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## ***Robotics 2***

# **Adaptive Trajectory Control**

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# Motivation and approach

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- need of adaptation in robot motion control laws
  - large uncertainty on the robot dynamic parameters
  - poor knowledge of the inertial payload
- characteristics of **direct** adaptive control
  - direct aim is to bring to zero the state trajectory error, i.e., position and velocity errors
  - no need to estimate on-line the true values of the dynamic coefficients of the robot (as opposed to **indirect** adaptive control)
- main tool and methodology
  - **linear parametrization** of robot dynamics
  - **nonlinear** control law of the **dynamic** type (the controller has its own 'states')



# Summary of robot parameters

- parameters assumed to be **known**
  - kinematic description based, e.g., on Denavit–Hartenberg parameters ( $\{\alpha_i, d_i, a_i, i = 1, \dots, N\}$  in case of all revolute joints), including link lengths (**kinematic calibration**)
- **uncertain** parameters that can be **identified** off-line
  - masses  $m_i$ , positions  $r_{ci}$  of CoMs, and inertia matrices  $I_i$  of each link, appearing in combinations (**dynamic coefficients**)  $\Rightarrow p \ll 10 \times N$
- parameters that are **(slowly) varying** during operation
  - viscous  $F_{Vi}$ , dry  $F_{Di}$ , and stiction  $F_{Si}$  friction at each joint  $\Rightarrow 1 \div 3 \times N$
- **unknown** and abruptly changing parameters
  - mass, CoM, inertia matrix of the **payload** (w.r.t. the tool center point)



when a payload is firmly **attached** to the robot E-E, only the 10 parameters of the last link are modified, influencing however most part of the robot dynamics



# Goal of adaptive control

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- given a twice-differentiable desired joint trajectory  $q_d(t)$ 
  - with known desired velocity  $\dot{q}_d(t)$  and acceleration  $\ddot{q}_d(t)$
  - possibly obtained by kinematic inversion + joint interpolation
- execute this trajectory under large dynamic uncertainties
  - with a trajectory tracking error vanishing **asymptotically**
$$e = q_d - q \rightarrow 0 \quad \dot{e} = \dot{q}_d - \dot{q} \rightarrow 0$$
  - guaranteeing **global stability**, no matter how far are the initial estimates of the unknown/uncertain parameters from their true values and how large is the initial trajectory error
- identification is **not** of particular concern: in general, the estimates of dynamic coefficients will not converge to the true ones!
- if this convergence is a specific extra requirement, then one should use (more complex) **indirect adaptive** schemes



# Linear parametrization

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$$M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) + F_V\dot{q} = u$$

- there exists always a ( $p$ -dimensional) **vector  $a$**  of **dynamic coefficients**, so that the robot model takes the **linear** form

$$Y(q, \dot{q}, \ddot{q}) a = u$$

- vector  **$a$**  contains only unknown or uncertain coefficients
- each component of  **$a$**  is in general a **combination** of the robot physical parameters (not necessarily all of them)
- the model **regression matrix  $Y$**  depends linearly on  $\ddot{q}$ , quadratically on  $\dot{q}$  (for the terms related to kinetic energy), and nonlinearly (trigonometrically) on  $q$



# Trajectory controllers based on model estimates

- inverse dynamics feedforward (**FFW**) + PD (**linear**) control

$$u = \underbrace{\hat{M}(q_d)\ddot{q}_d + \hat{S}(q_d, \dot{q}_d)\dot{q}_d + \hat{g}(q_d) + \hat{F}_V\dot{q}_d}_{\hat{u}_d} + K_P e + K_D \dot{e}$$

- (**nonlinear**) control based on feedback linearization (**FBL**)

$$u = \hat{M}(q)(\ddot{q}_d + K_P e + K_D \dot{e}) + \hat{S}(q, \dot{q})\dot{q} + \hat{g}(q) + \hat{F}_V\dot{q}$$

$$\boxed{\hat{M}, \hat{S}, \hat{g}, \hat{F}_V \quad \Longleftrightarrow \quad \text{estimate } \hat{a}}$$

