Dynamic model of robots: 
Lagrangian approach

Prof. Alessandro De Luca
Dynamic model

- provides the relation between generalized forces \( u(t) \) acting on the robot and robot motion, i.e., assumed configurations \( q(t) \) over time.

\[
\Phi(q, \dot{q}, \ddot{q}) = u
\]

a system of 2\(^{nd}\) order differential equations
Direct dynamics

- direct relation

\[ u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \Rightarrow \quad q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} \]

input for \( t \in [0, T] \) \( q(0), \dot{q}(0) \)

initial state at \( t = 0 \)

- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time \( T_c \))

- solution by simulation
  - use dynamic model and integrate numerically the differential equations (with simulation step \( T_s \leq T_c \))

resulting motion \( \Phi(q, \dot{q}, \ddot{q}) = u \)
Inverse dynamics

- inverse relation

\[ q_d(t), \dot{q}_d(t), \ddot{q}_d(t) \quad \Rightarrow \quad u_d(t) \]

- desired motion for \( t \in [0, T] \)

- required input for \( t \in [0, T] \)

- experimental solution
  - repeated motion trials of direct dynamics using \( u_k(t) \), with iterative learning of nominal torques updated on trial \( k + 1 \) based on the error in \([0, T]\) measured in trial \( k \):
    \[ \lim_{k \to \infty} u_k(t) \Rightarrow u_d(t) \]

- analytic solution
  - use dynamic model and compute algebraically the values \( u_d(t) \) at every time instant \( t \)

\[ \Phi(q, \dot{q}, \ddot{q}) = u \]
Approaches to dynamic modeling

Euler-Lagrange method (energy-based approach)
- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

Newton-Euler method (balance of forces/torques)
- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d’Alembert, of Hamilton, of virtual works, ...
Euler-Lagrange method (energy-based approach)

basic assumption: the $N$ links in motion are considered as **rigid bodies**
(+ later on, include also **concentrated elasticity** at the joints)

$q \in \mathbb{R}^N$ generalized coordinates (e.g., joint variables, but not only!)

Lagrangian

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

kinetic energy – potential energy

- principle of least action of Hamilton
- principle of virtual works

Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \quad i = 1, \ldots, N$$

non-conservative (external or dissipative) generalized forces performing work on $q_i$
Dynamics of an actuated pendulum

a first example

\[ \dot{\theta}_m = n_r \dot{\theta} \rightarrow \theta_m = n_r \theta + \theta_{m0} \]

\[ \tau = n_r \tau_m \]

\[ q = \theta \quad \text{(or} \ q = \theta_m) \]

\[ T = T_m + T_l \]

\[ T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \]

\[ T_l = \frac{1}{2} (I_l + m d^2) \dot{\theta}^2 \]

\[ T = \frac{1}{2} (I_l + m d^2 + I_m n_r^2) \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2 \]
Dynamics of an actuated pendulum (cont)

\[ U = U_0 - m g_0 d \cos \theta \]

potential energy

\[ L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos \theta - U_0 \]

\[ \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta} \]
\[ \frac{\partial L}{\partial \theta} = -m g_0 d \sin \theta \]

\[ u = n_r \tau_m - b_l \dot{\theta} - n_r b_m \dot{\theta}_m + J_x^T F_x = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta F_x \]

applied or dissipated torques on motor side are multiplied by \( n_r \)
when moved to the link side

equivalent joint torque due to force \( F_x \) applied to the tip at point \( p_x \)

“sum” of non-conservative torques

Robotics 2
Dynamics of an actuated pendulum (cont)

\[ I \ddot{\theta} + mg_0 d \sin \theta = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta \cdot F_x \]

dividing by \( n_r \) and substituting \( \theta = \theta_m/n_r \)

\[ \frac{I}{n_r^2} \ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin \frac{\theta_m}{n_r} = \tau_m - \left( \frac{b_l}{n_r^2} + b_m \right) \dot{\theta}_m + \frac{l}{n_r} \cos \frac{\theta_m}{n_r} \cdot F_x \]

dynamic model in \( q = \theta_m \)
Kinetic energy of a rigid body

mass density

\[ m = \int_B \rho(x, y, z) \, dx \, dy \, dz = \int_B dm \]

mass

\[ m = \int_B \rho(x, y, z) \, dx \, dy \, dz = \int_B dm \]

position of center of mass (CoM)

\[ r_c = \frac{1}{m} \int_B r \, dm \]

when all vectors are referred to a body frame \( RF_c \) attached to the CoM, then

\[ r_c = 0 \quad \Rightarrow \quad \int_B r \, dm = 0 \]

kinetic energy

\[ T = \frac{1}{2} \int_B v^T(x, y, z) \, v(x, y, z) \, dm \]

