



Robotics 2

Regulation in the Joint Space

(with an introduction to stability)

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Equilibrium states of a robot

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

$$\begin{aligned} \text{blue arrow} \quad \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1, x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u \\ \text{red arrow} \quad \dot{x} &= 0 \quad = f(x) + G(x_1)u \end{aligned}$$

$$x_e \text{ unforced equilibrium} \quad (u = 0) \quad \Rightarrow \quad f(x_e) = 0 \quad \Rightarrow \quad \begin{cases} x_{e2} = 0 \\ g(x_{e1}) = 0 \end{cases}$$

$$x_e \text{ forced equilibrium} \quad (u = u(x)) \quad \Rightarrow \quad f(x_e) + G(x_{e1})u(x_e) = 0 \quad \Rightarrow \quad \begin{cases} x_{e2} = 0 \\ u(x_e) = g(x_{e1}) \end{cases}$$

all equilibrium states of mechanical systems have zero velocity!

joint torques must balance gravity at the equilibrium!

Stability of dynamical systems

definitions - 1



$$\dot{x} = f(x)$$

e.g., a closed-loop system
(under feedback control)

$$x_e \text{ equilibrium: } f(x_e) = 0$$

(sometimes we consider as equilibrium state
 $x_e = 0$, e.g., when using errors as variables)

stability of x_e

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \|x(t_0) - x_e\| < \delta_\varepsilon \Rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$$

asymptotic stability of x_e

stability +

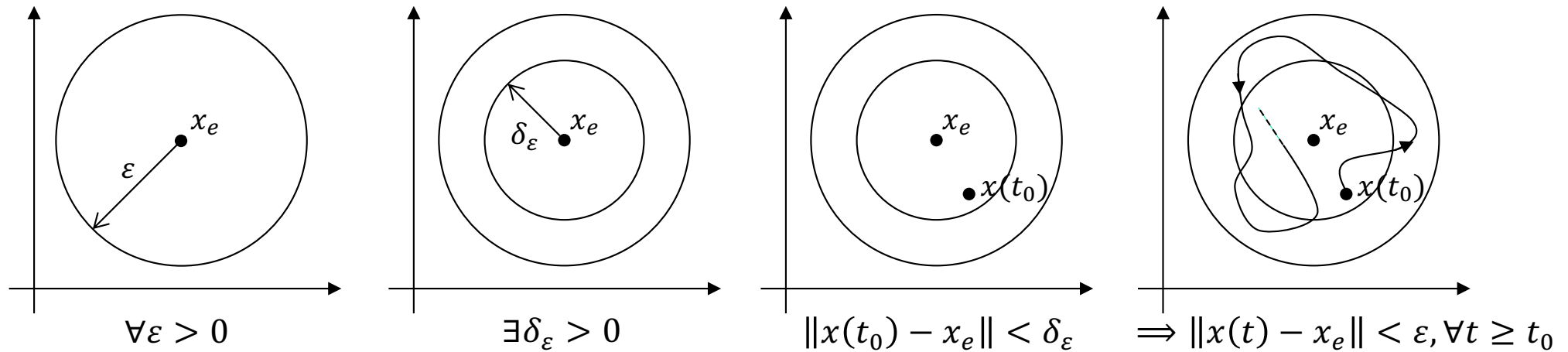
$$\exists \delta > 0: \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \rightarrow 0, \text{ for } t \rightarrow \infty$$

asymptotic stability may become **global** ($\forall \delta > 0$, finite)

note: these are definitions of stability “in the sense of **Lyapunov**”

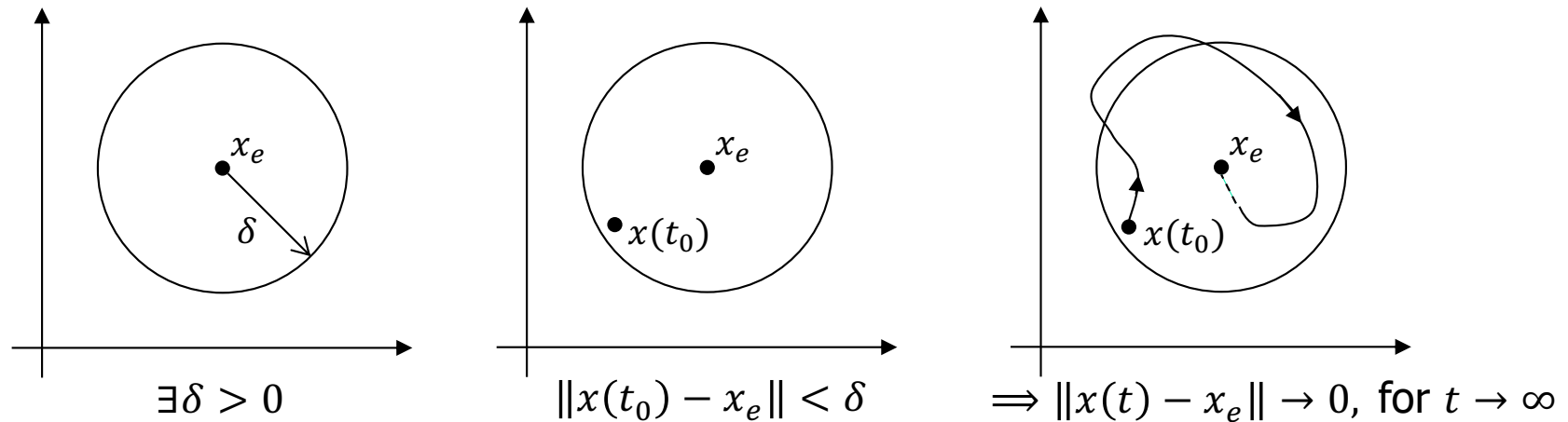
Stability vs. asymptotic stability

whiteboard...



equilibrium state x_e is **stable**

+



equilibrium state x_e is **asymptotically stable**

Stability of dynamical systems

definitions - 2



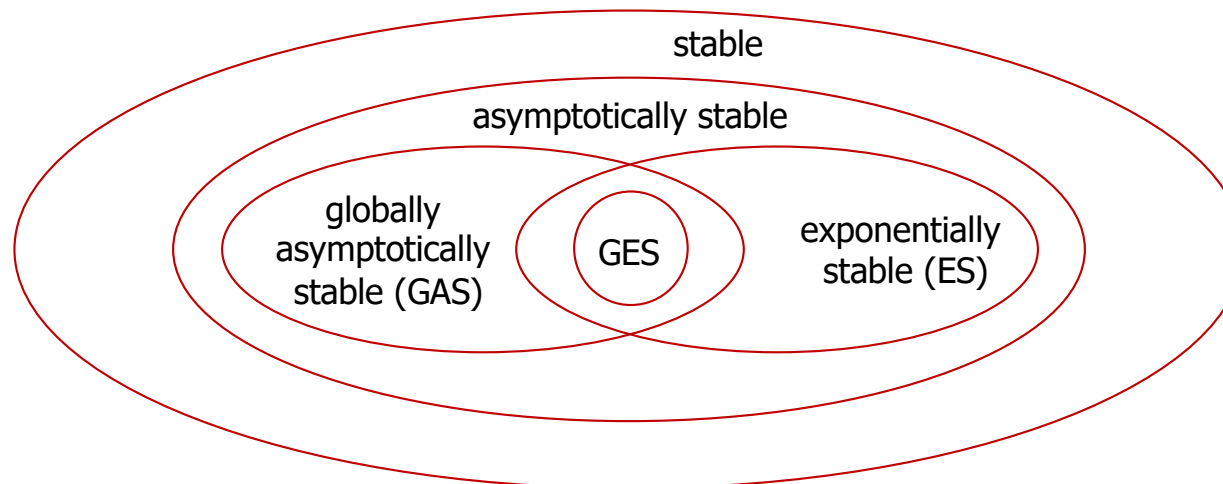
exponential stability of x_e

exponential rate λ

$$\exists \delta, c, \lambda > 0: \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \leq c e^{-\lambda(t-t_0)} \|x(t_0) - x_e\|$$

- allows to estimate the time needed to “approximately” converge: for $c = 1$, in $t - t_0 = 3 \times$ the **time constant** $\tau = 1/\lambda$, the initial error is reduced to 5%
- typically, this is a **local** property only (within some maximum **finite** radius δ)
 \Rightarrow such “domain of attraction” is hard to be estimated accurately

taxonomy
of stability
definitions



a **necessary**
condition for
 x_e to be GAS
is that
it is the **only**
equilibrium state
of the system

The need for analysis and criteria

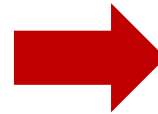
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a nonlinear system $\dot{x} = f(x)$ in \mathbb{R}^2

two equilibria $f(x_e) = 0$

$$\begin{cases} \dot{x}_1 = 1 - x_1^3 \\ \dot{x}_2 = x_1 - x_2^2 \end{cases}$$

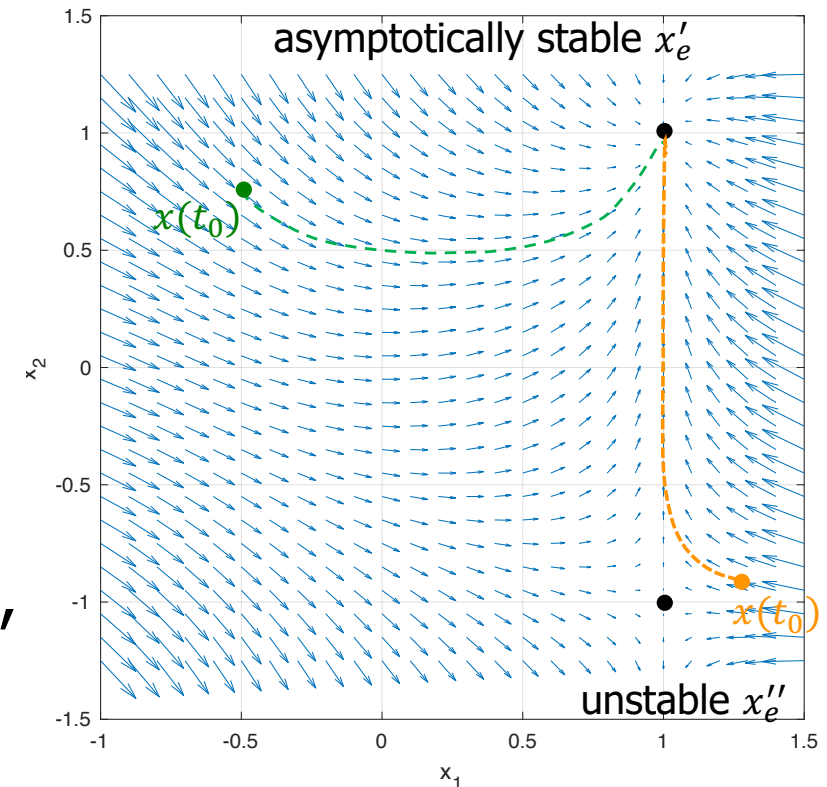


$$x'_e = (1, 1), \quad x''_e = (1, -1)$$

to assess (asymptotic) stability [or not] of equilibria, do we need to compute all system trajectories, starting from all possible initial states $x(t_0)$?



rather, we may be able to just look at the time evolution of **a scalar function V** , evaluated **analytically** along the state trajectories of the system (even in \mathbb{R}^n !)