- approximate estimates of dynamic coefficients may lead to **instability** with **FBL** due to temporary 'non-positive' PD gains (e.g.,  $\hat{M}(q)K_P < 0!$ )
- **not easy** to turn these laws in **adaptive** schemes: inertia inversion/use of acceleration (FBL); bounds on PD gains (FFW)



## A control law more easily made 'adaptive'

- nonlinear trajectory tracking control (without cancellations) having global asymptotic stabilization properties

$$u = \hat{M}(q)\ddot{q}_d + \hat{S}(q, \dot{q})\dot{q}_d + \hat{g}(q) + \hat{F}_V\dot{q}_d + K_P e + K_D \dot{e}$$

- a natural **adaptive version** would require ...

$\dot{\hat{a}}$  = designing a suitable **update law**  
(in continuous time)

- **without extra assumptions**, it can be shown that joint velocities become eventually "clamped" to those of the **desired** trajectory (zero **velocity** error), but a residual **position** error may be left
- idea: **on-line modification** with a **reference velocity**

$$\dot{q}_d \rightarrow \boxed{\dot{q}_r = \dot{q}_d + \Lambda(q_d - q)} \quad \Lambda > 0$$

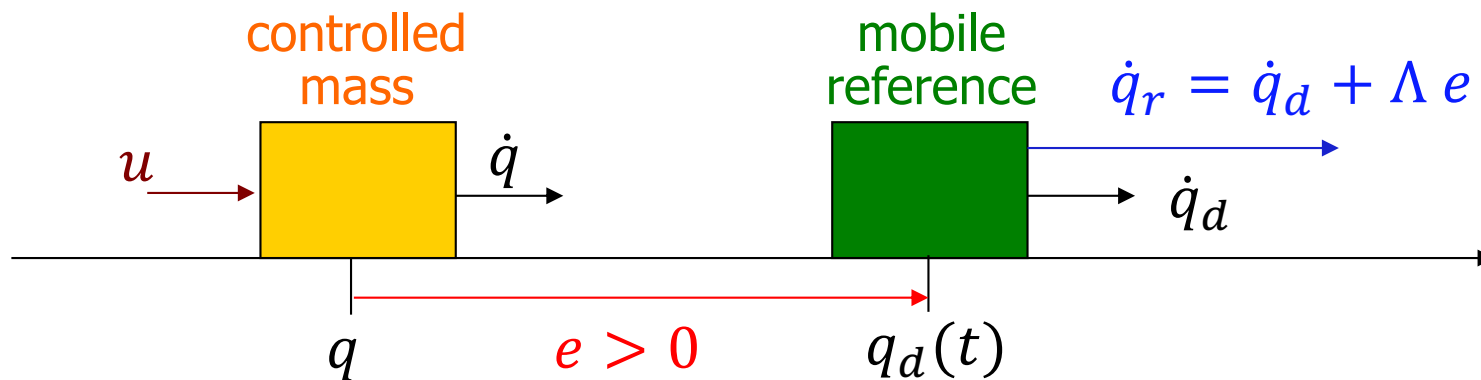
typically,  $\Lambda = K_D^{-1} K_P$  (all matrices will be chosen **diagonal**)



# Intuitive interpretation of $\dot{q}_r$

## ■ elementary case

- a mass 'lagging behind' a mobile reference ( $e > 0$ ) at constant speed



➡ 'enhanced' velocity error  $s = \dot{q}_r - \dot{q} > \dot{q}_d - \dot{q} = \dot{e}$

$$u = K_D s = K_D (\dot{q}_r - \dot{q}) = K_D (\dot{q}_d + \Lambda e - \dot{q}) = K_D \dot{e} + \underbrace{K_D \Lambda}_{K_P} e$$

- a mass 'leading in front' of its mobile reference ( $e < 0$ )

➡ in a symmetric way, a 'reduced' velocity error will appear ( $s < \dot{e}$ )





# Adaptive control law design

- substituting  $\dot{q}_r = \dot{q}_d + \Lambda e$ ,  $\ddot{q}_r = \ddot{q}_d + \Lambda \dot{e}$  in the previous nonlinear controller for trajectory tracking

$$\begin{aligned} u &= \hat{M}(q)\ddot{q}_r + \hat{S}(q, \dot{q})\dot{q}_r + \hat{g}(q) + \hat{F}_V\dot{q}_r + K_P e + K_D \dot{e} \\ &= Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\hat{a} + K_P e + K_D \dot{e} \end{aligned}$$

dynamic parametrization of  
control law using current estimates  
(note here the 4 arguments in  $Y(\cdot)$ !)

