Robotics II

September 12, 2011

1. A 2R robot moving in the vertical plane and having link lengths $l_1 = l_2 = 1$ [m] is *self-balanced* with respect to gravity in the absence of a payload (see Fig. 1). Provide the mechanical conditions under which the gravity term in the dynamics of this robot is identically zero for any configuration $\boldsymbol{q} = (q_1, q_2)$.

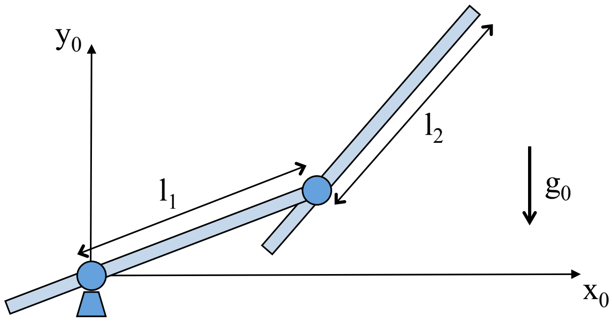

Figure 1: A 2R robot

2. For a regulation task to a desired $\boldsymbol{q}_d$, the following control law (PD + constant gravity compensation)

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d), \qquad \boldsymbol{K}_P > \boldsymbol{0},\ \boldsymbol{K}_D > \boldsymbol{0}, \tag{1}$$

is applied to this robot, now in the presence of a point-wise payload of known mass M located at the end of the second link.

a) Derive the correct expression of the gravity term $\boldsymbol{g}(\boldsymbol{q}_d)$ in (1).

b) Given the fixed choice of PD gain matrices

$$\boldsymbol{K}_P = \mathrm{diag}\{100, 100\}, \qquad \boldsymbol{K}_D = \mathrm{diag}\{25, 25\}, \tag{2}$$

provide an *upper bound* for the value of M such that the control law (1), with the gains chosen as in (2), certainly guarantees global asymptotic stability of the equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$. It is suggested to perform an approximate (conservative) analysis.

[120 minutes; open books]

Solution

September 12, 2011

In the absence of a payload, the potential energy U due to gravity is

$$U = U_1 + U_2, \quad U_1 = m_1 g_0 \, d_1 \sin q_1, \quad U_2 = m_2 g_0 \left(l_1 \sin q_1 + d_2 \sin(q_1 + q_2) \right),$$

where m_1 and m_2 are the masses of the two links, $g_0 = 9.81$ [m/s^2] is the gravity acceleration, and d_1 and d_2 are the (oriented) distances of the the center of mass of link 1 and, respectively, of link 2 from the associated joint axes (d_1 and d_2 can be positive, zero, or negative). The gravity term is then

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} (m_1 d_1 + m_2 l_1) g_0 \cos q_1 + m_2 d_2 g_0 \cos(q_1 + q_2) \\ m_2 d_2 g_0 \cos(q_1 + q_2) \end{pmatrix},$$

from which the requested mechanical conditions are:

$$\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0} \quad \Longleftrightarrow \quad \begin{cases} d_2 = 0, \\ d_1 = -\dfrac{m_2}{m_1} \, l_1 < 0. \end{cases}$$

The center of mass of the second link is then on the axis of joint 2, while the center of mass of the first link is located 'in opposition' to the second link 2 with respect to the joint 2 axis so that the center of gravity of the two masses of the first and second link lies always on the axis of joint 1. In these conditions, it follows that $U \equiv 0$ (or constant).

In the presence of a payload, the additional potential energy U_M due to the point-wise mass M at the end-effector is

$$U_M = M g_0 \left(l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \right),$$

and so the gravity term in the robot dynamics becomes

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_M}{\partial \boldsymbol{q}} \right)^T = M g_0 \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) \end{pmatrix}.$$

Evaluating this at $\boldsymbol{q} = \boldsymbol{q}_d$ yields the term $\boldsymbol{g}(q_d)$ in the control law (1).