Stability of dynamical systems

results - 1



Lyapunov candidate

$V(x): \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$V(x_e) = 0, V(x) > 0, \forall x \neq x_e$$

positive
definite
function

typically, **quadratic** (e.g., $\frac{1}{2}(x - x_e)^T P(x - x_e)$ with level surfaces = ellipsoids)
may also be a **local** candidate only ($\forall x \neq x_e: \|x - x_e\| < \delta$)

sufficient condition of stability

$\exists V$ candidate: $\dot{V}(x) \leq 0$, along the trajectories of $\dot{x} = f(x)$

negative
semi-definite
function

sufficient condition of asymptotic stability

$\exists V$ candidate: $\dot{V}(x) < 0$, along the trajectories of $\dot{x} = f(x)$, $x \neq x_e$

negative
definite
function

sufficient condition of instability

$\exists V$ candidate: $\dot{V}(x) > 0$, along the trajectories of $\dot{x} = f(x)$

Stability of dynamical systems

results - 2

LaSalle Theorem

if $\exists V$ candidate: $\dot{V}(x) \leq 0$ along the trajectories of $\dot{x} = f(x)$

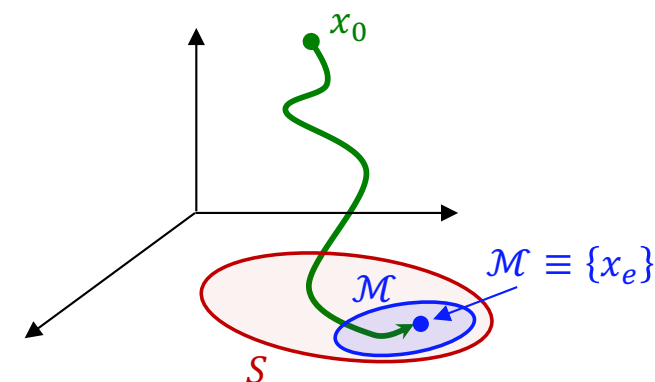
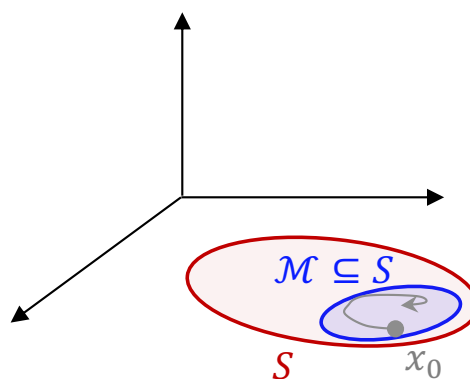
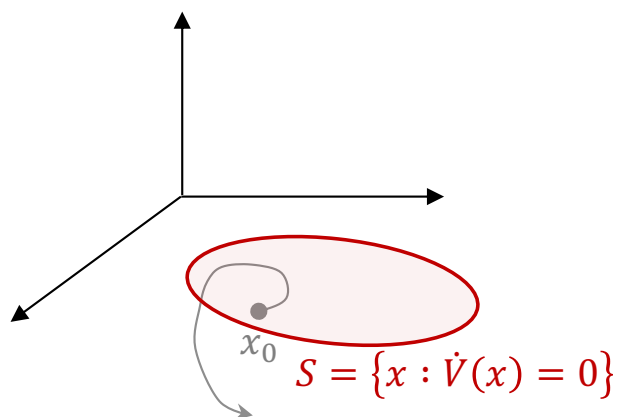


then system trajectories asymptotically converge to the
largest invariant set $\mathcal{M} \subseteq S = \{x \in \mathbb{R}^n: \dot{V}(x) = 0\}$

\mathcal{M} is invariant if $x(t_0) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M}, \forall t \geq t_0$

Corollary

$\mathcal{M} \equiv \{x_e\} \Rightarrow$ asymptotic stability



Bird-eye view on Lyapunov analysis

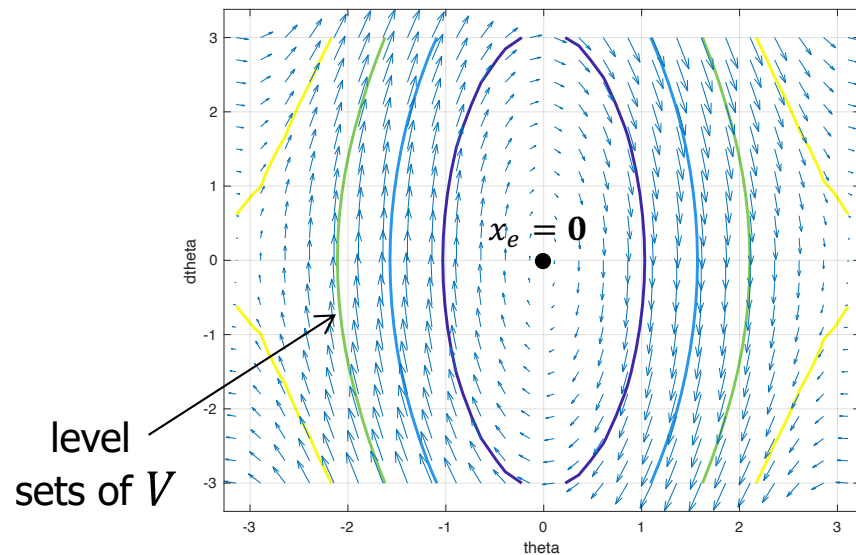
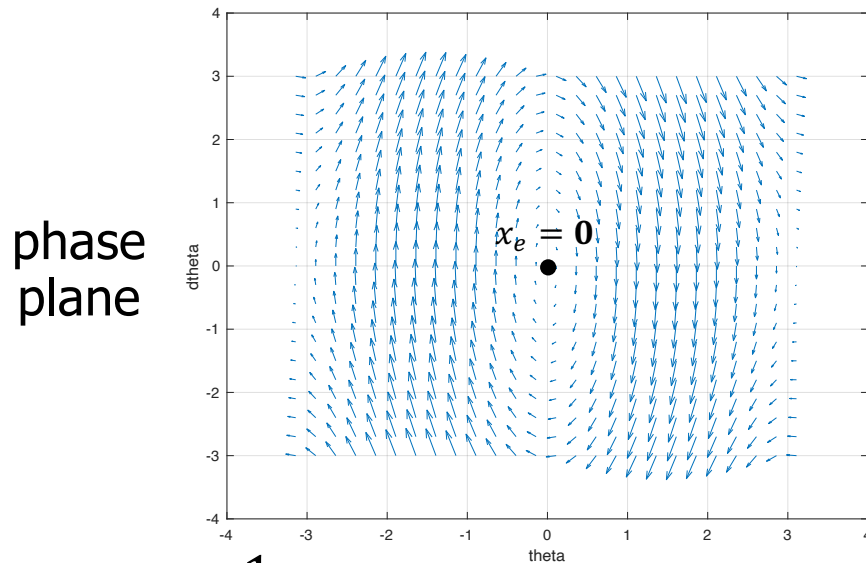
whiteboard...



a mass m at the end of an unforced (passive) pendulum of length l

$$ml^2\ddot{\theta} + b\dot{\theta} + mlg_0 \sin \theta = 0 \Rightarrow x = (x_1, x_2) = (\theta, \dot{\theta}) \in \mathbb{R}^2 \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{g_0}{l}\right) \sin x_1 - \left(\frac{b}{ml^2}\right) x_2 \end{cases}$$

lower **equilibrium at $\theta_e = 0$**



$$V = E = \frac{1}{2} ml^2 \dot{\theta}^2 + mlg_0 (1 - \cos \theta) \geq 0 \quad V = 0 \Leftrightarrow x_e = (\theta_e, \dot{\theta}_e) = (0,0)$$

$$\dot{V} = \dot{\theta}(ml^2\ddot{\theta} + mlg_0 \sin \theta) = -b\dot{\theta}^2 \leq 0 \Rightarrow \text{stability of equilibrium } x_e = 0 \text{ (... at least!)}$$

$$\Rightarrow \text{use LaSalle: } \dot{V} = 0 \Leftrightarrow \dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\left(\frac{g_0}{l}\right) \sin \theta \neq 0 \text{ unless } \theta = \theta_e = 0 \text{ (or } \pi\text{!)}$$

\Rightarrow local asymptotic stability

Stability of dynamical systems

results - 3



- previous results are also valid for **periodic** time-varying systems

$$\dot{x} = f(x, t) = f(x, t + T_p) \Rightarrow V(x, t) = V(x, t + T_p)$$

- for general **time-varying** systems (e.g., in robot **trajectory tracking** control)

$$\dot{x} = f(x, t)$$

Barbalat Lemma

if i) a function $V(x, t)$ is lower bounded

ii) $\dot{V}(x, t) \leq 0$

then $\Rightarrow \exists \lim_{t \rightarrow \infty} V(x, t)$ (but this does **not** imply that $\lim_{t \rightarrow \infty} \dot{V}(x, t) = 0$)

if in addition iii) $\ddot{V}(x, t)$ is bounded

then $\Rightarrow \lim_{t \rightarrow \infty} \dot{V}(x, t) = 0$

Corollary

if a Lyapunov candidate $V(x, t)$ satisfies Barbalat Lemma along the trajectories of $\dot{x} = f(x, t)$, **then** the conclusions of LaSalle Theorem hold



Stability of dynamical systems

additional definition and result (for robust control)

“practical” stability of a set S

$$\exists T(x(t_0), S) \in \mathbb{R}: x(t) \in S, \forall t \geq t_0 + T(x(t_0), S)$$

a finite time

also known as **u.u.b. stability**

⇒ trajectories $x(t)$ are “uniformly ultimately bounded” (use in **robust control**)

sufficient condition of u.u.b. stability of a set S

∃ V candidate: **i)** S is a level set of V for a given c_0

$$S = S(c_0) = \{x \in \mathbb{R}^n: V(x) \leq c_0\}$$

ii) $\dot{V}(x) < 0$ along trajectories of $\dot{x} = f(x)$, $x \notin S$

Stability of linear systems

time-invariant case



$$\dot{x} = Ax$$

$x_e = 0$ is always an equilibrium state

- I. asymptotic stability
- II. **global** asymptotic stability
- III. **exponential** stability
- IV. $\sigma(A) \subset \mathbb{C}^-$ (all eigenvalues of A have **negative real part**)
- V. $\forall Q > 0$ (positive definite), $\exists! P > 0: A^T P + PA = -Q$
Lyapunov equation $\Rightarrow \frac{1}{2} x^T P x$ is a Lyapunov candidate

ALL EQUIVALENT !!

