Consider the robot in Figure 1, having four revolute joints. The Denavit-Hartenberg frames are already placed, with frame 0 located at the intersection of the first and second joint axis. The configuration shown corresponds (approximately) to \( \theta \simeq (0 \ 6\pi/10 \ \pi \ 6\pi/10)^T \) [rad] (or, equivalently, \( \theta \simeq (0 \ 108 \ 180 \ 108)^T \) [deg]).

Let the robot be in the configuration \( \theta^* = (0 \ 3\pi/4 \ \pi \ \pi)^T \) [rad], and set \( L = 1 \) [m] in the following if you plan to work in a numerical way.

1. Obtain the \( 6 \times 4 \) geometric Jacobian \( J(\theta^*) \).

2. Show that the following Cartesian linear/angular velocity vector is feasible:
\[
\begin{pmatrix}
 v_d^T \\
 \omega_d^T
\end{pmatrix} =
\begin{pmatrix}
 0 & 0 & -L & -\sqrt{2}/2 & 0
\end{pmatrix}.
\]

3. Determine the minimum norm joint velocity vector \( \dot{\theta} \) realizing the above Cartesian velocity.

4. Compute the joint torque vector \( \tau \) that keeps the robot in static equilibrium when the following Cartesian force/torque vector is applied from the environment to the end-effector:
\[
\begin{pmatrix}
 F^T \\
 M^T
\end{pmatrix} =
\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

5. Consider only the velocity \( v \) of point \( P \). Verify whether the associated \( 3 \times 4 \) Jacobian \( J_L(\theta) \) is singular or not in the configuration \( \theta^* \).

[120 minutes; open books]
Solution

December 17, 2009

The 4R spatial manipulator is made by the subset of first four joints of the DLR manipulator considered in the textbook (p. 79, Fig. 2.29)\(^1\). However, the fourth (and last) reference frame is different, due to the missing axes 5, 6, and 7. The Denavit-Hartenberg parameters are given in Table 1 (the first three rows are those of Table 2.7 in the textbook, with \(d_3 = L\)).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\alpha_i)</th>
<th>(a_i)</th>
<th>(d_i)</th>
<th>(\theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\pi/2)</td>
<td>0</td>
<td>0</td>
<td>(\theta_1)</td>
</tr>
<tr>
<td>2</td>
<td>(\pi/2)</td>
<td>0</td>
<td>0</td>
<td>(\theta_2)</td>
</tr>
<tr>
<td>3</td>
<td>(\pi/2)</td>
<td>0</td>
<td>(L)</td>
<td>(\theta_3)</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>(L)</td>
<td>0</td>
<td>(\theta_4)</td>
</tr>
</tbody>
</table>

Table 1: Denavit-Hartenberg parameters

The associated homogeneous transformation matrices are:

\[ ^0A_{1}(\theta_1) = \begin{pmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ -\sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ^0R_{1}(\theta_1) & 0 \\ 0^T & 1 \end{pmatrix}, \]

\[ ^1A_{2}(\theta_2) = \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ^1R_{2}(\theta_2) & 0 \\ 0^T & 1 \end{pmatrix}, \]

\[ ^2A_{3}(\theta_3) = \begin{pmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0 \\ -\sin \theta_3 & 0 & \cos \theta_3 & 0 \\ 0 & 1 & 0 & \(L\) \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ^2R_{3}(\theta_3) & ^2p_{23} \\ 0^T & 1 \end{pmatrix}, \]

\[ ^3A_{4}(\theta_4) = \begin{pmatrix} \cos \theta_4 & -\sin \theta_4 & 0 & \(L\cos \theta_4\) \\ \sin \theta_4 & \cos \theta_4 & 0 & \(L\sin \theta_4\) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ^3R_{4}(\theta_4) & ^3p_{34}(\theta_4) \\ 0^T & 1 \end{pmatrix}. \]

The 6 × 4 geometric Jacobian

\[ J(\theta) = \begin{pmatrix} J_L(\theta) \\ J_A(\theta) \end{pmatrix} \]

can be computed symbolically or numerically for a given configuration. We present first the general symbolic derivation, and then a more direct numerical approach.