(fundamental) kinematic relation for a rigid body

\[ v = v_c + \omega \times r = v_c + S(\omega) r \]

skew-symmetric matrix

Robotics 2
Kinetic energy of a rigid body (cont)

\[ T = \frac{1}{2} \int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r) \, dm \]

\[ = \frac{1}{2} \int_B v_c^T v_c \, dm + \int_B v_c^T S(\omega) r \, dm + \frac{1}{2} \int_B r^T S^T(\omega) S(\omega) r \, dm \]

\[ = \frac{1}{2} m v_c^T v_c \]

translational kinetic energy (point mass at CoM)

König theorem

rotational kinetic energy (of the whole body)

body inertia matrix (around the CoM)

Euler matrix

\[ = \frac{1}{2} \omega^T I_c \omega \]

\[ = \frac{1}{2} \omega^T I_c \omega \]

Homework #1:
provide the expressions of the elements of Euler matrix \( J_c \)

Homework #2:
prove last equality and provide the expressions of the elements of inertia matrix \( I_c \)
Examples of body inertia matrices
homogeneous bodies of mass \( m \), with axes of symmetry

parallelepiped with sides \( a \) (length/height), \( b \) and \( c \) (base)

\[
I_c = \begin{pmatrix}
I_{xx} & I_{yy} \\
I_{yy} & I_{zz}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{12}m(b^2 + c^2) \\
\frac{1}{12}m(a^2 + c^2) \\
\frac{1}{12}m(a^2 + b^2)
\end{pmatrix}
\]

empty cylinder with length \( h \), and external/internal radius \( a \) and \( b \)

\[
I_c = \begin{pmatrix}
\frac{1}{2}m(a^2 + b^2) \\
\frac{1}{12}m(3(a^2 + b^2) + h^2) \\
I_{zz}
\end{pmatrix}
\]

\( I_{zz} = I_{yy} \)

\[
l_{zz}' = l_{zz} + m\left(\frac{h}{2}\right)^2
\]

(parallel) axis translation theorem

Steiner theorem

\[
l = l_c + m(r^T r \cdot E_{3 \times 3} - rr^T) = l_c + mS^T(r)S(r)
\]

body inertia matrix relative to the CoM
identity matrix
Homework #3: prove the last equality

... its generalization: changes on body inertia matrix due to a pure translation \( r \) of the reference frame
Robot kinetic energy

\[ T = \sum_{i=1}^{N} T_i \]

\[ T_i = T_i(q_j, \dot{q}_j; j \leq i) \]

- \( N \) rigid bodies (+ fixed base)
- Open kinematic chain

König theorem

\[ T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]

- Absolute velocity of the center of mass (CoM)
- Absolute angular velocity of whole body

i-th link (body) of the robot
Kinetic energy of a robot link

\[ T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]

\( \omega_i, I_{ci} \) should be expressed in the same reference frame, but the product \( \omega_i^T I_{ci} \omega_i \) is invariant w.r.t. any chosen frame in frame \( RF_{ci} \) attached to (the center of mass of) link \( i \)

\[ i I_{ci} = \begin{pmatrix}
\int (y^2 + z^2) dm & - \int xy \, dm & - \int xz \, dm \\
\int (x^2 + z^2) dm & \int (x^2 + y^2) dm & - \int yz \, dm \\
\text{symm} & \int (x^2 + y^2) dm & \int (x^2 + y^2) dm
\end{pmatrix} \]

constant!
Dependence of $T$ from $q$ and $\dot{q}$

\[
v_{ci} = J_{Li}(q)\dot{q} = \begin{pmatrix} 1 & : & i & | & 0 & : & 0 \end{pmatrix} \dot{q}
\]

\[
\omega_i = J_{Ai}(q)\dot{q} = \begin{pmatrix} 1 & : & i & | & 0 & : & 0 \end{pmatrix} \dot{q}
\]
Final expression of $T$

$T = \frac{1}{2} \sum_{i=1}^{N} \left( m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci} \omega_i \right)$

$= \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} m_i J_{Li}(q) J_{Li}(q) + J_{Ai}(q) I_{ci} J_{Ai}(q) \right) \dot{q}$

NOTE 1: in practice, NEVER use this formula (or partial Jacobians) for computing $T$ ⇒ a better method is available...