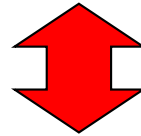
if $x_e = 0$ is an asymptotically stable equilibrium,
then it is necessarily the **unique equilibrium**



Stability of the linear approximation

Let $\Delta x = x - x_e$ and let $\dot{\Delta x} = \frac{df}{dx} \big|_{x=x_e} (x - x_e) = A \Delta x$ be the linear approximation of $\dot{x} = f(x)$ around the equilibrium x_e

A asymptotically stable ($\sigma(A) \subset \mathbb{C}^-$)



the original nonlinear system is **exponentially** stable at the origin

this is only a **local** result
(used also to estimate the domain of attraction)



PD control

(proportional + derivative action on the error)

robot $M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$

goal: asymptotic stabilization (= **regulation**)
of the closed-loop equilibrium state

$$q = q_d, \dot{q} = 0$$

possibly obtained from kinematic inversion: $q_d = f^{-1}(r_d)$

control law $u = K_P(q_d - q) - K_D\dot{q}$

$$K_P > 0, K_D > 0 \text{ (positive definite), symmetric}$$



Asymptotic stability with PD control

Theorem 1

In the absence of gravity ($g(q) \equiv 0$), the robot state $(q_d, 0)$ under the given PD joint control law is globally asymptotically stable

Proof

let $e = q_d - q$ (q_d constant)

Lyapunov candidate

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e \geq 0$$

$$V = 0 \Leftrightarrow e = \dot{e} = 0$$

$$\begin{aligned} \dot{V} &= \dot{q}^T M \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M} \dot{q} - e^T K_P \dot{q} = \dot{q}^T \left(u - \underbrace{S \dot{q} + \frac{1}{2} \dot{M} \dot{q}}_{= 0, \text{ due to energy conservation}} \right) - e^T K_P \dot{q} \\ &= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} - \cancel{e^T K_P \dot{q}} = -\dot{q}^T K_D \dot{q} \leq 0 \quad (K_D > 0, \text{ symmetric}) \end{aligned}$$

up to here, we proved
stability only

but

$$\dot{V} = 0 \Leftrightarrow \dot{q} = 0$$

continues ...
→



Asymptotic stability with PD control

$\dot{V} = 0 \Leftrightarrow \dot{q} = 0$ LaSalle ➔ system trajectories converge to the largest invariant set of states \mathcal{M} where $\dot{q} \equiv 0$ (that is $\dot{q} = \ddot{q} = 0$)

$$\dot{q} = 0 \quad \xrightarrow{\text{closed-loop dynamics}} \quad \underbrace{M(q)\ddot{q} = K_P e}_{\text{closed-loop dynamics}} \quad \xrightarrow{\text{invertible}} \quad \ddot{q} = \underbrace{M^{-1}(q)K_P e}_{\text{invertible}}$$

$$\dot{q} = 0, \ddot{q} = 0 \Leftrightarrow e = 0$$

➔ the only invariant state in $\dot{V} = 0$ is given by $q = q_d, \dot{q} = 0$

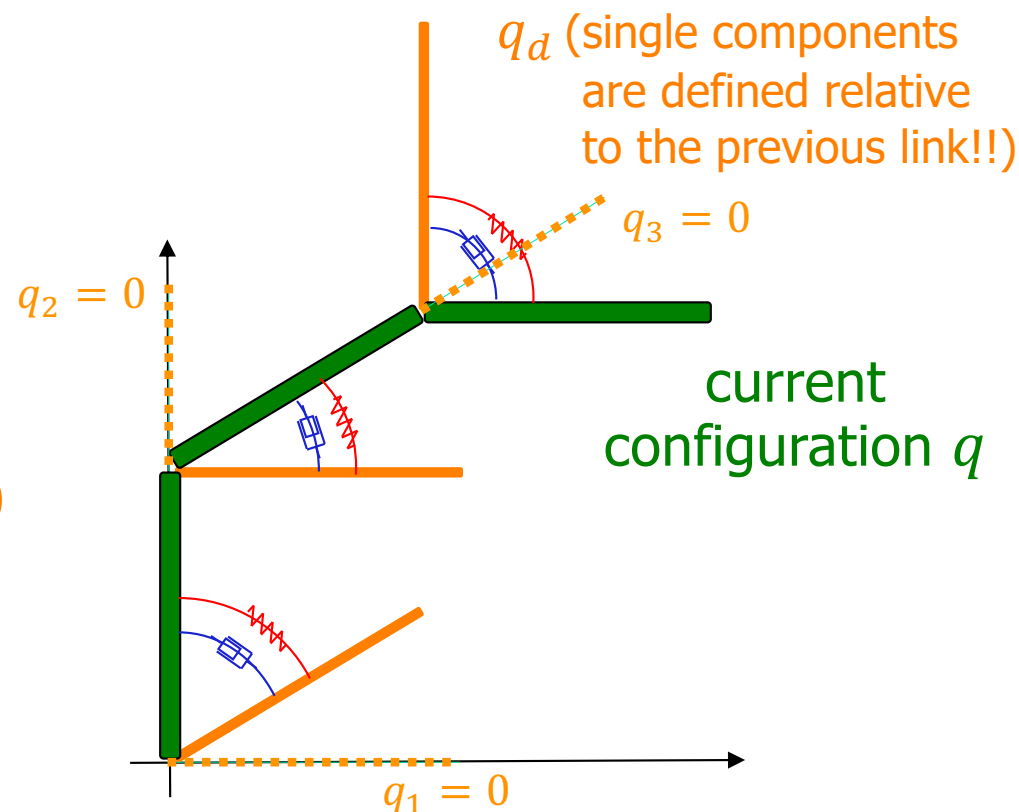
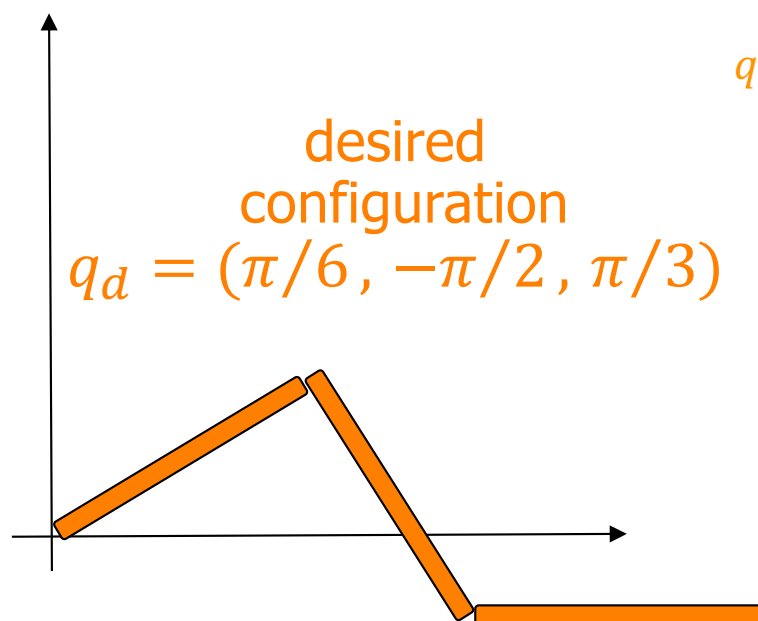
note: typically, $K_P = \text{diag}\{k_{Pi}\}$, $K_D = \text{diag}\{k_{Di}\}$,
➔ decentralized linear control (local to each joint)