\(^1\)Note that in Fig. 2.29 the \(x_1\), \(x_2\), and \(x_3\) axes are drawn in a wrong way. The associated Table 2.7 of DH parameters is instead correct for the full 7R arm.
The $3 \times 4$ upper part $J_L$ of the geometric Jacobian relates $\dot{\theta}$ to the velocity $v$ of point $P$. It can be obtained either by (analytic) differentiation of $p_04$, i.e., by computing this vector as

$$
\begin{pmatrix}
  p_{04}(\theta) \\
  1
\end{pmatrix} = 0 A_1(\theta_1)^4 A_2(\theta_2)^2 A_3(\theta_3)^3 A_4(\theta_4) \begin{pmatrix} 
  0 \\
  1
\end{pmatrix}
$$

and obtaining then

$$J_L(\theta) = \frac{\partial p_{04}(\theta)}{\partial \theta},$$

or by the geometric formula

$$J_L(\theta) = \left( z_0 \times p_{04} \ z_1 \times p_{04} \ z_2 \times p_{04} \ z_3 \times (p_{04} - p_{03}) \right),$$

where we used the fact that $p_{00} = p_{01} = p_{02} = 0$ (the origins of frames 0, 1, and 2 coincide).

Thus, for deriving its explicit symbolic form we need

$$p_{04} = L \begin{pmatrix} 
  \cos \theta_1 \sin \theta_2 + \cos \theta_1 \sin \theta_2 \sin \theta_4 + (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
  \sin \theta_1 \sin \theta_2 + \sin \theta_1 \sin \theta_2 \sin \theta_4 - (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
  - \cos \theta_2 - \cos \theta_2 \sin \theta_4 + \sin \theta_2 \cos \theta_3 \cos \theta_4
\end{pmatrix},$$

and, when following the geometric construction, also

$$p_{04} - p_{03} = L \begin{pmatrix} 
  \cos \theta_1 \sin \theta_2 \sin \theta_4 + (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
  \sin \theta_1 \sin \theta_2 \sin \theta_4 - (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
  - \cos \theta_2 \sin \theta_4 + \sin \theta_2 \cos \theta_3 \cos \theta_4
\end{pmatrix}$$

as well as

$$z_0 = \begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix}
$$

$$z_1 = 0 R_1(\theta_1) \begin{pmatrix} 
  0 \\
  0 \\
  1
\end{pmatrix} = \begin{pmatrix} 
  \sin \theta_1 \\
  - \cos \theta_1 \\
  0
\end{pmatrix}
$$

$$z_2 = 0 R_1(\theta_1)^1 R_2(\theta_2) \begin{pmatrix} 
  0 \\
  0 \\
  1
\end{pmatrix} = \begin{pmatrix} 
  \cos \theta_1 \sin \theta_2 \\
  \sin \theta_1 \sin \theta_2 \\
  - \cos \theta_2
\end{pmatrix}
$$

$$z_3 = 0 R_1(\theta_1)^1 R_2(\theta_2)^2 R_3(\theta_3) \begin{pmatrix} 
  0 \\
  0 \\
  1
\end{pmatrix} = \begin{pmatrix} 
  - \sin \theta_1 \cos \theta_3 + \cos \theta_1 \cos \theta_2 \sin \theta_3 \\
  \cos \theta_1 \cos \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3 \\
  \sin \theta_2 \sin \theta_3
\end{pmatrix}.$$

