Robust Trajectory Control

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Problem formulation

- given the real robot, modeled by
  \[ B(q)\ddot{q} + n(q, \dot{q}) = u \]
- assuming an estimated feedback linearization control
  \[ u = \hat{B}(q)a + \hat{n}(q, \dot{q}) \]
- we would like to design \( a \) so as to obtain
  - asymptotic stability of the closed-loop system
  - the best possible trajectory tracking performance
- the linear feedback choice is not enough...
  \[ a = \ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q) \]
- questions:
  - which should be the conditions on the estimates?
  - can we guarantee stability/performance, based on known bounds on the uncertainties?
Closed-loop equations - 1

- under uncertain conditions (estimated ≠ real dynamic coefficients), feedback linearization is only approximate and the closed-loop equations are still nonlinear

\[
\ddot{q} = B^{-1}(q)(\hat{B}(q)a + \hat{n}(q, \dot{q}) - n(q, \dot{q})) \\
= a + (B^{-1}(q)\hat{B}(q) - I)a \\
+ B^{-1}(q)(\hat{n}(q, \dot{q}) - n(q, \dot{q})) \\
= a + E(q)a + B^{-1}(q)\Delta n(q, \dot{q}) \\
= a + \eta(a, q, \dot{q})
\]

where \( \eta \) depends on the amount of uncertainty

\[
E(q) = B^{-1}(q)\Delta B(q) = B^{-1}(q)(\hat{B}(q) - B(q)) \\
\Delta n(q, \dot{q}) = \hat{n}(q, \dot{q}) - n(q, \dot{q})
\]
Closed-loop equations

- closed-loop state equations are written as

\[ \dot{x} = A x + B (a + \eta) \]

\[
A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}
\]

- closed-loop error equations with respect to a desired $q_d(t)$ are rewritten as

\[
e_1 = x_1 - x_{1d} = q - q_d \quad e_2 = x_2 - x_{2d} = \dot{q} - \dot{q}_d
\]

\[
\dot{e}_1 = e_2 \\
\dot{e}_2 = \ddot{q} - \ddot{q}_d = a + \eta(a, e_1, e_2, q_d, \dot{q}_d) - \ddot{q}_d
\]

\[ \Rightarrow \dot{e} = A e + B (a + \eta - \ddot{q}_d) \]

Note that errors are defined here with opposite signs w.r.t. usual.
Solution approach

- add an external robust control term/loop
  - based on computable bounds on the uncertainties
- based on the theory of guaranteed stability for nonlinear uncertain system
- Lyapunov-based analysis
- a discontinuous control law will result
  - difficult to implement because of chattering effects
  - smoothed version with only uniformly ultimate boundedness (u.u.b. stability) of the tracking error
Working assumptions

1. bound on the desired trajectory
   \[ \sup_{t \geq 0} \|\ddot{q}_d\| < Q_{\text{max}} < \infty \]

2. bound on the estimate of the robot inertia matrix
   \[ \|E(q)\| = \|B^{-1}(q)\hat{B}(q) - I\| \leq \alpha < 1 \]
   with \( \alpha \geq 0 \), holding for all configurations \( q \)

3. bound on the estimate of nonlinear dynamic terms
   \[ \|\Delta n(q, \dot{q})\| \leq \phi(e, t) \]
   with a known function \( \phi \), bounded for all \( t \)
   - as a general rule, exploiting the model structure (e.g., its linear parameterization) may lead to more stringent bounds
Bound on the inertia matrix

- assumption 2. can always be satisfied, knowing some upper and lower bounds (that always exist due to the positive definiteness) on the inverse of the inertia matrix

\[ 0 < m \leq \| B^{-1}(q) \| \leq M < \infty \]

- it is then sufficient to choose as estimate

\[ \hat{B} = \frac{1}{c} I \quad \text{with} \quad c = \frac{M + m}{2} \]

- in fact, using the SVD factorization of the inverse inertia matrix, it can be shown that (see Appendix A)

\[ \| B^{-1} \hat{B} - I \| \leq \frac{M - m}{M + m} = \alpha < 1 \]
Control design – step 1