Mechanical interpretation

- for **diagonal** positive definite gain matrices K_P and K_D (thus, with **positive** diagonal elements), such values correspond to stiffness of “virtual” **springs** and viscosity of “virtual” **dampers** placed at the joints

 stiffness $k_{Pi} > 0$

 viscosity $k_{Di} > 0$



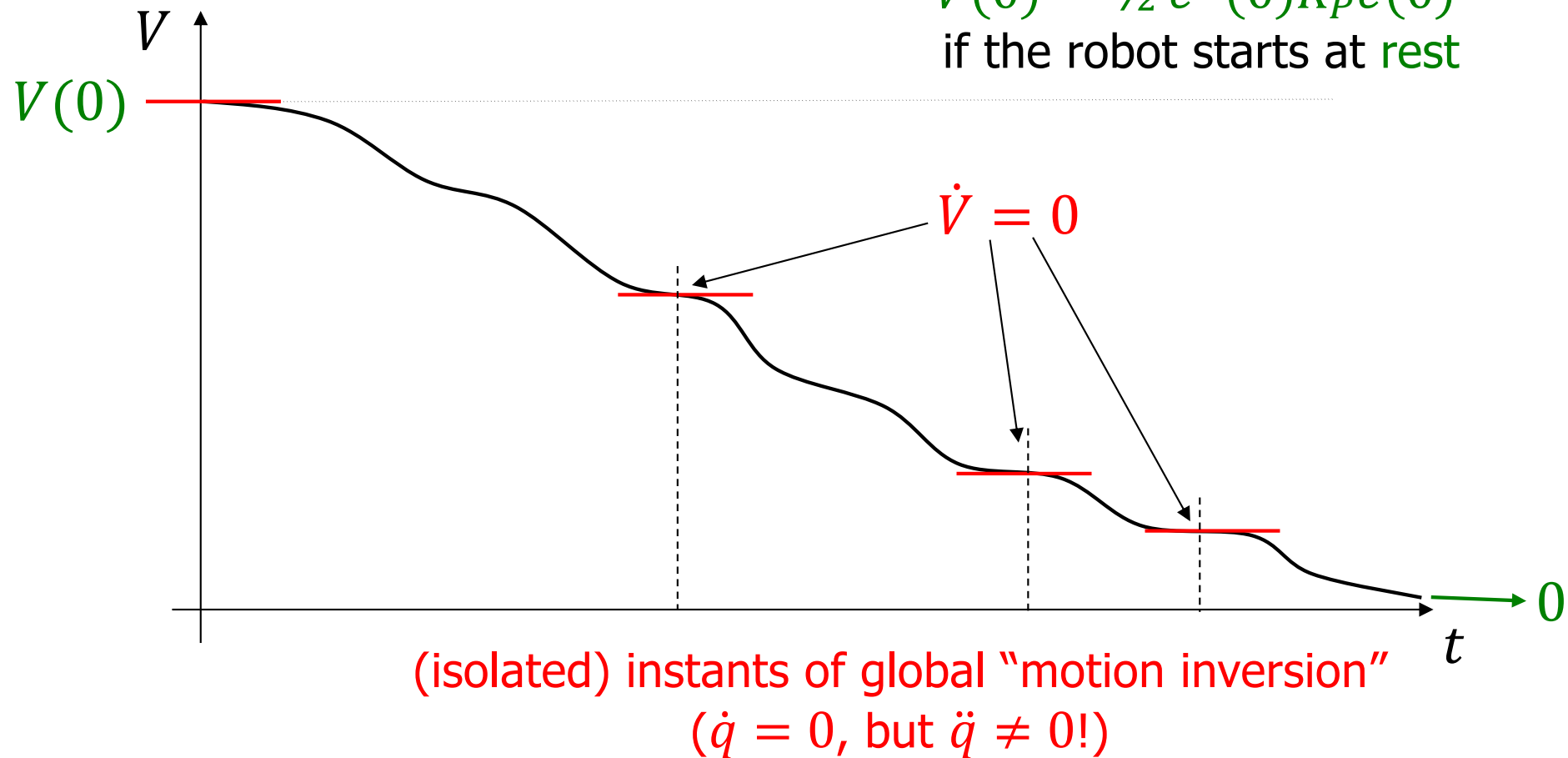


Plot of the Lyapunov function V

- time evolution of the Lyapunov candidate

$$V(0) = \frac{1}{2} e^T(0) K_P e(0)$$

if the robot starts at rest





Comments on PD control - 1

- choice of control gains affects robot evolution during transients and practical settling times
 - hard to define values that are “optimal” in the whole workspace
 - “full” K_P and K_D gain matrices allow to assign desired eigenvalues to the linear approximation of the robot dynamics around the final desired state $(q_d, 0)$
- when (joint) viscous friction is present, the derivative term in the control law is not strictly necessary
 - $-F_V \dot{q}$ in the robot model acts similarly to $-K_D \dot{q}$ in the control law, but the latter can be modulated at will
- in the absence of tachometers, the actual realization of the derivative term in the feedback law requires some processing of joint position data measured by digital encoders (or analog resolvers/potentiometers)



Comments on PD control - 2

- **analog** or **digital** implementation of derivative action in the control law when **only position is measured** at the joints (e.g., through **encoders**)

continuous-time
control law (design)

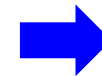
$$u(t) = K_P e(t) + K_D \dot{e}(t)$$

$$e = q_d - q, \dot{e} = -\dot{q}$$

representation in
the Laplace domain

$$u(s) = (K_P + K_D s) e(s)$$

not realizable as such
(non-proper transfer function)



$$u(s) = \left(K_P + \frac{K_D s}{1 + \tau s} \right) e(s)$$

derivative action **limited**
in bandwidth (up to $\omega \leq 1/\tau$)

transformation in
the Zeta-domain
(e.g., via **backward**
differentiation rule on
samples, every T_c sec)

$$u(z) = \left(K_P + K_D \frac{1 - z^{-1}}{T_c} \right) e(z)$$

$$u(z) = \left(K_P + K_D \frac{\frac{1 - z^{-1}}{T_c}}{1 + \tau \frac{1 - z^{-1}}{T_c}} \right) e(z)$$

discrete-time
implementations

$$u_k = K_P e_k + K_D \frac{e_k - e_{k-1}}{T_c}$$

both realizable

$$u_k = K_P e_k + \frac{K_D}{\tau + T_c} (e_k - e_{k-1}) + \frac{\tau}{\tau + T_c} (u_{k-1} - K_P e_{k-1})$$



Inclusion of gravity

- in the presence of gravity, the same previous arguments (and proof) show that the control law

$$u = K_P(q_d - q) - K_D\dot{q} + g(q)$$

$$K_P > 0, K_D > 0$$

will make the equilibrium state $(q_d, 0)$ globally asymptotically stable (**nonlinear** **cancellation of gravity**)

- if gravity is **not** cancelled or only **approximately** cancelled

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q)$$

$$\hat{g}(q) \neq g(q)$$

it is $q \rightarrow q^* \neq q_d, \dot{q} \rightarrow 0$, with a **steady-state** position error

- q^* is not unique in general, except when K_P is chosen large enough
- explanation in terms of linear systems: there is **no integral action** **before** the point of access of the **constant "disturbance"** acting on the system

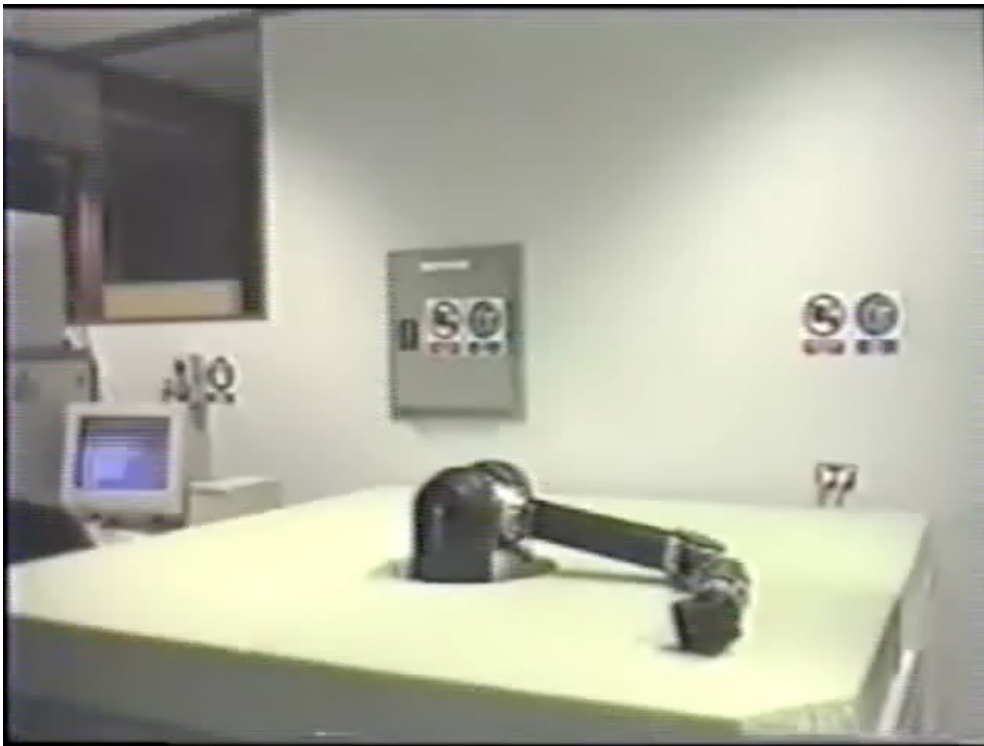
Approximate cancellation of gravity

WAM Barrett
(with some viscous friction)

$$u = \hat{g}(q)$$

$$\hat{g}(q) = g(q)$$

$$\hat{g}(q) \neq g(q)$$



two-part video

<http://handbookofrobotics.org/view-chapter/69/videodetails/611>

PD control + constant gravity compensation



if the robot potential energy $U(q)$ is bounded for all q , then its partial derivative $g(q)$ is also **bounded** everywhere and the following **structural property** holds

finite \curvearrowright

$$\exists \alpha > 0: \left\| \frac{\partial^2 U}{\partial q^2} \right\| = \left\| \frac{\partial g}{\partial q} \right\| \leq \alpha, \forall q$$

consequence \Rightarrow

$$\|g(q) - g(q_d)\| \leq \alpha \|q - q_d\|$$

note: induced
norm of
a matrix

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)} \triangleq A_M \geq A_m \triangleq \sqrt{\lambda_{\min}(A^T A)}$$

LINEAR CONTROL law

$$u = K_P(q_d - q) - K_D \dot{q} + g(q_d)$$

$K_P, K_D > 0$
symmetric

linear feedback + **constant** feedforward



More on the basic assumption ...

(in PD control + gravity compensation)

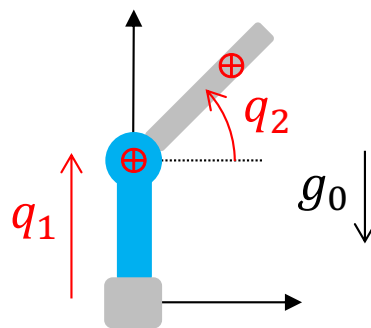
when is the (non-zero) gravity term $g(q)$ bounded for all q ?