Performing symbolic computations\(^2\), and factoring out the length $L$, we obtain

$$J_L(\theta) = L \cdot \left( J_{L,1} \ J_{L,2} \ J_{L,3} \ J_{L,4} \right),$$

\(^2\)When using the Matlab Symbolic Toolbox, take advantage of the simplify instruction to reduce the length/complexity of terms.
where:

\[
J_{L,1} = \begin{pmatrix}
-\sin \theta_1 \sin \theta_2 - \sin \theta_1 \sin \theta_2 \sin \theta_4 + (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
\cos \theta_1 \sin \theta_2 + \cos \theta_1 \sin \theta_2 \sin \theta_4 + (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3) \cos \theta_4 \\
0
\end{pmatrix}
\]

\[
J_{L,2} = \begin{pmatrix}
\cos \theta_1 (\cos \theta_2 + \cos \theta_2 \sin \theta_4 - \sin \theta_2 \cos \theta_3 \cos \theta_4) \\
\sin \theta_1 (\cos \theta_2 + \cos \theta_2 \sin \theta_4 - \sin \theta_2 \cos \theta_3 \cos \theta_4) \\
\sin \theta_2 + \sin \theta_2 \sin \theta_4 + \cos \theta_2 \cos \theta_3 \cos \theta_4
\end{pmatrix}
\]

\[
J_{L,3} = \begin{pmatrix}
\sin \theta_1 \cos \theta_3 - \cos \theta_1 \cos \theta_2 \sin \theta_3 \cos \theta_4 \\
-(\cos \theta_1 \cos \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3) \cos \theta_4 \\
-\sin \theta_2 \sin \theta_3 \cos \theta_4
\end{pmatrix}
\]

\[
J_{L,4} = \begin{pmatrix}
\cos \theta_1 \sin \theta_2 \cos \theta_4 - (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3) \sin \theta_4 \\
\sin \theta_1 \sin \theta_2 \cos \theta_4 + (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \cos \theta_3) \sin \theta_4 \\
-\cos \theta_2 \cos \theta_4 - \sin \theta_2 \cos \theta_3 \sin \theta_4
\end{pmatrix}
\]

The 3 × 4 lower part \(J_A\) of the geometric Jacobian, relating \(\dot{\theta}\) to the angular velocity \(\omega\) of frame 4, is given instead by

\[
J_A(\theta) = \begin{pmatrix}
z_0 \\
z_1 \\
z_2 \\
z_3
\end{pmatrix},
\]

where the previous symbolic expressions for \(z_i, i = 0,1,2,3\), are used.

At this stage, the elements of the Jacobian matrix \(J(\theta)\) should be evaluated at the given configuration

\[
\theta^* = \begin{pmatrix} 0 & 3\pi/4 & \pi & \pi \end{pmatrix}^T.
\]

In this configuration, the end-effector (the origin of frame 4) is positioned along the axis of joint 1.

Alternatively (and in a much faster way for the problem at hand!), we may first evaluate numerically the homogeneous transformations at the configuration \(\theta^*\), using in this case also \(L = 1\), and then perform all the required operations, including products of matrices and (vector) cross products, so as to obtain the numerical value of the geometric Jacobian. The Matlab code is:

\[
\text{th1}=0; \\
\text{th2}=3*pi/4; \\
\text{th3}=pi; \\
\text{th4}=pi; \\
\text{L}=1; \\
\text{A1} = [\text{cos(th1)} 0 \text{sin(th1)} 0; 0 1 0 0; 0 0 0 1]; \\
\text{A2} = [\text{cos(th2)} 0 \text{sin(th2)} 0; \text{sin(th2)} 0 -\text{cos(th2)} 0];
\]

% configuration data

\[
\text{th1}=0; \\
\text{th2}=3*pi/4; \\
\text{th3}=pi; \\
\text{th4}=pi; \\
\text{L}=1; 
\]

% homogeneous transformations

\[
\text{A1} = [\text{cos(th1)} 0 \text{sin(th1)} 0; 0 1 0 0; 0 0 0 1]; \\
\text{A2} = [\text{cos(th2)} 0 \text{sin(th2)} 0; \text{sin(th2)} 0 -\text{cos(th2)} 0];
\]
\[
A_3 = \begin{bmatrix}
\cos(\theta_3) & 0 & \sin(\theta_3) & 0 \\
\sin(\theta_3) & 0 & -\cos(\theta_3) & 0 \\
0 & 1 & 0 & L \\
0 & 0 & 0 & 1
\end{bmatrix};
\]

\[
A_4 = \begin{bmatrix}
\cos(\theta_4) & -\sin(\theta_4) & 0 & L \cos(\theta_4) \\
\sin(\theta_4) & \cos(\theta_4) & 0 & L \sin(\theta_4) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix};
\]

\[
A_{12} = A_1 \cdot A_2;
\]

\[
A_{13} = A_{12} \cdot A_3;
\]

\[
A_{14} = A_{13} \cdot A_4;
\]