PD stabilization  
(diagonal matrices,  $>0$ )

- update law for the estimates of the dynamic coefficients ( $\hat{a}$  becomes the  $p$ -dimensional state of the dynamic controller)

$$\dot{\hat{a}} = \Gamma Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)(\dot{q}_r - \dot{q})$$

$\Gamma > 0$  (diagonal)

estimation gains  
(variation rate of estimates)

'modified' velocity error



# Asymptotic stability of trajectory error

## Theorem

The introduced adaptive controller makes the **tracking error** along the desired trajectory **globally asymptotically stable**

$$e = q_d - q \rightarrow 0, \dot{e} = \dot{q}_d - \dot{q} \rightarrow 0$$

## Proof

- a **Lyapunov candidate** for the closed-loop system (robot + dynamic controller) is given by

$$V = \frac{1}{2} s^T M(q) s + \frac{1}{2} e^T R e + \frac{1}{2} \tilde{a}^T \Gamma^{-1} \tilde{a} \geq 0$$

$$s = \dot{q}_r - \dot{q} (= \dot{e} + \Lambda e)$$

modified velocity error

$$R > 0$$

constant matrix  
(to be specified later)

$$\tilde{a} = a - \hat{a}$$

error in parametric  
estimation

$$V = 0 \iff \hat{a} = a, \quad q = q_d, \quad s = 0 \quad (\Rightarrow \dot{q} = \dot{q}_d)$$



## Proof (cont)

- the **time derivative** of  $V$  is

$$\dot{V} = \frac{1}{2} s^T \dot{M}(q) s + s^T \underbrace{M(q) \dot{s}} + e^T R \dot{e} - \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}}$$

since  $\dot{\tilde{a}} = -\dot{\hat{a}}$  ( $\dot{a} = 0$ )

- the **closed-loop** dynamics is given by

$$\begin{aligned} M(q) \ddot{q} + S(q, \dot{q}) \dot{q} + g(q) + F_V \dot{q} &= \\ &= \hat{M}(q) \ddot{q}_r + \hat{S}(q, \dot{q}) \dot{q}_r + \hat{g}(q) + \hat{F}_V \dot{q}_r + K_P e + K_D \dot{e} \end{aligned}$$

**subtracting** the two sides **from**  $M(q) \ddot{q}_r + S(q, \dot{q}) \dot{q}_r + g(q) + F_V \dot{q}_r$  leads to

$$\begin{aligned} \underbrace{M(q) \dot{s}} + (S(q, \dot{q}) + F_V) s &= \\ &= \tilde{M}(q) \ddot{q}_r + \tilde{S}(q, \dot{q}) \dot{q}_r + \tilde{g}(q) + \tilde{F}_V \dot{q}_r - K_P e - K_D \dot{e} \end{aligned}$$

with  $\tilde{M} = M - \hat{M}$ ,  $\tilde{S} = S - \hat{S}$ ,  $\tilde{g} = g - \hat{g}$ ,  $\tilde{F}_V = F_V - \hat{F}_V$



## Proof (cont)

- from the property of **linearity in the dynamic coefficients**, it follows

$$\boxed{M(q)\dot{s}} + (S(q, \dot{q}) + F_V)s = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\tilde{a} - K_P e - K_D \dot{e}$$

- substituting in  $\dot{V}$** , together with  $\hat{a} = \Gamma Y^T s$ , and using the skew-symmetry of matrix  $\dot{M} - 2S$  we obtain

$$\begin{aligned}\dot{V} &= \frac{1}{2} s^T [\dot{M}(q) - 2S(q, \dot{q})] s - s^T F_V s + \cancel{s^T Y \tilde{a}} \\ &\quad - s^T (K_P e + K_D \dot{e}) + e^T R \dot{e} - \cancel{\tilde{a}^T Y^T s} \\ &= -s^T F_V s - s^T (K_P e + K_D \dot{e}) + e^T R \dot{e}\end{aligned}$$