- the robot has **all revolute** joints
 - all terms in $U(q)$ and thus in $g(q)$ are trigonometric (bounded)
- the robot has both types of joints, but **no prismatic variables in $g(q)$**
 - potential energy $U(q)$ may still be unbounded!
- all **prismatic** joints of the robot have a **limited range**
 - ... ok, but one should take these limits into account in the control analysis

$$U = g_0(m_1 q_1 + m_2(q_1 + d_{c2} \sin q_2)) + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 + m_2) \\ m_2 d_{c2} \cos q_2 \end{pmatrix}$$

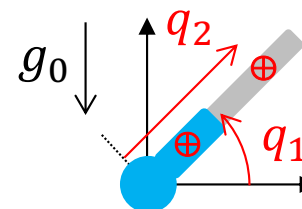
$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha = m_2 d_{c2} g_0$$



PR robot

$$U = g_0(m_1 d_{c1} + m_2 q_2) \sin q_1 + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 d_{c1} + m_2 q_2) \cos q_1 \\ m_2 \sin q_1 \end{pmatrix}$$



$$\left\| \frac{\partial g}{\partial q} \right\| \leq ??$$

RP robot

PD control + constant gravity compensation

stability analysis



Theorem 2

If $K_{P,m} > \alpha$, the state $(q_d, 0)$ of the robot under joint-space PD control + constant gravity compensation at q_d is **globally asymptotically stable**

Proof

1. $(q_d, 0)$ is the unique closed-loop equilibrium state

in fact, for $\dot{q} = 0$ and $\ddot{q} = 0$, it is $K_P e = g(q) - g(q_d)$

which can hold only for $q = q_d$, because when $q \neq q_d$

$$\|K_P e\| \geq K_{P,m} \|e\| > \alpha \|e\| \geq \|g(q) - g(q_d)\|$$

$$M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = K_P e - K_D \dot{q} + g(q_d)$$



PD control + constant gravity compensation

stability analysis

with $e = q_d - q$, $g(q) = \left(\frac{\partial U}{\partial q}\right)^T$, consider as **Lyapunov candidate**

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

2. V is convex in \dot{q} and e , and zero only for $e = \dot{q} = 0$

$$\left(\frac{\partial V}{\partial \dot{q}}\right)^T = M(q) \dot{q} = 0 \text{ only for } \dot{q} = 0$$

$$\frac{\partial^2 V}{\partial \dot{q}^2} = M(q) > 0$$

$(q_d, 0)$ is a
global minimum
of $V \geq 0$

$$\left(\frac{\partial V|_{\dot{q}=0}}{\partial e}\right)^T = K_P e - \left(\frac{\partial U}{\partial q}\right)^T + g(q_d) = K_P e + g(q_d) - g(q) = 0$$

$\partial e / \partial q = -I$ only for $q = q_d$

$$\frac{\partial^2 V|_{\dot{q}=0}}{\partial e^2} = K_P + \frac{\partial^2 U}{\partial q^2} > 0, \text{ since } \|K_P\| = K_{P,M} \geq K_{P,m} > \alpha$$



PD control + constant gravity compensation

stability analysis

differentiating

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

$$\begin{aligned} \dot{V} &= \dot{q}^T \left(M(q) \ddot{q} + \frac{1}{2} \dot{M}(q) \dot{q} \right) - e^T K_P \dot{q} + \frac{\partial U(q)}{\partial q} \dot{q} - \dot{q}^T g(q_d) \\ &= \dot{q}^T \left(\underbrace{u - S(q, \dot{q}) \dot{q} + \frac{1}{2} \dot{M}(q) \dot{q} - g(q)}_{=0} \right) - e^T K_P \dot{q} + \dot{q}^T (g(q) - g(q_d)) \\ &= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} + \dot{q}^T (g(q_d) - g(q)) - \cancel{e^T K_P \dot{q}} + \dot{q}^T (g(q) - g(q_d)) \\ &= -\dot{q}^T K_D \dot{q} \leq 0 \end{aligned}$$

for $\dot{V} = 0 (\Leftrightarrow \dot{q} = 0)$, we have in the closed-loop system

$$M(q) \ddot{q} + g(q) = K_P e + g(q_d)$$

$$\Rightarrow \ddot{q} = M^{-1}(q) (K_P e + g(q_d) - g(q)) = 0 \Leftrightarrow e = 0$$

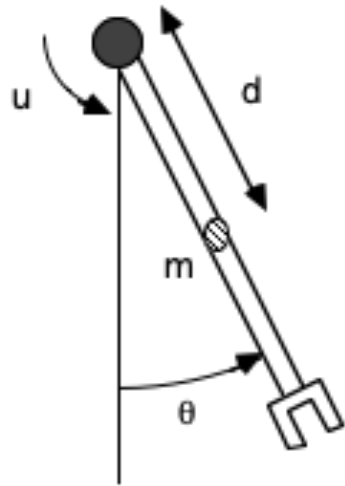
by LaSalle Theorem, the thesis follows





Example of a single-link robot

stability analysis



task: regulate the link position to the **upward equilibrium**

$$\theta_d = \pi \rightarrow g(\theta_d) = 0$$

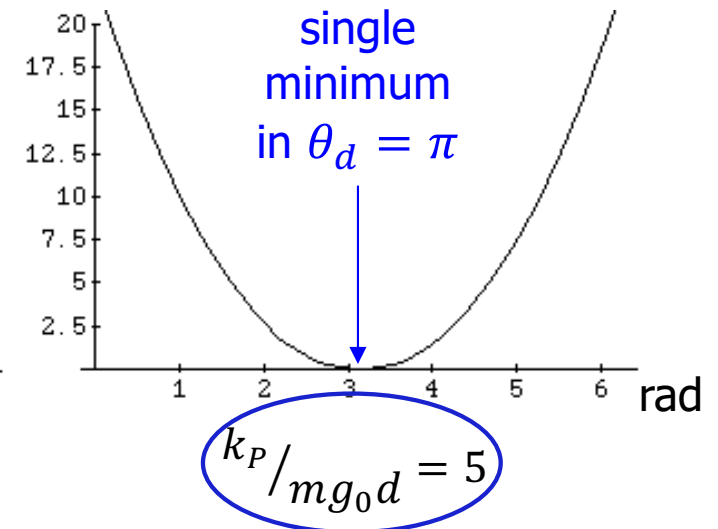
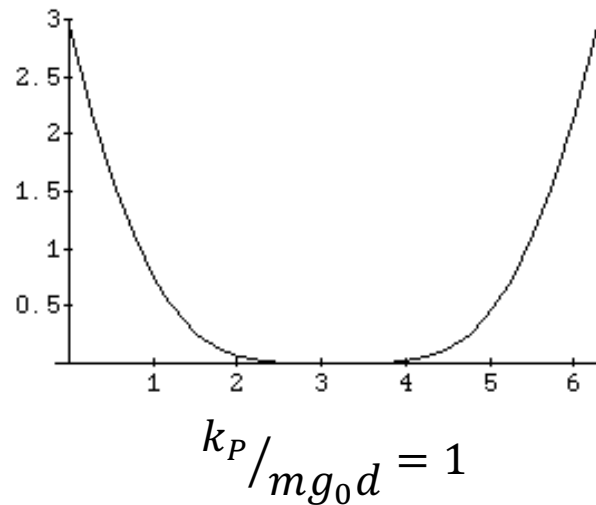
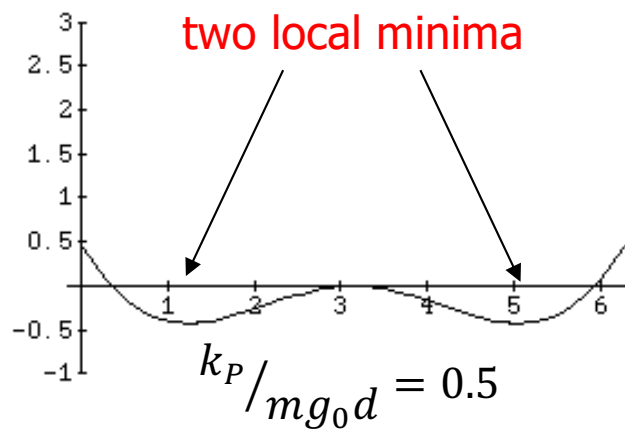
PD control + constant gravity compensation (here, **zero!**)

$$u = k_P(\pi - \theta) - k_D\dot{\theta}$$

by Theorem 2, it is **sufficient** (here, also **necessary***) to choose

$$k_P > \alpha = mg_0d, \quad k_D > 0$$

$$I\ddot{\theta} + mg_0d \sin \theta = u$$



plots of $V(\theta)$ (for $\dot{\theta} = 0$)

