Robotics II

October 27, 2014

Exercise 1

The collision detection and isolation method based on the use of residuals that monitor the robot generalized momentum has been presented only for open chain manipulators with *revolute* joints. Consider the planar PRR robot in Fig. 1, moving on a *horizontal* plane. Assuming that the robot at rest at time $t = 0$, provide the explicit expressions of the contributions to the residual vector $\boldsymbol{r} \in \mathbb{R}^3$ in terms of the robot dynamic model components[1]. Comment on analogies or differences that may result due to the presence of *prismatic* joints in the chain, only one in the present case, or one or more in the general case.

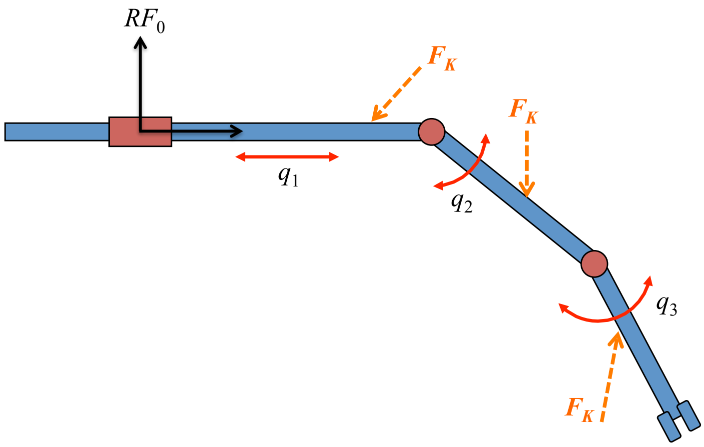

Figure 1: A planar PRR robot that may undergo collisions at any point along its structure (only non-simultaneous collision forces $\boldsymbol{F}_K$ in the plane of motion are considered)

Exercise 2

For the same robot of the previous exercise, assume that second and third links have unitary lengths. Moreover, the force (for joint 1) and torques (for joints 2 and 3) that can be delivered by the actuators at the joints are bounded as follows:

$$|\tau_1| \leq 5 \;\; [\text{N}], \qquad |\tau_2| \leq 1 \;\; [\text{Nm}], \qquad |\tau_3| \leq 2 \;\; [\text{Nm}].$$

The robot should be able to sustain in static conditions contact forces $\boldsymbol{F}$ that are applied to its end-effector in various planar directions. Determine the maximum norm of a contact force that can be applied in *any* planar direction and sustained by the robot when kept always in a fixed configuration. Provide at least one *non-singular* configuration in which the robot achieves this optimal result.

[180 minutes; open books]

[1] For dynamic analysis, you may use whatever generalized coordinates you find most convenient.

Solution

October 27, 2014

Exercise 1

When considering the Lagrangian dynamics of a robot with $\boldsymbol{q} \in \mathbb{R}^n$ that is possibly subject to collision forces, we have

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_K, \qquad \boldsymbol{\tau}_K = \boldsymbol{J}_K^T(\boldsymbol{q})\boldsymbol{F}_K,$$

where the left-hand side of the dynamic equation contains the usual inertia terms, Coriolis and centrifugal terms (with a factorization such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix), and gravity terms, while the non-conservative terms on the right-hand side are the motor torque $\boldsymbol{\tau}$ and the joint torque $\boldsymbol{\tau}_K \in \mathbb{R}^n$ resulting from a collision/contact on the robot structure (i.e., on one of the links). Moreover, $\boldsymbol{F}_K \in \mathbb{R}^m$ is the generalized force at the contact and $\boldsymbol{J}_K^T(\boldsymbol{q})$ is the transpose of the Jacobian of the contact point. Both these quantities are unknown.

To detect the presence of a joint torque $\boldsymbol{\tau}_K$ due to a collision, we use a residual vector based on the robot generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$. With the robot starting at rest at time $t = 0$, the residual $\boldsymbol{r} \in \mathbb{R}^n$ is defined as

$$\boldsymbol{r} = \boldsymbol{K}_I \left[\boldsymbol{p} - \int_0^t (\boldsymbol{\tau} + \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r}) ds \right], \qquad \boldsymbol{r}(0) = \boldsymbol{0},$$

with $\boldsymbol{K}_I > 0$ and diagonal. Therefore, for its computation we need the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, the factorization matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Coriolis and centrifugal terms (obtained using Christoffel's coefficients), and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ of the specific robot.

