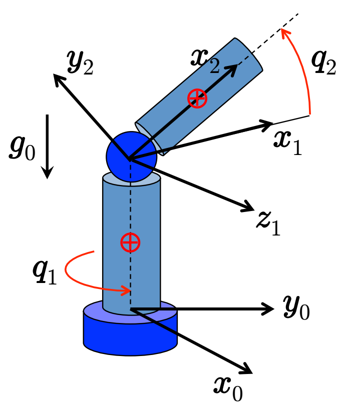

Figure 2: The reference frames used for defining link inertial parameters of the 2R polar robot

We start by computing the various terms in the robot dynamic model, following a Lagrangian approach.

The kinetic energy of the robot is $T = T_1 + T_2$. For the first link,

$$T_1 = \frac{1}{2} I_{1y}\, \dot{q}_1^2.$$

For the second link, the position of its center of mass is

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_2 \cos q_2 \cos q_1 \\ d_2 \cos q_2 \sin q_1 \\ \ell_1 + d_2 \sin q_2 \end{pmatrix},$$

where ℓ_1 is the length of link 1 (an irrelevant kinematic parameter for what follows). Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_2 \cos q_2 \sin q_1 \, \dot{q}_1 - d_2 \cos q_1 \sin q_2 \, \dot{q}_2 \\ d_2 \cos q_2 \cos q_1 \, \dot{q}_1 - d_2 \sin q_1 \sin q_2 \, \dot{q}_2 \\ d_2 \cos q_2 \, \dot{q}_2 \end{pmatrix},$$

and its squared norm simplifies to

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = d_2^2 \left(\dot{q}_2^2 + \cos^2 q_2 \, \dot{q}_1^2 \right).$$

The angular velocity of link 2, when expressed in frame 0, is computed as[1]

$${}^0\boldsymbol{\omega}_2 = {}^0\boldsymbol{z}_0 \, \dot{q}_1 + {}^0\boldsymbol{z}_1 \, \dot{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{q}_1 + \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix} \dot{q}_2.$$

In order to use the constant diagonal inertia matrix of link 2, we need to express the angular velocity in frame 2. Since

$${}^1\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1^T(q_1) \, {}^0\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{z}_0 = {}^1\boldsymbol{R}_2^T(q_1) \, {}^1\boldsymbol{z}_0 = \begin{pmatrix} \sin q_2 \\ \cos q_2 \\ 0 \end{pmatrix}$$

and

$${}^1\boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad {}^2\boldsymbol{z}_1 = {}^1\boldsymbol{R}_2^T(q_1) \, {}^1\boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we have

$${}^2\boldsymbol{\omega}_2 = {}^2\boldsymbol{z}_0 \, \dot{q}_1 + {}^2\boldsymbol{z}_1 \, \dot{q}_2 = \begin{pmatrix} \sin q_2 \, \dot{q}_1 \\ \cos q_2 \, \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}.$$

As a result,

$$T_2 = \frac{1}{2} m_2 \, \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} + \frac{1}{2} \, {}^2\boldsymbol{\omega}_2^T \, {}^2\boldsymbol{I}_2 \, {}^2\boldsymbol{\omega}_2 = \frac{1}{2} \left(I_{2x} \sin^2 q_2 + \left(I_{2y} + m_2 d_2^2 \right) \cos^2 q_2 \right) \dot{q}_1^2 + \left(I_{2z} + m_2 d_2^2 \right) \dot{q}_2^2.$$

We note that in general $I_{2x} \neq I_{2y}$, whereas it is $I_{2y} = I_{2z}$, due to the cylindric form and uniform mass distribution of the links. Therefore,

$$T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} I_{1y} + I_{2x} \sin^2 q_2 + \left(I_{2y} + m_2 d_2^2 \right) \cos^2 q_2 & 0 \\ 0 & I_{2z} + m_2 d_2^2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}}.$$

Using the following definition of dynamic coefficients

$$a_1' = I_{1y}, \qquad a_2' = I_{2x}, \qquad a_3' = I_{2y} + m_2 d_2^2 = I_{2z} + m_2 d_2^2,$$

we can write the inertia matrix as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1' + a_2' \sin^2 q_2 + a_3' \cos^2 q_2 & 0 \\ 0 & a_3' \end{pmatrix}.$$

[1] We use here the expression for revolute joints of the columns of the angular part of the geometric Jacobian.