- replacing  $s = \dot{e} + \Lambda e$**  and being  $F_V = F_V^T$  (diagonal)

$$\dot{V} = -e^T (\Lambda^T F_V \Lambda + \Lambda^T K_P) e$$

a complete  
quadratic form  
in  $e, \dot{e}$ !

$$-e^T (2\Lambda^T F_V + \Lambda^T K_D + K_P - \boxed{R}) \dot{e} - \dot{e}^T (F_V + K_D) \dot{e}$$



## Proof (end)

- defining now (all matrices are **diagonal!**)

$$\Lambda = K_D^{-1} K_P > 0 \quad \textcircled{R} = 2K_P (I + K_D^{-1} F_V) > 0$$

cancels the cross-term in  $e^T (\dots) \dot{e}$  and leads to

$$\begin{aligned} \dot{V} &= -e^T \Lambda^T (F_V + K_D) \Lambda e - \dot{e}^T (F_V + K_D) \dot{e} \\ &= -e^T K_P K_D^{-1} (F_V + K_D) K_D^{-1} K_P e - \dot{e}^T (F_V + K_D) \dot{e} \leq 0 \end{aligned}$$

and thus

$$\dot{V} = 0 \iff e = \dot{e} = 0$$

the thesis follows from **Barbalat lemma** + LaSalle theorem

⇒ see slide 10  
in block 8

the maximal invariant set of states  $\subseteq \{\dot{V} = 0\}$  has **zero trajectory error** ( $e = \dot{e} = 0$ ) and **a constant value** for  $\hat{a}$ , not necessarily the true one ( $\tilde{a} \neq 0$ )

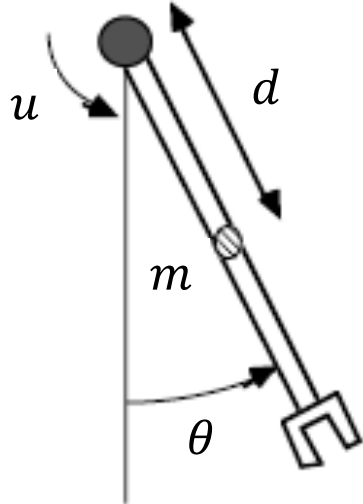


# Remarks

- the adaptive controller **does not require** the inverse of the inertia matrix (true or estimated), nor the actual robot acceleration (only the desired acceleration), nor further lower bounds on  $K_P > 0, K_D > 0$
- adaptation can also be used **only for a subset** of dynamic coefficients, with the others being known ( $Y a = Y_{adapt} \hat{a}_{adapt} + Y_{known} a_{known}$ )
- if the desired trajectory  $q_d(t)$  is **persistently exciting**, then also the estimates of the dynamic coefficients converge to their true values
- **condition** of persistent excitation
  - for **linear** systems: # of frequency components in the desired trajectory should be at least **twice as large** as # of unknown coefficients
  - for **nonlinear** systems: the condition can be checked only **a posteriori** (a squared motion integral should always be positively bounded from below)
- **minimal** linear parametrization of robot dynamics **not strictly necessary!** (but dynamic **parameters**  $\Rightarrow$  **coefficients** is useful to reduce control complexity)
- **non-adaptive version** using accurate estimates is a global tracking controller with static feedback based on the **passivity** of robot dynamics



# Case study: Single-link under gravity



model  $I\ddot{\theta} + mg_0d \sin \theta + f_V\dot{\theta} = u$  (with friction)

linear parametrization  $g_0$  may also be moved out ...