- linear control law with an added robust term
  \[ a = \ddot{q}_d - K_P e_1 - K_D e_2 + \Delta a \]
  where the PD gains are diagonal and positive matrices
- we obtain
  \[ \dot{e} = \bar{A} e + B (\Delta a + \bar{\eta}) \]
  being
  \[ \bar{A} = A - BK \quad K = [K_P \quad K_D] \]
  where \( \bar{A} \) has all eigenvalues with negative real part, and
  \[ \bar{\eta} = E (\ddot{q}_d - K e + \Delta a) + B^{-1} \Delta n \]
Control design – step 2

- (same) bound on nonlinear terms and added robust term

\[ \| \bar{\eta} \| < \rho(e, t) \quad \| \Delta a \| < \rho(e, t) \]

- we can use the previous data and implicitly define the bound \( \rho(e, t) \) from

\[
\| \bar{\eta} \| = \| E \Delta a + E(\dot{q}_d - K e) + B^{-1} \Delta n \| \\
\leq \alpha \rho(e, t) + \alpha(Q_{\text{max}} + \| K \| \cdot \| e \|) + M \phi(e, t) \\
=: \rho(e, t)
\]

yielding the well-defined (since \( 0 < \alpha < 1 \)), limited and possibly time-varying function

\[
\rho(e, t) = \frac{1}{1 - \alpha} \left[ \alpha(Q_{\text{max}} + \| K \| \cdot \| e \|) + M \phi(e, t) \right]
\]
Control design – step 3

- solve an associated (linear) Lyapunov equation, for any given symmetric $Q > 0$ matrix

$$\overline{A}^T P + P \overline{A} + Q = 0$$

finding the unique (symmetric) solution matrix $P > 0$

- finally, define the discontinuous robust term as

$$\Delta a = \begin{cases} 
-\rho(e, t) \frac{B^T Pe}{\|B^T Pe\|} & \text{if } \|B^T Pe\| \neq 0 \\
0 & \text{if } \|B^T Pe\| = 0
\end{cases}$$

that also satisfies, by its own structure, $\|\Delta a\| < \rho(e, t)$
Solving the Lyapunov equation

- in general, using `lyap` in Matlab (only once, in advance)
- closed-form solution in an interesting scalar case (one robot joint/link), to get a “feeling”...

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad K = [k_P \ k_D] \quad \bar{A} = A - BK = \begin{bmatrix} 0 & 1 \\ -k_P & -k_D \end{bmatrix} \quad k_P > 0, k_D > 0 \]

choose, e.g., \( Q = q \cdot I_{2 \times 2} > 0 \) \( \Rightarrow \) find \( P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0 \)

\[
\bar{A}^T P + P \bar{A} + q \cdot I = \\
= \begin{bmatrix} -2p_{12}k_P + q & p_{11} - p_{12}k_D - p_{22}k_P \\ p_{11} - p_{12}k_D - p_{22}k_P & 2(p_{12} - k_Dp_{22}) + q \end{bmatrix} = 0 \\
\]

\[
p_{12} = \frac{q}{2k_P} \quad p_{22} = \frac{q}{2} \left( \frac{1 + k_P}{k_P k_D} \right) > 0 \\
p_{11} = \frac{q}{2} \left( \frac{k_D}{k_P} + \frac{1 + k_P}{k_D} \right) > 0 \\
\Rightarrow p_{11}p_{22} - p_{12}^2 > 0
\]

so that (also in the n-dof case) \( B^T P e = \) block \( \left\{ \frac{q}{2} \left( \frac{e_1}{k_P} + \frac{1 + k_P}{k_P} \frac{e_2}{k_D} \right) \right\} \)
Stability analysis

**Theorem 1**

Defining \( \dot{V}(e) = e^T P e \), the presented robust control law with the discontinuous term is such that \( \dot{V}(e) < 0 \) along the trajectories of the closed-loop error system.