* by a local analysis of the linear approximation at π



Example of a single-link robot

simulations with data: $I = 0.9333$, $mg_0d = 19.62 (= \alpha)$

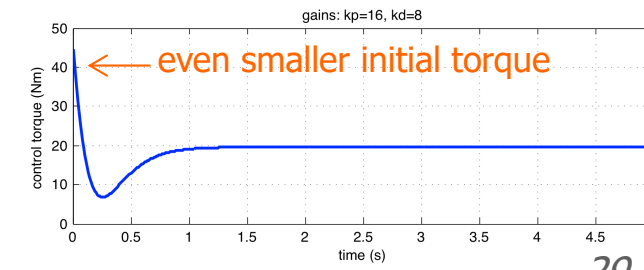
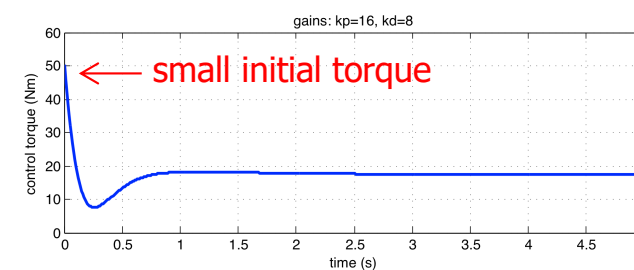
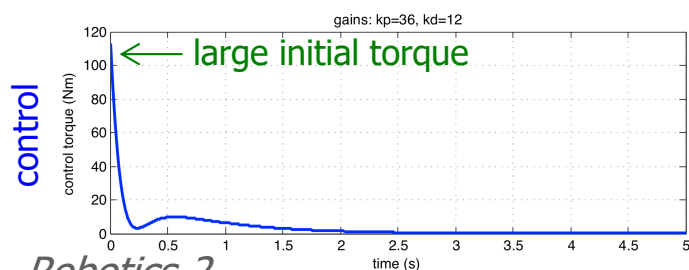
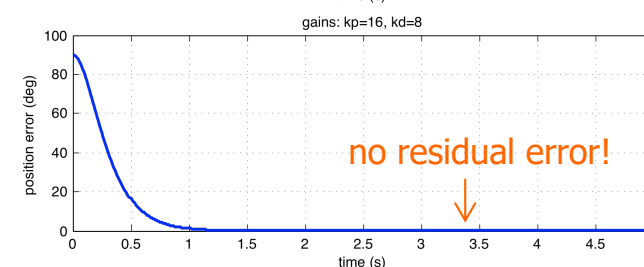
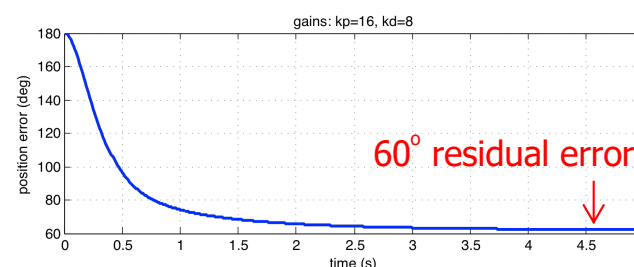
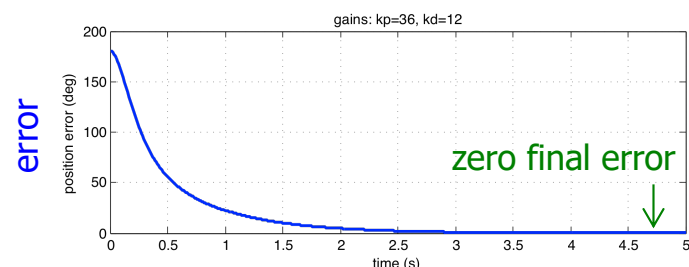
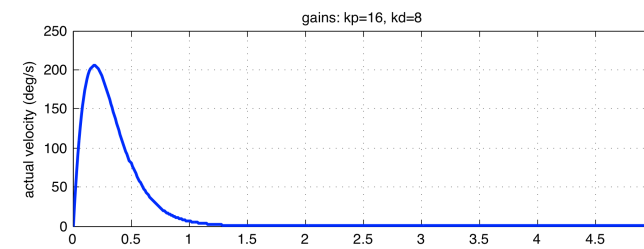
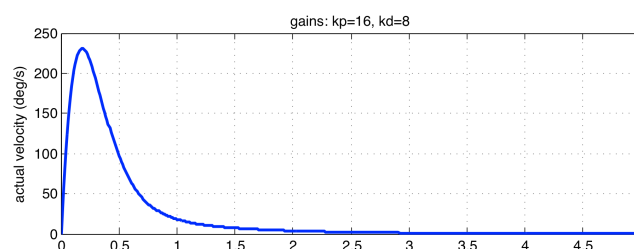
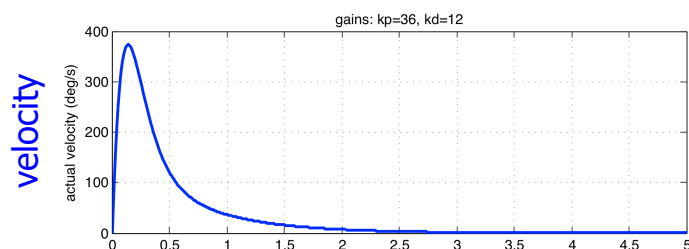
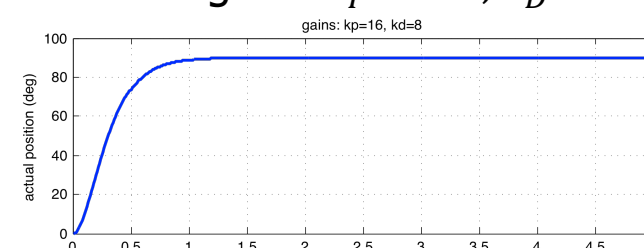
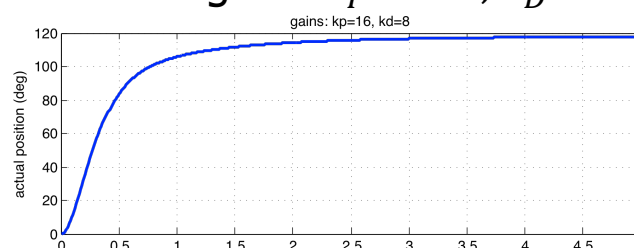
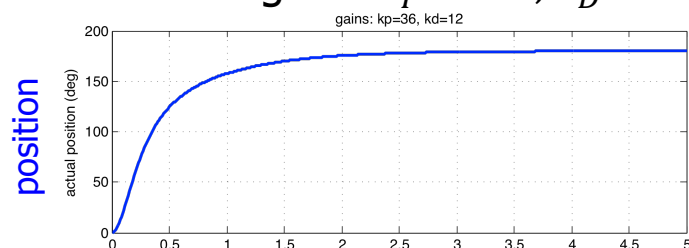
$\theta_d = 180^\circ \rightarrow g(\theta_d) = 0$

sufficient P gain: $k_P = 36, k_D = 12$

low P gain: $k_P = 16, k_D = 8$

$\theta_d = 90^\circ \rightarrow g(\theta_d) = mg_0d$

low P gain: $k_P = 16, k_D = 8$





Approximate gravity compensation

the **approximate** control law

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q_d)$$

leads, under similar hypotheses, to a closed-loop equilibrium q^*

- its uniqueness is not guaranteed (unless K_P is large enough)
- for $K_P \rightarrow \infty$, one has $q^* \rightarrow q_d$

conclusion: in the presence of gravity, the previous regulation control laws require an **accurate knowledge** of the **gravity term** in the dynamic model to guarantee the zeroing of the position error (since we can only use “finite” control gains \Rightarrow in practice, not too large)



PID control

- in linear systems, the addition of an integral control action is used to eliminate a constant error in the step response at steady state
- in robots, a PID may be used to recover such a position error due to an incomplete (or absent) gravity compensation/cancellation

➡ the control law
$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t (q_d - q(\tau)) d\tau - K_D \dot{q}(t)$$

- is independent from any robot dynamic model term
- if the desired closed-loop equilibrium is asymptotically stable under PID control, the integral term is “loaded” at steady state to the value

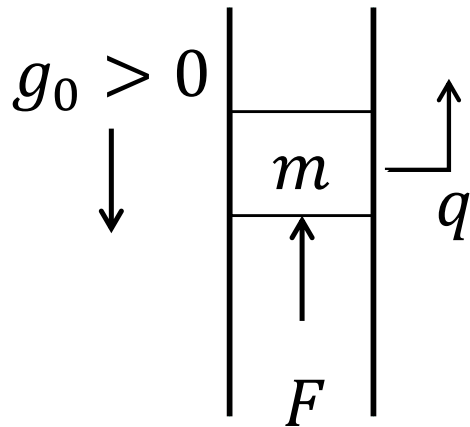
$$K_I \int_0^\infty (q_d - q(\tau)) d\tau = g(q_d)$$

- however, one can show only local asymptotic stability of this law, i.e., for $q(0) \in \Delta(q_d)$, under complex conditions on K_P, K_I, K_D and $e(0)$



Linear example with PID control

whiteboard...



$$m\ddot{q} + mg_0 = F \quad (\text{no friction})$$

$$e(t) = q_d - q(t)$$
$$\dot{e}(t) = -\dot{q}(t)$$

$$F = k_P(q_d - q) - k_D\dot{q}$$

(PD \Rightarrow steady-state error $e = q_d - \bar{q}$, with $\bar{q} = q_d - \frac{mg_0}{k_P}$)

$$F = k_P(q_d - q) - k_D\dot{q} + mg_0$$

(PD + gravity cancellation \Rightarrow regulation $\forall k_P > 0, k_D > 0$)

$$F = k_P(q_d - q) - k_D\dot{q} + k_I \int_0^t (q_d - q(\tau)) d\tau$$

(PID \Rightarrow regulation $\forall k_I > 0, k_D > 0, k_P > \frac{mk_I}{k_D} > 0$)

with global
exponential
stability!

Laplace domain analysis: $e(s) = \mathcal{L}[e(t)]$, $d(s) = \mathcal{L}[mg_0]$ + Routh criterion

$$\frac{e(s)}{d(s)} = W_d(s) = \frac{s}{ms^3 + k_Ds^2 + k_Ps + k_I}$$

3	m	k_P
2	k_D	k_I
1	$(k_Dk_P - mk_I)/k_D$	
0	k_I	



Saturated PID control

- more in general, one can prove **global** asymptotic stability of $(q_d, 0)$, under **lower bound limitations** for K_P, K_I, K_D (depending on suitable “bounds” on the terms in the dynamic model), for a **nonlinear PID law**

$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t \Phi(q_d - q(\tau)) d\tau - K_D \dot{q}$$

where $\Phi(q_d - q)$ is a **saturation-type** function, such as

$$\Phi(x) = \begin{cases} \sin x, & |x| \leq \pi/2 \\ 1, & x > \pi/2 \\ -1, & x < -\pi/2 \end{cases} \quad \text{or} \quad \Phi(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(see paper by R. Kelly, IEEE TAC, 1998; available as extra material on the course web)



Limits of robot regulation controllers

- **response times** needed for reaching the desired steady state are **not** easily **predictable** in advance
 - depend heavily on robot dynamics, on PD/PID gains, on the required total displacement, and on the interested area of the robot workspace
 - integral term (when present) needs some time to “unload” itself from the error history accumulated during transients
 - large initial errors are stored in the integral term
 - anti-windup schemes stop the integration when commands saturate
 - ... an intuitive explanation for the success of “saturated” PID law
- **control efforts in the few first instants** of motion typically exceed by far those required at steady state
 - especially for high positional gains
 - may lead to saturation (hard nonlinearity) of robot actuators