Although there will be no reduction in the number of dynamic coefficients, it is slightly more convenient to use the trigonometric identity $\cos^2 q_2 = 1 - \sin^2 q_2$ and obtain

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 & 0 \\ 0 & a_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1(q_2) & \boldsymbol{b}_2 \end{pmatrix},$$

with

$$\begin{aligned} a_1 &= (a_1' + a_3' =)\, I_{1y} + I_{2y} + m_2 d_2^2 \\ a_2 &= (a_2' - a_3' =)\, I_{2x} - I_{2y} - m_2 d_2^2 \\ a_3 &= (a_3' =)\, I_{2y} + m_2 d_2^2 = I_{2z} + m_2 d_2^2. \end{aligned}$$

For the Coriolis and centrifugal terms, we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}, \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad i = 1, 2, \qquad \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix},$$

where

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{ \frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right\}, \qquad i = 1, 2.$$

Note that with the above definition based on Christoffel symbols, the factorization matrix $\boldsymbol{S}$ satisfies automatically the property of skew-symmetry for $\dot{\boldsymbol{B}} - 2\boldsymbol{S}$.

Computing

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{ \begin{pmatrix} 0 & 2a_2 \sin q_2 \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2a_2 \sin q_2 \cos q_2 & 0 \end{pmatrix} - \boldsymbol{0} \right\} \\ &= \begin{pmatrix} 0 & a_2 \sin q_2 \cos q_2 \\ a_2 \sin q_2 \cos q_2 & 0 \end{pmatrix} \qquad, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \cdots = \begin{pmatrix} -a_2 \sin q_2 \cos q_2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2a_2 \sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_2 \\ -a_2 \sin q_2 \cos q_2\, \dot{q}_1^2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = a_2 \sin q_2 \cos q_2 \begin{pmatrix} \dot{q}_2 & \dot{q}_1 \\ -\dot{q}_1 & 0 \end{pmatrix}.$$

The potential energy of the robot is given by

$$U = U_1 + U_2, \qquad U_i = -m_i \boldsymbol{g}_0^T \boldsymbol{r}_{0,c_i}, \quad i = 1, 2.$$

Since

$$\boldsymbol{g}_0^T = \begin{pmatrix} 0 & 0 & -g_0 \end{pmatrix}, \qquad g_0 = 9.81\,[\mathrm{m/s^2}]$$

and the potential energy U_1 is constant, we only need the z-component of the position vector $\boldsymbol{r}_{0,c_2}$ of the center of mass of link 2. We have

$$U_1 = \text{cost}, \qquad U_2 = m_2 g_0 d_2 \sin q_2 = a_4 \sin q_2,$$

where we have introduced a fourth, and last, dynamic coefficient $a_4 = m_2 g_0 d_2$. Therefore,

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ a_4 \cos q_2 \end{pmatrix}.$$

The dynamic model of the robot can thus be written in its linear parametrized form,

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{a} = \boldsymbol{u},$$

with

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & \sin^2 q_2\, \ddot{q}_1 + 2\sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_2 & 0 & 0 \\ 0 & -\sin q_2 \cos q_2\, \dot{q}_1^2 & \ddot{q}_2 & \cos q_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Defining $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e} = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q})$, with a diagonal matrix $\boldsymbol{\Lambda} > 0$, two diagonal gain matrices $\boldsymbol{K}_D > 0$ and $\boldsymbol{K}_P = \boldsymbol{K}_D\boldsymbol{\Lambda}^{-1} > 0$, and a diagonal estimation gain matrix $\boldsymbol{\Gamma} > 0$, the adaptive controller will have dimension 4 (equal to the minimum number of dynamic coefficients to be estimated in this robot) and the expression

$$\begin{aligned} \boldsymbol{u} &= \hat{\boldsymbol{B}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{a}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma}\,\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}})\,, \qquad \hat{\boldsymbol{a}}(0) = \text{arbitrary}, \end{aligned}$$

where

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & \sin^2 q_2\, \ddot{q}_{r1} + \sin q_2 \cos q_2\, (\dot{q}_1 \dot{q}_{r2} + \dot{q}_{r1}\dot{q}_2) & 0 & 0 \\ 0 & -\sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_{r1} & \ddot{q}_{r2} & \cos q_2 \end{pmatrix}, \qquad \hat{\boldsymbol{a}} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \end{pmatrix}.$$

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