**Proof**

\[
\dot{V}(e) = e^T P e + e^T P \dot{e} \\
= e^T (\bar{A}^T P + P \bar{A}) e + 2e^T P B (\Delta a + \bar{\eta}) \\
= -e^T Q e + 2e^T P B (\Delta a + \bar{\eta}) \\
= -e^T Q e + 2w^T (\Delta a + \bar{\eta})
\]

if \( w = 0 \) \( \Rightarrow \) \( \dot{V} = -e^T Q e < 0 \)

if \( w \neq 0 \) \( \Rightarrow \) \( \Delta a = -\rho w / ||w|| \) \( \Rightarrow \)

\[
\begin{align*}
& w^T (-\rho \frac{w}{||w||} + \bar{\eta}) = -\rho \frac{w^T w}{||w||} + w^T \bar{\eta} \\
& \leq -\rho ||w|| + ||w|| \cdot ||\bar{\eta}|| \\
& = ||w||(-\rho + ||\bar{\eta}||) \leq 0
\end{align*}
\]

\( \Rightarrow \dot{V} < 0 \)

note: because of the discontinuity we cannot directly conclude on the (global) asymptotic stability of \( e=0 \)
A smoother robust controller

- for any given (small) $\epsilon > 0$, define the continuous robust term as

$$\Delta a = \begin{cases} 
-\rho(e, t) \frac{B^T P e}{\|B^T P e\|} & \text{if } \|B^T P e\| \geq \epsilon \\
-\rho(e, t) \frac{e}{\epsilon} B^T P e & \text{if } \|B^T P e\| < \epsilon
\end{cases}$$

**Theorem 2**

With the above continuous robust control law, any solution $e(t)$, with $e(0) = e_0$, of the closed-loop error system is uniformly ultimately bounded with respect to a suitable set $S$ (a neighborhood of the origin)

**Proof** in Appendix B
Case study: Single-link under gravity

Model

\[ I \ddot{\theta} + mgd \sin \theta = u \]  
(no friction)

Error equations

\[ \dot{e}_1 = e_2 \]

\[ \dot{e}_2 = \frac{1}{I} u - \frac{mgd}{I} \sin \theta - \ddot{\theta}_d \]

\[ = \frac{1}{I} [\hat{I} (a + \Delta a) + mgd \sin \theta] - \frac{mgd}{I} \sin \theta - \ddot{\theta}_d \]

\[ e_1 = \theta - \theta_d \]

\[ e_2 = \dot{\theta} - \dot{\theta}_d \]

Known bounds for control design

\[ 5 \leq I \leq 10 \quad 5 \leq mgd \leq 7 \]
Calculations for robust control

% real robot
I=5; mgd=7;
% initial robot state
th0=0; thp0=0;
% range of uncertainties
I_min=5; I_max=10;
mgd_min=5; mgd_max=7;
% linear tracking stabilizer gains
kp=25; kd=10; % two poles in -5

% robust control part
% Lyapunov matrix P and b^T P term
A=[0 1; -kp -kd];
q=1; Q=q*eye(2);
P=lyap(A',Q); % solve A'*P+P*A+Q=0
b=[0 1];
bP=b*P; % = [0.02 0.052]

% bounding dynamic terms
% inertia
m=1/I_max; M=1/I_min;
c=(M+m)/2;
alpha=(M-m)/(M+m);
Ihat=1/c; % = 6.6667
% nonlinear terms (only gravity)
Mphi=M*(mgd_max-mgd_min);
mgdhat=5;
% overall bounding
rho0=Mphi/(1-alpha) % = 0.6
rho1=alpha/(1-alpha) % = 0.5
% smoothed version
epsilon=5*10^-4;

red values are used in Simulink
Results
first trajectory – feedback linearization, no robust loop

\[ \theta_d(t) = -\sin t \]
\[ \theta(0) = 0, \quad \dot{\theta}(0) = 0 \]

non-zero initial error on velocity
Results
first trajectory – discontinuous robust control

\[ \theta_d(t) = - \sin t \]

position and velocity errors are largely reduced, but control chattering at high frequency (when error is close to zero)
Results
first trajectory – smoothed robust control