For part 2.b, one relies on the property of boundedness for the gradient of the gravity term for all configurations of the robot, i.e.,

$$\left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \leq \alpha, \qquad \forall \boldsymbol{q},$$

where α is a suitable (large enough) positive constant, and the norm of the matrix $\boldsymbol{A} = \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}$ is defined[1] as the square root of the largest eigenvalue of the (symmetric and positive semi-definite) matrix $\boldsymbol{A}^T \boldsymbol{A}$:

$$\|\boldsymbol{A}\| = \sqrt{\lambda_{\max} \left(\boldsymbol{A}^T \boldsymbol{A} \right)}.$$

[1] This matrix norm is the one induced by (or, naturally associated with) the standard Euclidean norm of vectors: $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$.

Indeed, both the configuration-dependent norm of matrix $\boldsymbol{A}$ and its constant upper bound α will be functions of (in particular, increase with) the mass M. In order to guarantee that the control law (1) globally asymptotically stabilizes any desired equilibrium state of the robot, with the PD gains chosen as in (2), it will be sufficient to have

$$\alpha < \boldsymbol{K}_{P,\min} = 100. \tag{3}$$

We compute then

$$\boldsymbol{A} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = M g_0 \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1+q_2) & -l_2 \sin(q_1+q_2) \\ -l_2 \sin(q_1+q_2) & -l_2 \sin(q_1+q_2) \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{A}^T\boldsymbol{A} = {} & (M g_0)^2 \cdot \\ & \begin{pmatrix} (l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 + (l_2 \sin(q_1+q_2))^2 & (l_1 \sin q_1 + l_2 \sin(q_1+q_2))\, l_2 \sin(q_1+q_2) \\ (l_1 \sin q_1 + l_2 \sin(q_1+q_2))\, l_2 \sin(q_1+q_2) & 2\,(l_2 \sin(q_1+q_2))^2 \end{pmatrix}. \end{aligned}$$

The two (real and non-negative) eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are the roots of

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T\boldsymbol{A}\right) = \lambda^2 - b(\boldsymbol{q}, M)\lambda + c(\boldsymbol{q}, M) = 0,$$

with

$$\begin{aligned} b(\boldsymbol{q}, M) &= (M g_0)^4 \left((l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 + 3\,(l_2 \sin(q_1+q_2))^2 \right) \geq 0, \\ c(\boldsymbol{q}, M) &= (M g_0)^4 \left[2\,(l_2 \sin(q_1+q_2))^4 + (l_2 \sin(q_1+q_2))^2\,(l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 \right] \geq 0, \end{aligned}$$

and where the inequalities on the right hold for all $\boldsymbol{q}$ and all positive values of M. For a given $M > 0$ and $\boldsymbol{q}$, the largest of the two eigenvalues is written compactly as

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q}, M)\boldsymbol{A}(\boldsymbol{q}, M)\right) = \frac{b(\boldsymbol{q}, M) + \sqrt{b^2(\boldsymbol{q}, M) - 4\,c(\boldsymbol{q}, M)}}{2}.$$

From the non-negativity of both b and c, an upper bound to this expression is obtained by simply neglecting c, so that

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q}, M)\boldsymbol{A}(\boldsymbol{q}, M)\right) \leq b(\boldsymbol{q}, M).$$

The maximum value for $b(\boldsymbol{q}, M)$ over all possible $\boldsymbol{q}$ is obtained when simultaneously

$$\sin q_1 = 1 \quad \text{AND} \quad \sin(q_1+q_2) = 1 \qquad (\text{e.g., for } q_1 = \pi/2, q_2 = 0).$$

Plugging the numerical values of the link lengths, $l_1 = l_2 = 1$, and setting $g_0 = 9.81$, we can set for the constant α to be used as an upper bound to $\|\partial \boldsymbol{g}/\partial \boldsymbol{q}\|$

$$\alpha^2 = b(\boldsymbol{q}, M)\big|_{q_1=\pi/2, q_2=0} = 7 \cdot (9.81)^4 \cdot M^4,$$

and thus

$$\alpha = \sqrt{7} \cdot (9.81)^2 \cdot M^2.$$

Therefore, the inequality (3) is satisfied for

$$M < \frac{10}{9.81}\,\frac{1}{\sqrt[4]{7}} \simeq 0.6267\,[\text{kg}].$$

* * * * *