$$Y(\theta, \dot{\theta}, \ddot{\theta})a = \begin{bmatrix} \ddot{\theta} & \sin \theta & \dot{\theta} \end{bmatrix} \begin{bmatrix} I \\ mg_0d \\ f_V \end{bmatrix} = u$$

adaptive controller

$$\begin{aligned} e &= \theta_d - \theta \\ \dot{\theta}_r &= \dot{\theta}_d + \frac{k_P}{k_D} e \\ \gamma_i &> 0, i = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} u &= \hat{I} \ddot{\theta}_r + \widehat{mg_0d} \sin \theta + \hat{f}_V \dot{\theta}_r + k_P e + k_D \dot{e} \\ \hat{a} &= \begin{pmatrix} \hat{I} \\ \widehat{mg_0d} \\ \hat{f}_V \end{pmatrix} = \begin{pmatrix} \gamma_1 \ddot{\theta}_r \\ \gamma_2 \sin \theta \\ \gamma_3 \dot{\theta}_r \end{pmatrix} (\dot{\theta}_r - \dot{\theta}) \end{aligned}$$



# Simulation data

- **real** dynamic coefficients

$$I = 7.5, \quad mg_0d = 6, \quad f_V = 1$$

- **initial** estimates

$$\hat{I} = 5, \quad \widehat{mg_0d} = 5, \quad \hat{f}_V = 2$$

- control parameters

$$k_P = 25, \quad k_D = 10, \quad \gamma_i = 5, \quad i = 1, 2, 3$$

- **motion tests** with robot starting at  $\theta(0) = 0, \dot{\theta}(0) = 0$

- **first trajectory**

$$\theta_d(t) = -\sin t$$

- **second trajectory**

$\ddot{\theta}_d(t)$  = (periodic) bang-bang acceleration profile with  
 $A = 1 \text{ rad/s}^2, \omega = 1 \text{ rad/s}$