\[ \theta_d(t) = -\sin t \]

position and velocity errors are similarly reduced, without control chattering

(using here \( \epsilon > 0 \))
Results
second trajectory – fbk linearization, no robust loop

bang-bang acceleration profile
at 1 rad/s frequency and
with $Q_{\text{max}} = 1 \text{ rad/s}^2$

zero initial tracking error
(matching state conditions)
Results
second trajectory – discontinuous robust control

position error

control torques

bang-bang acceleration profile

position and velocity errors again largely reduced, but control chattering and larger effort
Results
second trajectory – smoothed robust control

position error
control torques

velocity error

position and velocity errors are further reduced, without control chattering and same control effort as without robustifying term (using here $\epsilon > 0$)
Appendix A

Proof of bounds on the inertia matrix

- the SVD factorization of the (symmetric) inverse inertia matrix is

\[ B^{-1} = U \Sigma^{-1} U^T = U \text{diag}\left\{ \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n} \right\} U^T \]

so that, with the choice made for its estimate, it follows that

\[ \| B^{-1} \hat{B} - I \| = \| U \Sigma^{-1} U^T \hat{B} - I \| \]

\[ = \| U \Sigma^{-1} U^T \cdot \left( \frac{1}{\hat{c}} I \right) - I \| \]

\[ = \| U (\Sigma^{-1} \cdot \frac{1}{\hat{c}} - I) U^T \| \]

\[ \leq \| U \| \cdot \| \Sigma^{-1} \cdot \frac{1}{\hat{c}} - I \| \cdot \| U^T \| \]

\[ = \| \Sigma^{-1} \cdot \frac{1}{\hat{c}} - I \| \leq \frac{M}{\hat{c}} - 1 \]

\[ = \frac{M - \hat{c}}{\hat{c}} = \frac{M - m}{M + m} = \alpha < 1 \]
Appendix B
Proof of Theorem 2

- setting $w = B^T P e$, note that for the robust term in the control law it is

$$\|\Delta a\| = \begin{cases} \rho & \text{if } \|w\| \geq \epsilon \\ (\rho/\epsilon)\|w\| & \text{if } \|w\| < \epsilon \end{cases}$$

- defining as before $V(e) = e^T P e$, we have

$$\dot{V}(e) = -e^T Q e + 2w^T (\Delta a + \tilde{\eta})$$

$$\leq -e^T Q e + 2w^T \left(\Delta a + \rho \frac{w}{\|w\|}\right)$$

having used the chain of inequalities

$$w^T \tilde{\eta} \leq \|w^T \tilde{\eta}\| \leq \|w\| \cdot \|\tilde{\eta}\| \leq \|w\| \rho = w^T \rho \frac{w}{\|w\|}$$

- if $\|w\| \geq \epsilon$, the rest of the proof is the same as in Theorem 1
Appendix B

Proof of Theorem 2 (cont)

- if $\|w\| < \epsilon$, the second term in the derivative of $V$ is
  $$2w^T \left( -\frac{\rho}{\epsilon} w + \rho \frac{w}{\|w\|} \right) = 2\rho \left( -\frac{\|w\|^2}{\epsilon} + \|w\| \right)$$
  with a maximum value $\rho \frac{\epsilon}{2}$ attained for $\|w\| = \frac{\epsilon}{2}$

- therefore, it is
  $$\dot{V}(e) \leq -e^T Q e + \rho \frac{\epsilon}{2} \leq -\lambda_{\min}(Q) \|e\|^2 + \rho \frac{\epsilon}{2} < 0$$
  provided that
  $$\|e\| \geq \left[ \frac{\rho \epsilon}{2 \lambda_{\min}(Q)} \right]^{1/2} := \omega$$

- if $S$ is the smallest level set of $V = e^T P e$ (an ellipsoid) containing the hyper-sphere of radius $\omega$, then
  $$e \notin S \implies \dot{V}(e) < 0$$
  and u.u.b. is obtained for $S$
  (an upper bound for the time needed to reach $S$ can be given)