NOTE 2: I used previously the notation $B(q)$ for the robot inertia matrix ... (see past exams!)

robot (generalized) inertia matrix
- symmetric
- positive definite, $\forall q \Rightarrow$ always invertible
Robot potential energy

assumption: GRAVITY contribution only

\[ U = \sum_{i=1}^{N} U_i \]  \( N \) rigid bodies (+ fixed base)

\[ U_i = U_i(q_j; j \leq i) \] open kinematic chain

\[ U_i = -m_i g^T r_{0,ci} \]

\( \{ \) gravity acceleration vector \( \)

position of the center of mass of link \( i \)

\( \} \) typically expressed in \( RF_0 \)

dependence on \( q \)

\[
\begin{pmatrix}
  r_{0,ci} \\
  r_{i,ci}
\end{pmatrix} =
\begin{pmatrix}
  0 A_1(q_1) \ A_2(q_2) \cdots i^{-1} A_i(q_i)
\end{pmatrix}
\]

NOTE: need to work with homogeneous coordinates

constant in \( RF_i \)
Summarizing ...

**kinetic energy**

\[
T = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j
\]

**potential energy**

\[
U = U(q)
\]

**Lagrangian**

\[
L = T(q, \dot{q}) - U(q)
\]

**Euler-Lagrange equations**

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k \quad k = 1, \ldots, N
\]

**positive definite quadratic form**

\[
T \geq 0,
\]

\[
T = 0 \iff \dot{q} = 0
\]

**non-conservative (active/dissipative) generalized forces performing work on** \(q_k\) **coordinate**
Applying Euler-Lagrange equations
(the scalar derivation – see Appendix for vector format)

\[ L(q, \dot{q}) = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j - U(q) \]

\[ \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \dot{q}_j \quad \rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j \]

(dependences of elements on \( q \) are not shown)

\[ \frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial m_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j \quad - \quad \frac{\partial U}{\partial q_k} \]

LINEAR terms in ACCELERATION \( \ddot{q} \)

QUADRATIC terms in VELOCITY \( \dot{q} \)

NONLINEAR terms in CONFIGURATION \( q \)
The $k$-th dynamic equation is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

and

$$\sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \left( \frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

Exchanging "mute" indices $i, j$:

$$\cdots + \sum_{i,j} \frac{1}{2} \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \cdots$$

The Christoffel symbols of the first kind satisfy $c_{kij} = c_{kji}$.
... and interpretation of dynamic terms

\[
\sum_j m_{kj}(q) \ddot{q}_j + \sum_{i,j} c_{kij}(q) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k \quad k = 1, \ldots, N
\]

**INERTIAL** terms

**CENTRIFUGAL** \((i = j)\) and **CORIOLIS** \((i \neq j)\) terms

**GRAVITY** terms \(g_k(q)\)

\(m_{kk}(q) = \) inertia at joint \(k\) when joint \(k\) accelerates \((m_{kk} > 0!!)\)

\(m_{kj}(q) = \) inertia “seen” at joint \(k\) when joint \(j\) accelerates

\(c_{kii}(q) = \) coefficient of the centrifugal force at joint \(k\) when joint \(i\) is moving \((c_{iii} = 0, \forall i)\)

\(c_{kij}(q) = \) coefficient of the Coriolis force at joint \(k\) when joint \(i\) and joint \(j\) are both moving
Robot dynamic model
in vector formats

1. \[ M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u \]

\[ c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q} \]
\[ C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + \left( \frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right) \]

\[ k\text{-th component of vector } c \]

\[ k\text{-th column of matrix } M(q) \]

2. \[ M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = u \]

\[ s_{kj}(q, \dot{q}) = \sum_i c_{ki}q_j(q) \dot{q}_i \]

\[ \text{factorization of } c \text{ by } S \text{ is not unique!} \]

NOTE: the model is in the form \( \Phi(q, \dot{q}, \ddot{q}) = u \) as expected

Robotics 2
Dynamic model of a PR robot

\[ T = T_1 + T_2 \quad U = \text{constant} \Rightarrow g(q) \equiv 0 \]
(on horizontal plane)

\[ p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \quad \|v_{c1}\|^2 = p_{c1}^T \dot{p}_{c1} = \dot{q}_1^2 \]

\[ T_1 = \frac{1}{2} m_1 \dot{q}_1^2 \]

\[ T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2 \]

\[ p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \quad v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \dot{q}_2 \\ d_{c2} \cos q_2 \dot{q}_2 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} \]

\[ T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2 d_{c2} \sin q_2 \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2 \]
Dynamic model of a PR robot (cont)

where

\[
M(q) = \begin{pmatrix}
    m_1 + m_2 & -m_2d_{c2}\sin q_2 \\
    -m_2d_{c2}\sin q_2 & I_{c2,zz} + m_2d_{c2}^2
\end{pmatrix}
\]

\[
c(q, \dot{q}) = \begin{pmatrix}
c_1(q, \dot{q}) \\
c_2(q, \dot{q})
\end{pmatrix}
\]

\[
c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}
\]

\[
C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + (\frac{\partial M_k}{\partial q})^T - \frac{\partial M}{\partial q_k} \right)
\]