Note: same test trajectories  
used also for robust control

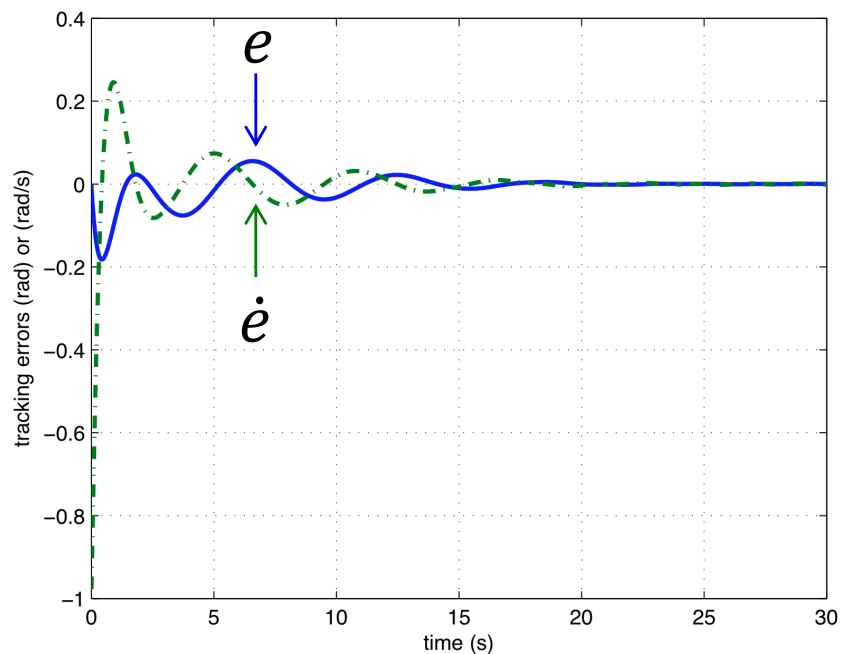


# Results

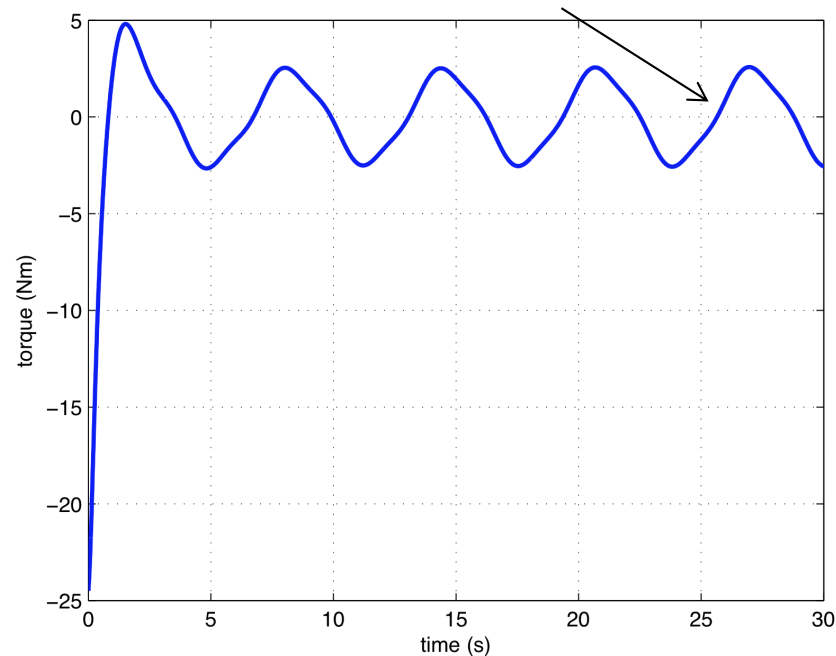
## first trajectory



no sinusoidal regime at steady state  
(due to nonlinear system dynamics!)