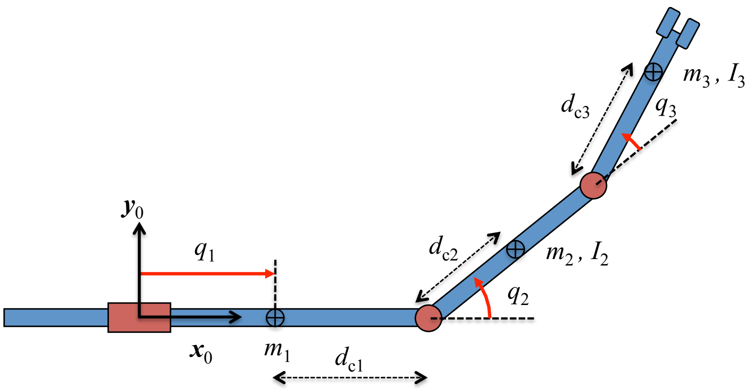

Figure 2: The definition of generalized coordinates and dynamic parameters for the planar PRR robot (the link length are l_2 and l_3)

We define the reference frame axes and the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ for the planar PRR robot as in Fig. 2 —note that these do not follow the DH convention. Since the robot is moving on a horizontal plane, $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$.

The kinetic energy of the robot

$$T = \sum_{i=1}^{3} T_i = \sum_{i=1}^{3} \frac{1}{2} \left[m_i \|\boldsymbol{v}_{c_i}\|^2 + \boldsymbol{\omega}_i^T \boldsymbol{I}_i \boldsymbol{\omega}_i \right] = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

is computed as follows[2]:

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_{c1} + d_{c2} c_2 \\ d_{c2} s_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} s_2 \dot{q}_2 \\ d_{c2} c_2 \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2 d_{c2} s_2 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} I_2 \dot{q}_2^2$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + d_{c1} + l_2 c_2 + d_{c3} c_{23} \\ l_2 s_2 + d_{c3} s_{23} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - l_2 s_2 \dot{q}_2 - d_{c3} s_{23} \left(\dot{q}_2 + \dot{q}_3 \right) \\ l_2 c_2 \dot{q}_2 + d_{c3} c_{23} \left(\dot{q}_2 + \dot{q}_3 \right) \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad T_3 &= \tfrac{1}{2} m_3 \Big(\dot{q}_1^2 + l_2^2 \dot{q}_2^2 + d_{c3}^2 \left(\dot{q}_2 + \dot{q}_3 \right)^2 - 2 l_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_{c3} s_{23} \dot{q}_1 \left(\dot{q}_2 + \dot{q}_3 \right) + 2 l_2 d_{c3} c_{23} \dot{q}_2 \left(\dot{q}_2 + \dot{q}_3 \right) \Big) \\ &\quad + \tfrac{1}{2} I_3 \left(\dot{q}_2 + \dot{q}_3 \right)^2 . \end{aligned}$$

As a result, the inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & -\left(m_2 d_{c2} + m_3 l_2 \right) s_2 - m_3 d_{c3} s_{23} & -m_3 d_{c3} s_{23} \\ symm & I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 + m_3 l_2^2 + 2 l_2 m_3 d_{c3} c_3 & I_3 + m_3 d_{c3}^2 + l_2 m_3 d_{c3} c_3 \\ symm & symm & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

Using the following parametrization[3]

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= m_2 d_{c2} + m_3 l_2 \\ a_3 &= m_3 d_{c3} \\ a_4 &= I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 + m_3 l_2^2 \\ a_5 &= I_3 + m_3 d_{c3}^2, \end{aligned}$$

the inertia matrix can be more compactly rewritten as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -a_2 s_2 - a_3 s_{23} & -a_3 s_{23} \\ -a_2 s_2 - a_3 s_{23} & a_4 + 2 l_2 a_3 c_3 & a_5 + l_2 a_3 c_3 \\ -a_3 s_{23} & a_5 + l_2 a_3 c_3 & a_5 \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) & \boldsymbol{m}_3(\boldsymbol{q}) \end{pmatrix}.$$

For the term $\boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$, we use the formula

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{C}_1^T(\boldsymbol{q}) \dot{\boldsymbol{q}} & \boldsymbol{C}_2^T(\boldsymbol{q}) \dot{\boldsymbol{q}} & \boldsymbol{C}_3^T(\boldsymbol{q}) \dot{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix},$$

with matrices $\boldsymbol{C}_i(\boldsymbol{q})$ defined as

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right) \right], \qquad \text{for } i = 1, 2, 3.$$

[2]Taking into account the planar nature of the problem, we work with two-dimensional vectors for linear quantities (in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$) and with scalars for angular quantities (components along $\boldsymbol{z}_0$).

[3]Here, the kinematic parameter l_2 is assumed to be known. This allows writing the product $m_3 l_2 d_{c3}$ as $a_3 l_2$, without the need of introducing a sixth dynamic coefficient.