Regulation in industrial robots

- in industrial robots, the planner generates a **reference trajectory** $q_r(t)$ even when the task requires **only** positioning/regulation of the robot
 - “smooth” enough, with a user-defined **transfer time** T
 - reference trajectory interpolates initial and final desired position

$$q_r(0) = q(0) \quad q_r(t \geq T) = q_d$$

- $q_r(t)$ is used within a control law of the form

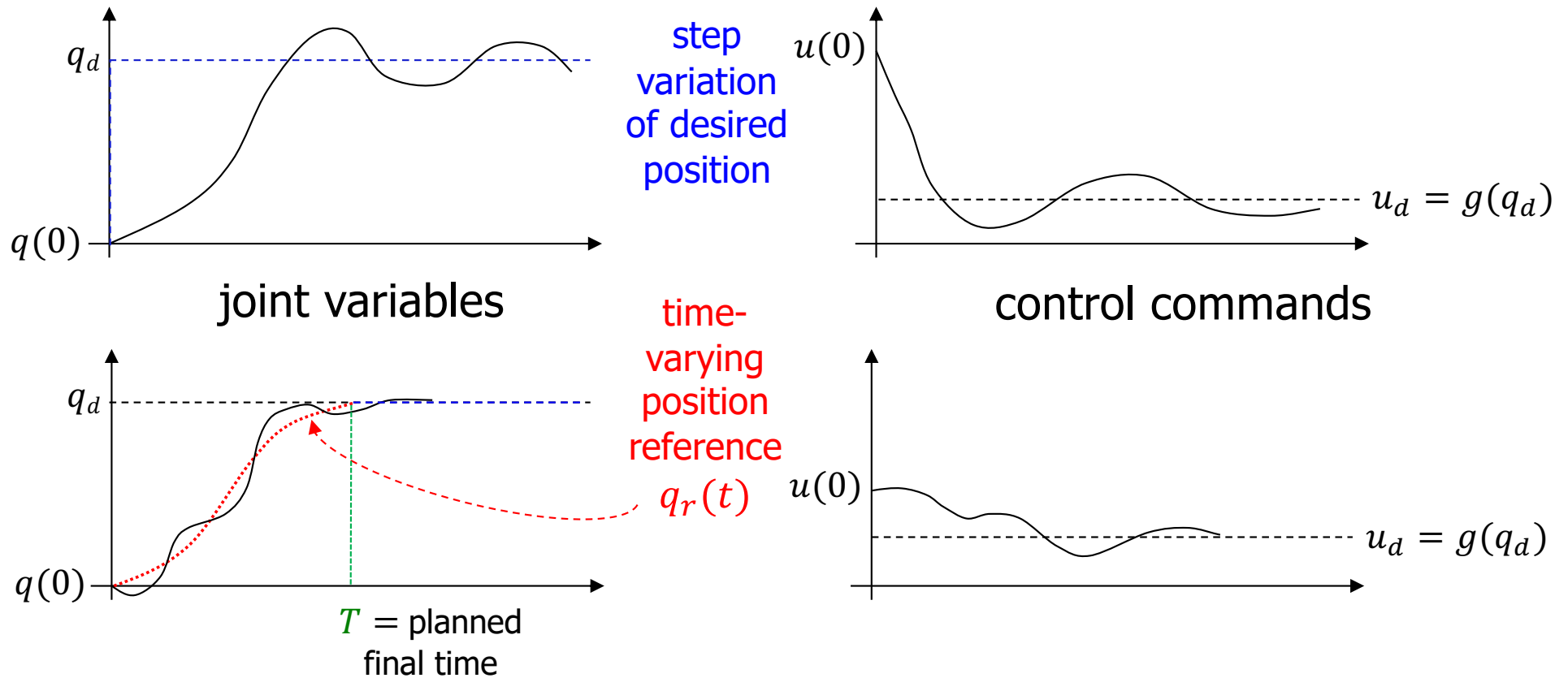
$$u = K_P(q_r(t) - q) + K_D(\dot{q}_r(t) - \dot{q}) + g(q)$$

e.g., PD with
gravity
cancellation

↑
often neglected

- in this way, the position error is **initially zero**
- robot motion stays only “in the vicinity” of the reference trajectory until $t = T$, typically with small position errors (gains can be **larger!**)
- **final** regulation is only a “local” problem ($e(T) = q_d - q(T)$ is small)

Qualitative comparison



- **no saturation** of commands: in principle, much larger gains can be used
- better **prediction of settling times**: local exponential convergence (designed on the linear approximation of the robot dynamics around $(q_d, 0)$)
- “fine tuning” of control gains made easier, but still a **tedious** and **delicate task**

Quantitative comparison

planar 2R arm

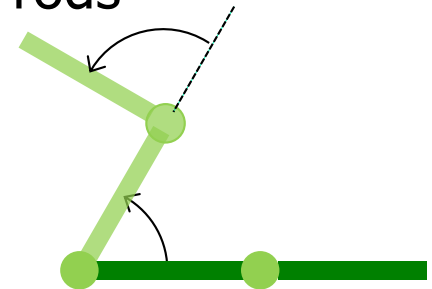
m_1	10 [kg]
m_2	5 [kg]
l_1	0.5 [m]
l_2	0.5 [m]
d_1	0.25 [m]
d_2	0.25 [m]
I_1	5/24 [kg m ²]
I_2	5/48 [kg m ²]

robot data: links are uniform thin rods

no gravity (horizontal plane)

rest-to-rest motion task:

$q(0) = (0, 0)$ (**straight**) $\rightarrow q_d = (\pi/3, \pi/2)$



interpolating trajectory: cubic polynomials

three case studies

a) low gains (overdamped) $K_P = \text{diag}\{80, 40\}, K_D = \text{diag}\{60, 30\}$

vs interpolating trajectory in $T = 2$ s

b) medium gains (**very** overdamped) $K_P = \text{diag}\{200, 100\}, K_D = \text{diag}\{200, 100\}$

vs interpolating trajectory in $T = 2$ s

c) high gains $K_P = \text{diag}\{1250, 180\}, K_D = \text{diag}\{200, 70\}$

vs interpolating trajectory in $T = 1$ s, with torque saturation $u_{1,\max} = 30$ Nm,

$u_{2,\max} = 10$ Nm

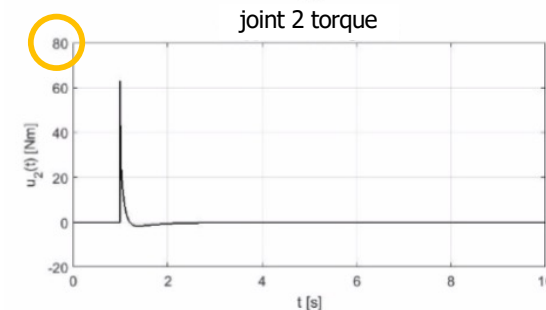
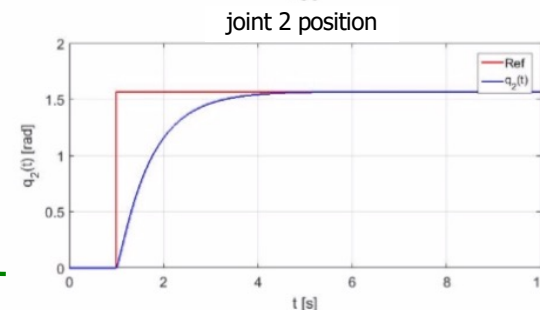
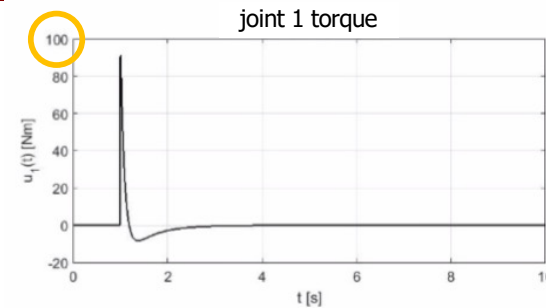
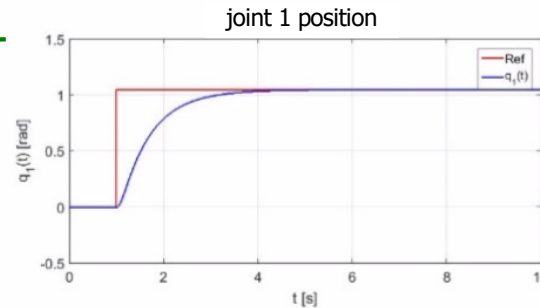
Comparison on a planar 2R arm – case a

PD with low gains

$$K_P = \text{diag}\{80, 40\}$$

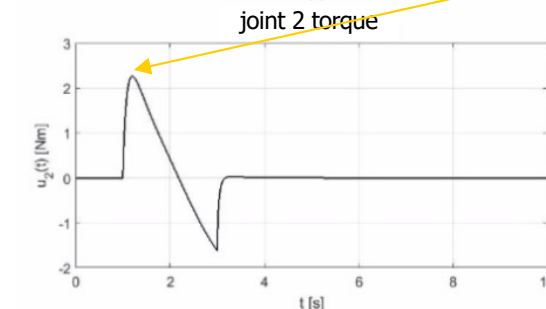
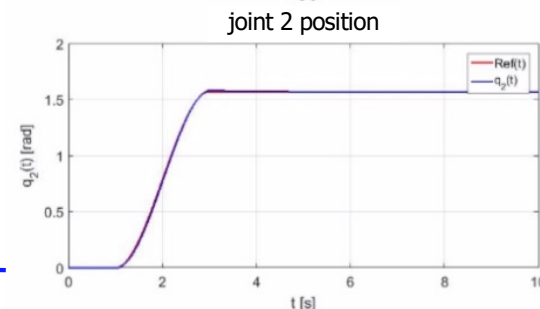
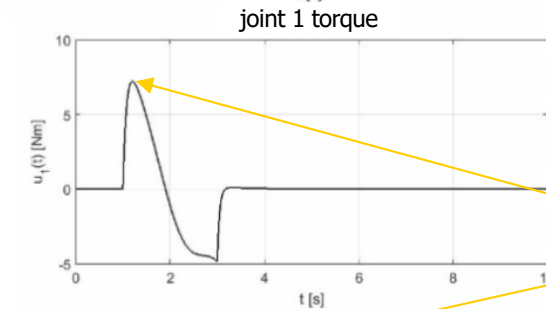
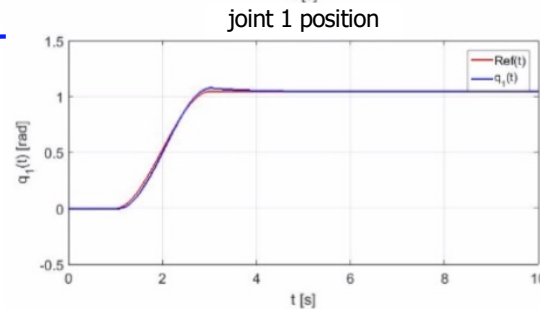
$$K_D = \text{diag}\{60, 30\}$$