position and velocity errors

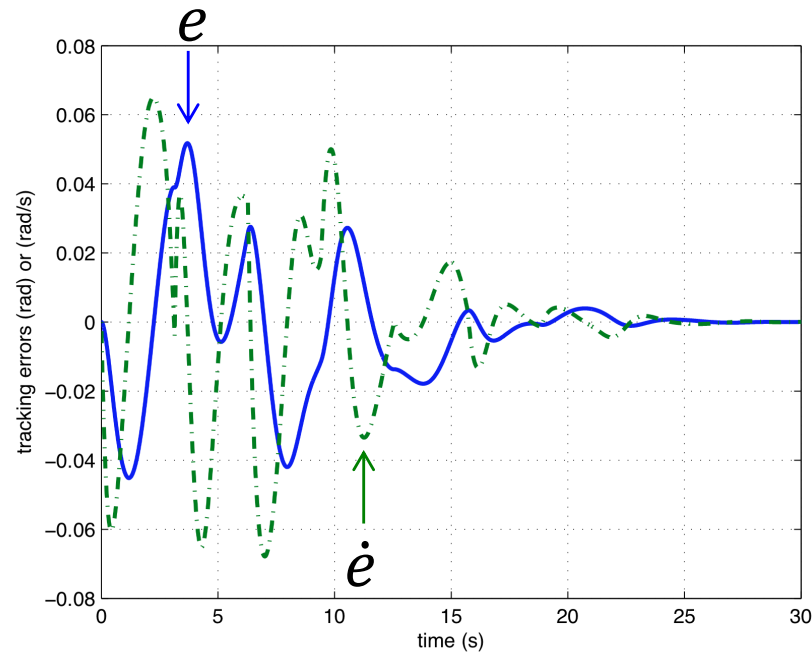


control torque

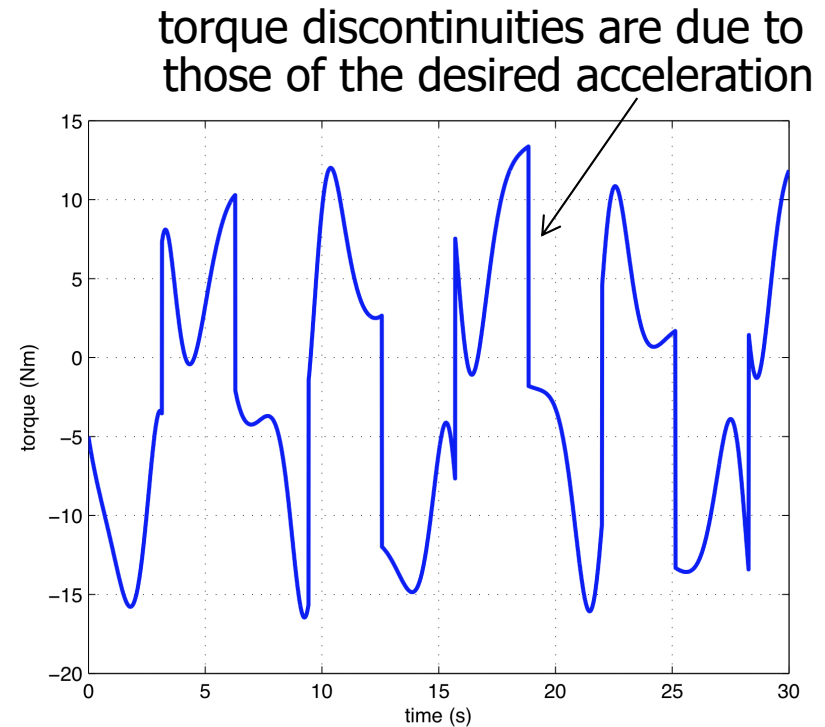
$$\theta_d(t) = -\sin t$$

# Results

## second trajectory



position and velocity errors

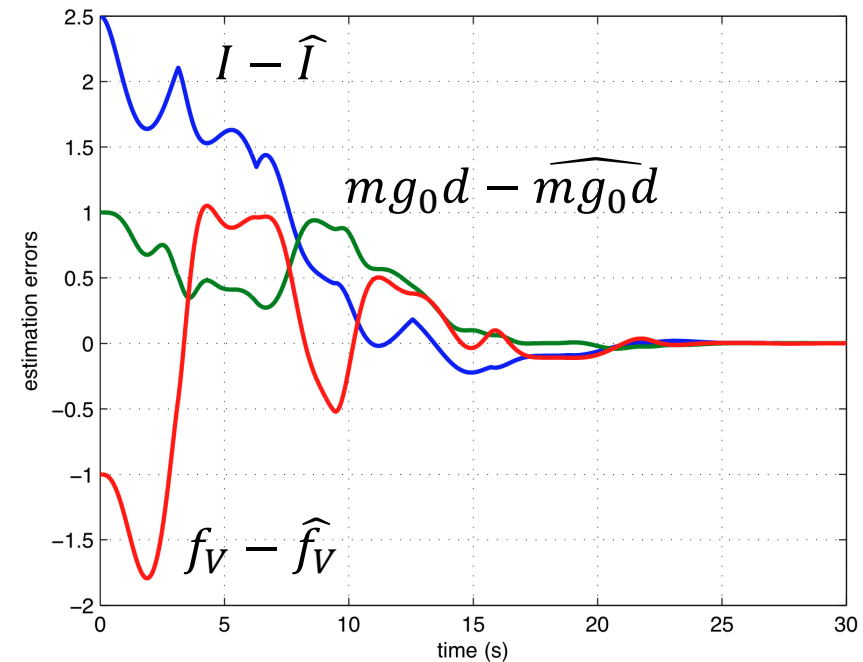
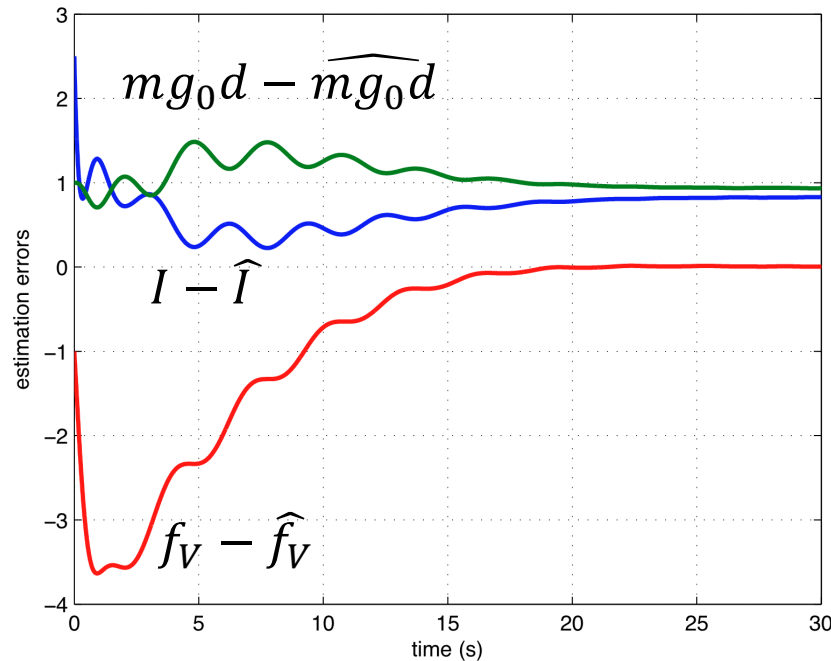