Performing computations yields finally

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(a_2 c_2 + a_3 c_{23}\right) & -a_3 c_{23} \\ 0 & -a_3 c_{23} & -a_3 c_{23} \end{pmatrix},$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_2 a_3 s_3 \\ 0 & -l_2 a_3 s_3 & -l_2 a_3 s_3 \end{pmatrix},$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -l_2 a_3 s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The presence of prismatic joints in the robot does not change the overall residual approach to collision detection. Indeed, there are slight changes in the detectability of contact forces. For instance, a force $\boldsymbol{F}_K$ applied to the first link of the PRR robot along a direction that is *normal* to the first joint axis will *not* be detected. For a robot with a first revolute joint, a force applied to the first link along a line *crossing* the first joint axis would *not* be detected. In both cases, note that such forces produce no motion anyway. When a prismatic joint is placed along the structure, as in a planar RPR robot, the same reasoning applies, although detectability becomes in any event easier, thanks to the role of the multiple joints preceding the link being hit.

In general, contact forces $\boldsymbol{F}_K \in \mathcal{N}\left\{\boldsymbol{J}_K^T(\boldsymbol{q})\right\}$ will never be recorded by the residual $\boldsymbol{r}$. In a dual fashion, the residual method will fully detect contact forces that are completely orthogonal to the null space of the contact Jacobian, no matter which is the prismatic or revolute nature of the robot joints.

Exercise 2

We use the same coordinates as in Exercise 1. From the direct kinematics

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_1 + l_2 c_2 + l_3 c_{23} \\ l_2 s_2 + l_3 s_{23} \end{pmatrix},$$

the end-effector Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -\left(l_2 s_2 + l_3 s_{23}\right) & -l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \end{pmatrix}.$$

Note that the Jacobian is singular (loses rank) when $c_{23} = c_2 = 0$, namely when the second and third links lie both (folded or stretched) along a line orthogonal to the first (prismatic) joint axis. In singular configurations, there are Cartesian directions along which arbitrary forces can be applied without the need of motor torques to keep the structure in static balance. While such configurations are indeed good candidates for the solution of the problem at hand, we should also take into account that even in singular configurations contact forces could be applied along arbitrary planar directions, and thus too large forces in norm may not be sustainable in view of the given actuator limits.

A generic contact force can be parametrized as

$$\boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \|\boldsymbol{F}\| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \qquad \alpha \in (-\pi.\pi], \tag{1}$$

where α is the angle of the force w.r.t. the $\boldsymbol{x}_0$ axis. For a given robot configuration $\boldsymbol{q}$ and a given value of $\|\boldsymbol{F}\|$, we should check if the actuator limits are still satisfied or not in correspondence to the 'worst case' angle α.

Setting then $l_2 = l_3 = 1$ [m], and using the given actuator bounds, the following inequalities need to be satisfied

$$\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} \le \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} = \begin{pmatrix} F_x \\ -\left(s_2 + s_{23}\right) F_x + \left(c_2 + c_{23}\right) F_y \\ -s_{23} F_x + c_{23} F_y \end{pmatrix} \le \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.$$

Or, using also eq. (1),

$$\begin{pmatrix} |F_x| \\ \|F\| \cdot |s_{2-\alpha} + s_{23-\alpha}| \\ \|F\| \cdot |s_{23-\alpha}| \end{pmatrix} \le \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}. \tag{2}$$

This form is more convenient for analysis. From the first inequality in (2), the norm of $\boldsymbol{F}$ should certainly not exceed 5 N. From the last inequality, no matter which value takes the sum $q_2 + q_3$, we can always select a direction (i.e., a suitable value of α) such that $|s_{23-\alpha}| = 1$, yielding then the worst case. Thus, the norm of $\boldsymbol{F}$ cannot exceed 2 N (which dominates the former condition). Finally, consider the second inequality. If we had a limit on the second torque larger than or equal to 4 Nm, since $|s_{2-\alpha} + s_{23-\alpha}| \le 2$ holds for any combination of q_2, q_3 and α, then this bound would anyway be dominated by the stricter condition on the third torque. Instead, with a maximum value of 1 Nm, we still need to carry the analysis a bit further.

It is easy to see that $c_3 = -1$ (namely, $q_3 = \pm\pi$) implies $s_{2-\alpha} + s_{23-\alpha} \equiv 0$, and so any force can be sustained by the actuator at joint 2 when folding the third link over the second, and independently of the value of q_2. This should not come unexpected, as the application line of any force applied to the end-effector would then always cross the axis of joint 2, producing thus no torque. Note that things would be quite different in the case $l_2 \neq l_3$.

Summarizing, the maximum sustainable contact force applied at the robot end-effector has norm $\|\boldsymbol{F}\| = 2$ N. A non-singular configuration where such a force can be sustained (with at least one torque in saturation) is given by $\boldsymbol{q}^* = (\text{any}, 3\pi/4, -\pi)$ —q_1 is irrelevant. When applying a force of 2 N in the direction normal to link 3 ($\alpha = \pi/4$), we would have as balancing joint torque

$$\boldsymbol{\tau}^* = -\boldsymbol{J}^T(\boldsymbol{q}^*)\boldsymbol{F} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 0 \\ -2 \end{pmatrix}.$$

Note that the Jacobian matrix at $\boldsymbol{q}^*$ has full rank, so this is *not* a singular configuration.

* * * * *