% geometric Jacobian

\[
z_0 = [0, 0, 1]';
\]

\[
z_1 = A_1(1:3,3);
\]

\[
z_2 = A_{12}(1:3,3);
\]

\[
z_3 = A_{13}(1:3,3);
\]

\[
p_0 = [0, 0, 0]';
\]

\[
p_1 = A_1(1:3,4);
\]

\[
p_2 = A_{12}(1:3,4);
\]

\[
p_3 = A_{13}(1:3,4);
\]

\[
p_4 = A_{14}(1:3,4);
\]

\[
J(1:3,1) = \text{cross}(z_0, p_4 - p_0);
\]

\[
J(1:3,2) = \text{cross}(z_1, p_4 - p_1);
\]

\[
J(1:3,3) = \text{cross}(z_2, p_4 - p_2);
\]

\[
J(1:3,4) = \text{cross}(z_3, p_4 - p_3);
\]

\[
J(4:6,1) = z_0;
\]

\[
J(4:6,2) = z_1;
\]

\[
J(4:6,3) = z_2;
\]

\[
J(4:6,4) = z_3;
\]

% end

Whatever approach is followed, one ends up with the following matrix (where \( L = 1 \), if we have
worked numerically):

\[
J(\theta^*) = \begin{bmatrix}
0 & -L\sqrt{2} & 0 & -L\frac{\sqrt{2}}{2} \\
0 & 0 & -L & 0 \\
0 & 0 & 0 & -L\frac{\sqrt{2}}{2} \\
0 & 0 & \frac{\sqrt{2}}{2} & 0 \\
0 & -1 & 0 & -1 \\
1 & 0 & \frac{\sqrt{2}}{2} & 0 \\
\end{bmatrix}.
\]

It can be seen that the rank of \(J_L(\theta^*)\) is 3, and thus the given configuration \(\theta^*\) is not singular for this sub-Jacobian. By inspection of this matrix, the desired linear/angular velocity vector \((v_d^T \ \omega_d^T)^T\) is realized by choosing

\[
\dot{\theta}_d = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}^T,
\]

obtaining in fact

\[
J(\theta^*)\dot{\theta}_d = \begin{bmatrix} 0 \\
0 \\
0 \\
-\sqrt{2} \\
0 \\
0 \end{bmatrix}.
\]

Moreover, one can see that the joint velocity vector \(\dot{\theta}_d\) is the only one providing the desired linear/angular velocity. Therefore, \(\dot{\theta}_d\) is the minimum norm solution (with \(\|\dot{\theta}_d\| = 1.5811\)). As a check, it can be verified that \(J^\#(\theta^*)\left(\begin{bmatrix} v_d^T \\
\omega_d^T \end{bmatrix}\right) = \dot{\theta}_d\), where the pseudoinverse \(J^\#(\theta^*)\) can be computed either by using the Matlab function \texttt{pinv} or by its explicit expression in case of a full (column) rank matrix \(J\) with more rows than columns,

\[
J^\# = (J^T J)^{-1} J^T,
\]

which applies to the present case since the rank of \(J(\theta^*)\) is 4. Finally, the joint torque vector \(\tau\) that balances the specified Cartesian force/torque vector \((F^T \ \ M^T)^T\) is computed as

\[
\tau = -J^T(\theta^*)\begin{bmatrix} 1 \\
0 \\
0 \\
0 \\
0 \\
L \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\
L \sqrt{2} \\
0 \end{bmatrix},
\]

i.e., it is given by the transpose of the first row of \(J(\theta^*)\), changed of sign (the usual convention holds also for joint torques: positive torques are counterclockwise).