control torque

$\ddot{\theta}_d(t)$  = (periodic) bang-bang acceleration profile



# Estimates of dynamic coefficients



errors  $\tilde{a} = a - \hat{a}$

first trajectory

only the estimate of the viscous friction coefficient converges to the true value

second trajectory

all three estimates of the dynamic coefficients converge to their true values



# A special case: Adaptive regulation

- adaptation in case  $q_d$  is **constant**
- **no special simplifications** for the presented adaptive control law (designed for the general tracking case...)

$$u = \hat{M}(q)\ddot{q}_r + \hat{S}(q, \dot{q})\dot{q}_r + \hat{g}(q) + \hat{F}_v\dot{q}_r + K_P e + K_D \dot{e}$$

$$\dot{\hat{a}} = \Gamma Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)(\dot{q}_r - \dot{q})$$

since  $\dot{q}_r = \Lambda(q_d - q)$  and  $\ddot{q}_r = -\Lambda\dot{q}$  **do not** vanish!

- a **different** case would be the availability of an adaptive version of the trajectory tracking controller

$$u = \hat{M}(q)\ddot{q}_d + \hat{S}(q, \dot{q})\dot{q}_d + \hat{g}(q) + \hat{F}_v\dot{q}_d + K_P e + K_D \dot{e}$$

since, when  $q_d$  collapses to a constant, **only adaptation of gravity term** would be left over (which is also what one would naturally expect...)



# An efficient adaptive regulator

- use a linear parametrization of the **gravity term** only

$$g(q) = G(q)a_g$$

with a  **$p_g$ -dimensional** vector  **$a_g$**

- an adaptive regulator yielding **global asymptotic stability** of the equilibrium state  $(q_d, 0)$  is provided by

$$u = G(q)\hat{a}_g + K_P(q_d - q) - K_D\dot{q}$$

$$\dot{\hat{a}}_g = \gamma G^T(q) \left( \frac{2e}{1 + 2\|e\|^2} - \beta\dot{q} \right), \quad \gamma > 0$$

where  $e = q_d - q$ ,  $K_P > 0$ ,  $K_D > 0$  (symmetric), and  $\beta > 0$  is chosen sufficiently **large**

(see paper by P. Tomei, IEEE TRA, 1991; available as extra material on the course web)



# An adaptive regulator

## Sketch of asymptotic stability analysis

- use the function

$$V = \frac{\beta}{2} (\dot{q}^T M(q) \dot{q} + e^T K_P e) - \frac{2\dot{q}^T M(q) e}{1 + 2\|e\|^2} + \frac{1}{2} (\hat{a}_g - a_g)^T (\hat{a}_g - a_g)$$

- a sufficient condition for  $V$  to be a **Lyapunov candidate** is that

$$\beta > \frac{2M_M}{\sqrt{M_m K_{P,m}}}$$

- a sufficient condition which guarantees **also** that

$$\dot{V} = \dots \leq -a\|e\|^2 - b\|\dot{q}\|^2 \leq 0, \quad a > 0, b > 0$$

is

$$\beta > \max \left\{ \frac{2M_M}{\sqrt{M_m K_{P,m}}}, \frac{1}{K_{D,m}} \left( \frac{K_{D,m}^2}{2K_{P,m}} + 4M_M + \frac{\alpha_s}{\sqrt{2}} \right) \right\} \quad \|S(q, \dot{q})\| \leq \alpha_s \dot{q}$$

**Note:** for any **symmetric, positive definite** matrix  $A$

$$A_M = \lambda_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)} = \|A\| \quad \text{and thus, e.g., } \frac{1}{2} \dot{q}^T M(q) \dot{q} \geq \frac{1}{2} M_m \|\dot{q}\|^2$$
$$A_m = \lambda_{\min}(A)$$