Robotics II

February 6, 2013

The end-effector of the RP robot in Fig. 1 is constrained to move on the Cartesian line $x = k$, with $k > 0$.

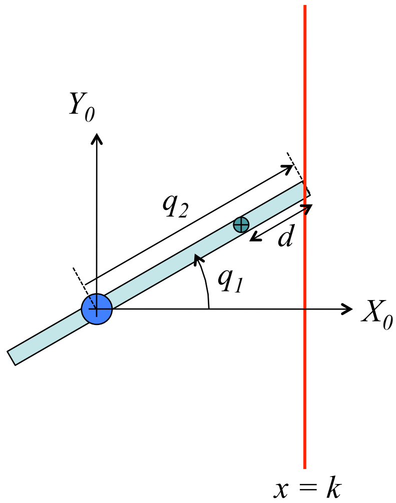

Figure 1: A RP robot moving on a horizontal plane with its end-effector constrained on a line

- Derive the expression of the *reduced* robot dynamics (in this case, a single first-order differential equation), written in terms of pseudoacceleration and automatically satisfying the constraint. Try to provide global validity to this model.
- Design a control law that regulates to constant desired values v_d and λ_d, respectively the tangent velocity and the normal force to the end-effector constraint.

[120 minutes; open books]

Solution

February 6, 2013

Following the Lagrangian approach for a robot with N, with multipliers $\boldsymbol{\lambda}$ used to weigh the M-dimensional holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \qquad s.t. \quad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}, \tag{1}$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q})/\partial \boldsymbol{q}$.

The *reduced* dynamic model is obtained by restricting the motion to a R-dimensional configuration space, with $R = N - M$, that is automatically compatible with the constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$. The constraints are thus discarded from the formulation. At the same time, it is also possible to eliminate the appearance of the multipliers (i.e., of the forces that arise when attempting to violate the constraints) in the resulting dynamic equations.

We provide first the terms that appear in (1), namely the robot inertia matrix $\boldsymbol{B}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, and the matrix $\boldsymbol{A}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2}\left(I_2 \dot{q}_1^2 + m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2}\right).$$

Since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix},$$

it follows

$$T = \frac{1}{2}\left(I_1 + I_2 + m_2(q_2 - d)^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

From the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2(q_2 - d)\dot{q}_1 \dot{q}_2 \\ -m_2(q_2 - d)\dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 - k = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \end{pmatrix}.$$

The reduction of the dynamics proceeds then as follows. In the present case, it is $N = 2$ and $M = 1$, so that $R = 1$. Define the matrix $\boldsymbol{D}(\boldsymbol{q})$ (a row in our case) such that

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ d_1(\boldsymbol{q}) & d_2(\boldsymbol{q}) \end{pmatrix} \quad \text{is } \textit{nonsingular}.$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

A good choice is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = -q_2.$$

Since the end-effector is constrained to live on $x = k > 0$, it will always be $q_2 \neq 0$. Thus, the requested non-singularity of the matrix holds globally as long as the constraint is enforced. This implies that the following derivations will lead also to a globally defined reduced model.

Define then the pseudovelocity v (a scalar) and its derivative $\dot{v}$ (the pseudoacceleration) as

$$\begin{aligned} v &= \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} = q_2 \cos q_1 \dot{q}_1 + \sin q_1 \dot{q}_2, \\ \dot{v} &= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{D}}(\boldsymbol{q})\dot{\boldsymbol{q}} = q_2 \cos q_1 \ddot{q}_1 + \sin q_1 \ddot{q}_2 - q_2 \sin q_1 \dot{q}_1^2 + 2 \cos q_1 \dot{q}_1 \dot{q}_2. \end{aligned}$$

To invert these relations, define

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{\sin q_1}{q_2} & \dfrac{\cos q_1}{q_2} \\ \cos q_1 & \sin q_1 \end{pmatrix}.$$

We obtain then

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{F}(\boldsymbol{q})\, v = \begin{pmatrix} \dfrac{\cos q_1}{q_2} \\ \sin q_1 \end{pmatrix} v, \\ \ddot{\boldsymbol{q}} &= \boldsymbol{F}(\boldsymbol{q})\, \dot{v} + \dot{\boldsymbol{F}}(\boldsymbol{q}) v = \begin{pmatrix} \dfrac{\cos q_1}{q_2} \\ \sin q_1 \end{pmatrix} \dot{v} + \begin{pmatrix} -\left(\dfrac{\sin q_1 \dot{q}_1}{q_2} + \dfrac{\cos q_1 \dot{q}_2}{q_2^2}\right) \\ \cos q_1 \dot{q}_1 \end{pmatrix} v. \end{aligned} \tag{2}$$

Based on their definitions, the matrix relations $\boldsymbol{AF} = \boldsymbol{O}$ and $\boldsymbol{AE} = \boldsymbol{I}$ hold for all $\boldsymbol{q}$. Premultiplying (1) by $\boldsymbol{F}^T(\boldsymbol{q})$ and substituting the acceleration $\ddot{\boldsymbol{q}}$ from (2) yields

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\, \dot{v} = \boldsymbol{F}^T(\boldsymbol{q}) \left(\boldsymbol{u} - \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})v - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right), \tag{3}$$

which is the reduced dynamics that we were looking for —in fact, a scalar first-order differential equation in v. All terms in (3) have been defined, but we can write more explicitly the leading scalar (a pseudoinertia)

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \left(I_1 + I_2 + m_2(q_2 - d)^2\right) \frac{\cos^2 q_1}{q_2^2} + m_2 \sin^2 q_1.$$

Similarly, by premultiplying (1) by $\boldsymbol{E}^T(\boldsymbol{q})$ (a row) we can isolate the scalar multiplier λ. Substituting the acceleration $\ddot{\boldsymbol{q}}$ from (2) yields

$$\lambda = \boldsymbol{E}^T(\boldsymbol{q}) \left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\dot{v} + \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})v + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{u} \right). \tag{4}$$

The control task is defined at the end-effector level, and requires the stabilization to zero of the error $e_v = v_d - v$ for the tangential velocity and of the error $e_\lambda = \lambda_d - \lambda$ for the normal force.

The tangential and normal directions are indeed referred to the geometry of the constraint. This hybrid control task is easily achieved in a linear and decoupled way by using a preliminary *feedback linearization* law for the joint-space control input $\boldsymbol{u} \in \mathbb{R}^2$. Defining in eqs. (3) and (4)

$$\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})v + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})u_1 - \boldsymbol{A}^T(\boldsymbol{q})u_2$$

leads simultaneously to

$$\dot{v} = u_1, \qquad \lambda = u_2.$$

The control design is completed by choosing

$$u_1 = k_1\, e_v, \qquad u_2 = \lambda_d + k_2 \int_0^t e_\lambda(\tau) d\tau,$$

with $k_1 > 0$, $k_2 > 0$. Beside the feedforward term (λ_d in the force loop, while $\dot{v}_d = 0$ in the velocity loop for constant desired velocity), we have respectively a proportional action on the velocity error and an integral action on the force error. The two error dynamics will be

$$\dot{e}_v + k_1 e_v = 0, \qquad e_\lambda + k_2 \int e_\lambda = 0,$$

both exponentially converging to zero.

* * * * *