\[
C_1(q) = \frac{1}{2} \left( \begin{pmatrix} 0 & 0 \\ 0 & -m_2d_{c2}\cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2d_{c2}\cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)
\]

\[
C_1(q, \dot{q}) = -m_2d_{c2}\cos q_2 \dot{q}_2^2
\]

\[
C_2(q) = \frac{1}{2} \left( \begin{pmatrix} 0 & -m_2d_{c2}\cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2d_{c2}\cos q_2 & 0 \end{pmatrix} \right) = 0
\]

\[
c_2(q, \dot{q}) = 0
\]
Dynamic model of a PR robot (cont)

\[ M(q)\ddot{q} + c(q, \dot{q}) = u \]

\[
\begin{pmatrix}
  m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\
  -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 
\end{pmatrix}
\begin{pmatrix}
  \ddot{q}_1 \\
  \ddot{q}_2 
\end{pmatrix}
+
\begin{pmatrix}
  -m_2 d_{c2} \cos q_2 \dot{q}_2^2 \\
  0 
\end{pmatrix}
=
\begin{pmatrix}
  u_1 \\
  u_2 
\end{pmatrix}
\]

**NOTE:** the \( m_{NN} \) element (here, for \( N = 2 \)) of \( M(q) \) is always **constant**!

**Q1:** why does variable \( q_1 \) not appear in \( M(q) \)? ... this is a general property!

**Q2:** why Coriolis terms are not present?

**Q3:** when applying a force \( u_1 \), does the second joint accelerate? ... always?

**Q4:** what is the expression of a factorization matrix \( S \)? ... is it unique here?

**Q5:** which is the configuration with “maximum inertia”?
A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix $S$)

Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \quad 2s_{kj} = \sum_i 2c_{kij} \dot{q}_i = \sum_i \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i$$

$$\dot{m}_{kj} - 2s_{kj} = \sum_i \left( \frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left( \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k} \right) \dot{q}_i = -n_{kj}$$

using the symmetry of $M$

$$x^T(\dot{M} - 2S)x = 0, \forall x$$
Energy conservation

- total robot energy
  \[ E = T + U = \frac{1}{2} \dot{q}^T M(q) \dot{q} + U(q) \]
- its evolution over time (using the dynamic model)
  \[ \dot{E} = \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \ddot{M}(q) \dot{q} + \frac{\partial U}{\partial q} \dot{q} \]
  \[ = \dot{q}^T (u - S(q, \dot{q}) \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \ddot{M}(q) \dot{q} + \dot{q}^T g(q) \]
  \[ = \dot{q}^T u + \frac{1}{2} \dot{q}^T \left( \ddot{M}(q) - 2S(q, \dot{q}) \right) \dot{q} \]
- if \( u \equiv 0 \), total energy is constant (no dissipation or increase)
  \[ \dot{E} = 0 \quad \Rightarrow \quad \dot{q}^T \left( \ddot{M}(q) - 2S(q, \dot{q}) \right) \dot{q} = 0, \forall q, \dot{q} \]

here, any factorization of vector \( c \) by a matrix \( S \) can be used

weaker property than skew-symmetry, as the external vector in the quadratic form is the same velocity \( \dot{q} \) that appears also inside the two internal matrices \( \ddot{M} \) also \( S \)

in general, the variation of the total energy is equal to the work of non-conservative forces
Appendix

dynamic model: alternative vector format derivation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T - \left( \frac{\partial L}{\partial q} \right)^T = u
\]

\[L = \frac{1}{2} \dot{q}^T M(q) \dot{q} - U(q)\]

\[M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^{N} M_i(q) e_i^T\]

\[\left( \frac{\partial L}{\partial \dot{q}} \right)^T = (\dot{q}^T M(q))^T = M(q) \ddot{q}\]

\[\text{dyadic expansion}\]

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T = M(q) \ddot{q} + \dot{M}(q) \dot{q} = M(q) \ddot{q} + \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} \right) \dot{q} \dot{q}_i\]

\[\left( \frac{\partial L}{\partial q} \right)^T = \left( \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} \frac{\partial M_i(q)}{\partial q} e_i^T \right) \dot{q} - \frac{\partial U(q)}{\partial q} \right)^T = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} \right)^T \dot{q}_i \dot{q} - \left( \frac{\partial U}{\partial q} \right)^T\]

this construction gives to $\dot{M} - 2S$

skew-symmetry

\[M(q) \ddot{q} + \left( \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} - \frac{1}{2} \left( \frac{\partial M_i}{\partial q} \right)^T \right) \dot{q}_i \right) \dot{q} + \left( \frac{\partial U}{\partial q} \right)^T = u\]

\[S_k^T(q, \dot{q}) = \dot{q}^T C_k(q)\]

\[S(q, \dot{q}) \rightarrow g(q)\]

Robotics 2