(overdamped)



a reduction of the
peak control effort
by a factor of 100
on joint 1 &
by a factor of 30
on joint 2

PD with same gains
on interpolating
trajectory of $T = 2$ s



max torques
of 7 and 2.3 Nm

Comparison on a planar 2R arm – case b

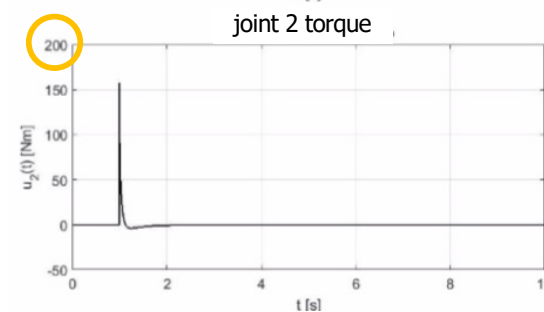
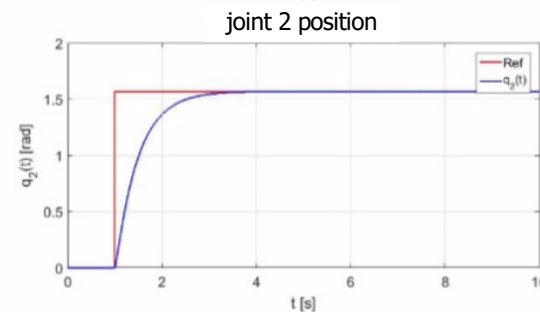
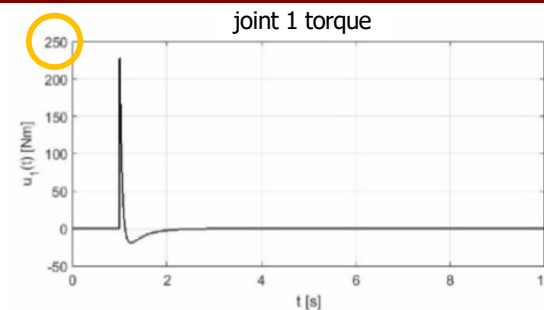
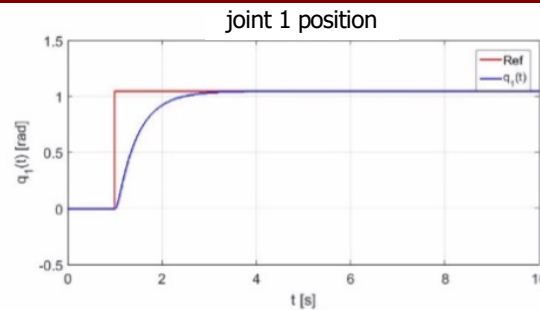


PD with medium gains

$$K_P = \text{diag}\{200, 100\}$$

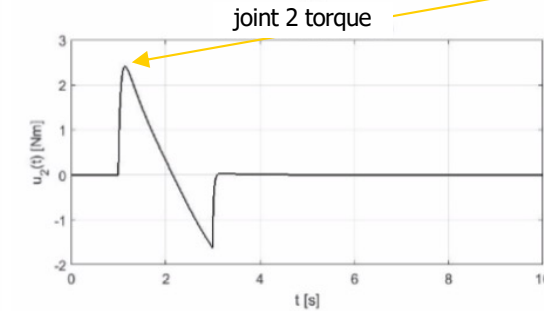
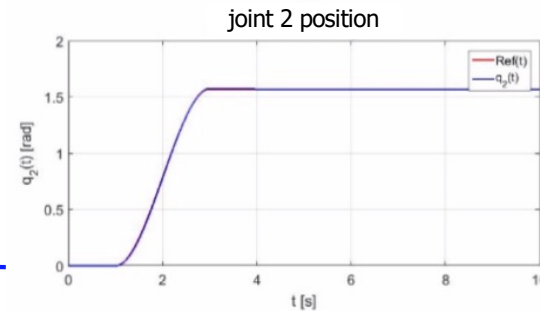
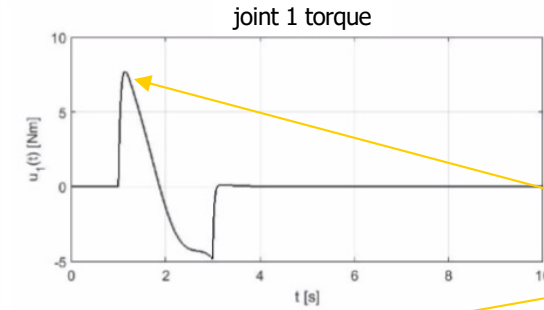
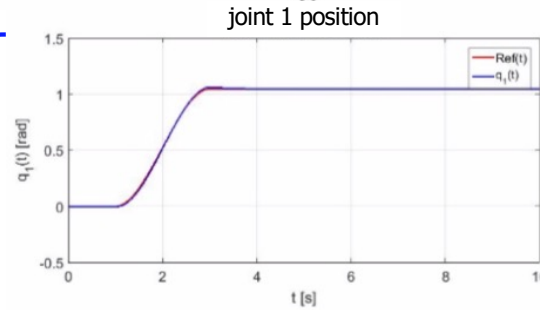
$$K_D = \text{diag}\{200, 100\}$$

(very overdamped)



even stronger
peak reduction,
with similar total
control effort,
plus improved
tracking of
reference trajectory
on both joints

PD with same gains
on interpolating
trajectory of $T = 2$ s



max torques
of 7.5 and 2.4 Nm

Comparison on a planar 2R arm – case c

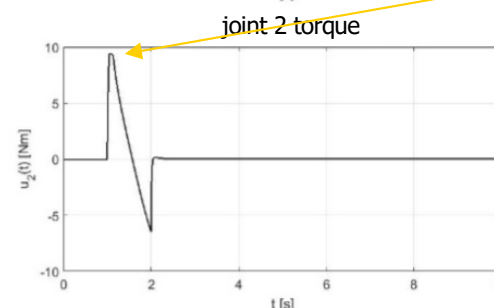
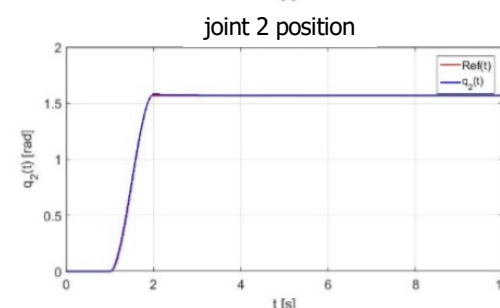
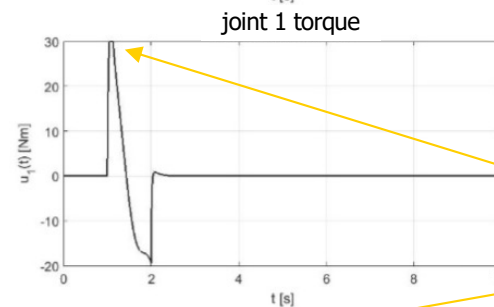
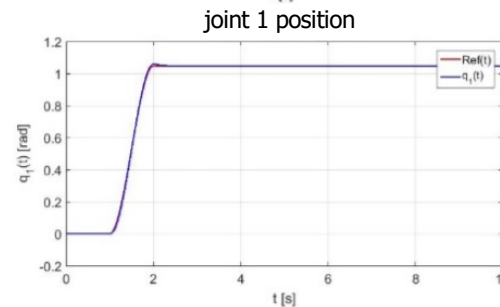
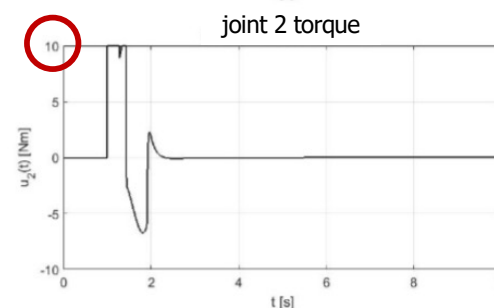
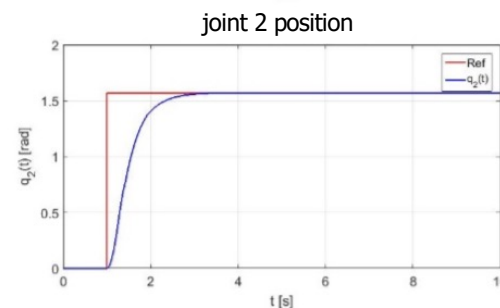
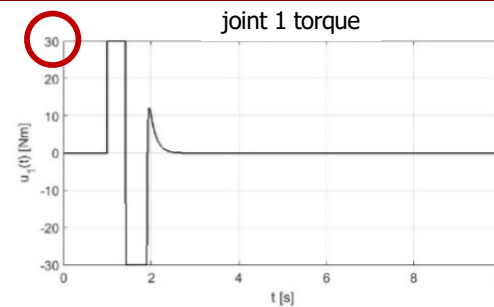
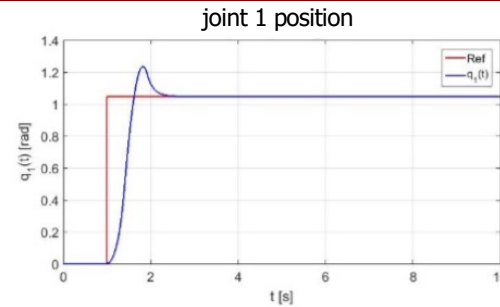
PD with high gains
 $K_P = \text{diag}\{1250, 180\}$
 $K_D = \text{diag}\{200, 70\}$

torque saturation

$$u_{1,\max} = 30 \text{ Nm}$$

$$u_{2,\max} = 10 \text{ Nm}$$

PD with same gains
 on interpolating
 trajectory of $T = 1 \text{ s}$



position overshoot
 and long saturations
 are avoided,
 with very good
 tracking of the
 faster reference
 trajectory

max torques
 of 30 and 9.5 Nm