Robotics 1

January 12, 2026

[students with midterm]

Exercise 1

For the RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ defined in the YZX sequence around fixed axes, compute the map $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity vector $\boldsymbol{\omega} \in \mathbb{R}^3$ and find the singularities of the matrix $\boldsymbol{T}(\boldsymbol{\phi})$.

Exercise 2

For the velocity transformation $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$, where $\boldsymbol{J}(\boldsymbol{q})$ is the 2×2 analytic Jacobian of a 2R planar robot with link lengths $l_1 = 1$, $l_2 = 0.5$ [m] and $\boldsymbol{p} \in \mathbb{R}^2$ is its end-effector position, build three case studies of the pair $(\boldsymbol{q}, \dot{\boldsymbol{p}})$ for which: *i)* the solution $\dot{\boldsymbol{q}}$ is unique; *ii)* there are infinite solutions $\dot{\boldsymbol{q}}$ and you choose the one with minimum norm $\|\dot{\boldsymbol{q}}\|$; *iii)* there is no solution $\dot{\boldsymbol{q}}$ and you choose the joint velocity with minimum norm that minimizes also the error norm $\|\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}}\|$. Sketch graphically the associated situations in the joint velocity plane $(\dot{q}_1, \dot{q}_2)$ and provide the numerical values of $\dot{\boldsymbol{q}}$ for the three case studies.

Exercise 3

Compute the 6×3 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of a 3-dof robot whose DH parameters are given in Tab. 1. Find all configurations $\boldsymbol{q}_s$ at which the Jacobian matrix loses rank. In one of these configurations, determine a basis for the null space $\mathcal{N}\{\boldsymbol{J}_s\}$ of $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$ and for the range space $\mathcal{R}\{\boldsymbol{J}_s^T\}$ of $\boldsymbol{J}_s^T$. Provide a graphical sketch of the robot in this situation and give a corresponding physical interpretation in terms of joint velocities $\dot{\boldsymbol{q}}$ and torques $\boldsymbol{\tau}$.

i	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	$\frac{\pi}{2}$	0	0	q_2
3	0	0	q_3	0

Table 1: Table of DH parameters for a 3-dof robot

Exercise 4

Consider a trapezoidal speed profile for a joint that has to move in minimum time from q_i to q_f, under the bounds $|\dot{q}| \leq V$ and $|\ddot{q}| \leq A$. Draw the position, velocity, and acceleration profiles and compute the relevant parameters for the data $q_i = \pi$, $q_f = \pi/4$ [rad] and the bounds $V = 3$ rad/s and $A = 5$ rad/s^2. Assume now that the same displacement should occur under the same bounds in a motion time $\bar{T}$ that is twice as long as the minimum feasible time T, by using a trapezoidal speed profile that has the same duration T_r of the acceleration/deceleration phases of the minimum time motion. Compute the associated velocity $\bar{V}$ and acceleration $\bar{A}$ of the new trajectory and draw the corresponding position, velocity, and acceleration profiles.

[210 minutes (3,5 hours); open books]

Solution

January 12, 2026
[students with midterm]

Exercise 1

The elementary rotation matrices involved with the given RPY-type sequence $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ are:

$$\boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \boldsymbol{R}_z(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{R}_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}.$$

Since RPY-type rotations are defined around fixes axes, the final orientation for the YZX sequence is obtained by the product of the elementary rotation matrices in the reverse order of definition:

$$\boldsymbol{R}_{YZX}(\boldsymbol{\phi}) = \boldsymbol{R}_x(\gamma)\boldsymbol{R}_z(\beta)\boldsymbol{R}_y(\alpha).$$

For the map $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity vector $\boldsymbol{\omega} \in \mathbb{R}^3$, the algebraic construction provides the sum of three contributions

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\gamma}} + \boldsymbol{\omega}_{\dot{\beta}} + \boldsymbol{\omega}_{\dot{\alpha}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\gamma} + \boldsymbol{R}_x(\gamma) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\beta} + \boldsymbol{R}_x(\gamma)\boldsymbol{R}_z(\beta) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\alpha},$$

which is reorganized in matrix form as

$$\boldsymbol{\omega} = \begin{pmatrix} -\sin\beta & 0 & 1 \\ \cos\beta\cos\gamma & -\sin\gamma & 0 \\ \cos\beta\sin\gamma & \cos\gamma & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

The determinant of the transformation matrix is $\det \boldsymbol{T}(\boldsymbol{\phi}) = \cos\beta$, so that this matrix is singular at $\beta = \pm\pi/2$ (exactly when the inverse problem of representing orientation by the sequence YZX of RPY-type angles has infinite solutions). When in a singularity, angular velocities of the form $\boldsymbol{\omega} = k\,(0, \cos\gamma, \sin\gamma)$, with $k \neq 0$, cannot be generated by any $\dot{\boldsymbol{\phi}}$.

Exercise 2

At a given configuration $\boldsymbol{q}$ and for an assigned $\dot{\boldsymbol{p}}$, the linear system of equations $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$ in the unknown $\dot{\boldsymbol{q}}$, with $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q})$ being a square matrix:

i) has the unique solution $\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}}$ if $\det \boldsymbol{J} \neq 0$;

ii) if $\det \boldsymbol{J} = 0$ and $\dot{\boldsymbol{p}} \in \mathcal{R}(\boldsymbol{J})$, has infinite solutions and $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}}$ is the solution with minimum norm, being $\boldsymbol{J}^{\#}$ the (unique) pseudoinverse of $\boldsymbol{J}$;

iii) if $\det \boldsymbol{J} = 0$ and $\dot{\boldsymbol{p}} \notin \mathcal{R}(\boldsymbol{J})$, has no solutions and $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}}$ is the joint velocity with minimum norm among all those that minimize the norm of the velocity error $\dot{\boldsymbol{e}} = \boldsymbol{J}\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}}$.

For a 2×2 matrix $\boldsymbol{J}$, let

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \end{pmatrix} \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}.$$

Then, the two linear equations

$$\boldsymbol{J}_1\dot{\boldsymbol{q}} = J_{11}\dot{q}_1 + J_{12}\dot{q}_2 = \dot{p}_x \qquad\qquad \boldsymbol{J}_2\dot{\boldsymbol{q}} = J_{21}\dot{q}_1 + J_{22}\dot{q}_2 = \dot{p}_y$$

can be represented in the plane $(\dot{q}_1, \dot{q}_2)$ as two lines that may or may not intersect. The above three cases *i)–iii)* are shown in Fig. 1. When $\det \boldsymbol{J} = 0$, the two rows are linearly dependent, i.e., $\boldsymbol{J}_2 = k\boldsymbol{J}_1$ for some $k \neq 0$; moreover, if $\dot{p}_y \neq k\dot{p}_x$ the two equations are inconsistent and there is no exact solution for both. The dashed line of the right picture in Fig. 1 is equidistant from the other two and represents the set of joint velocities $\dot{\boldsymbol{q}}$ yielding the minimum norm of the velocity error.

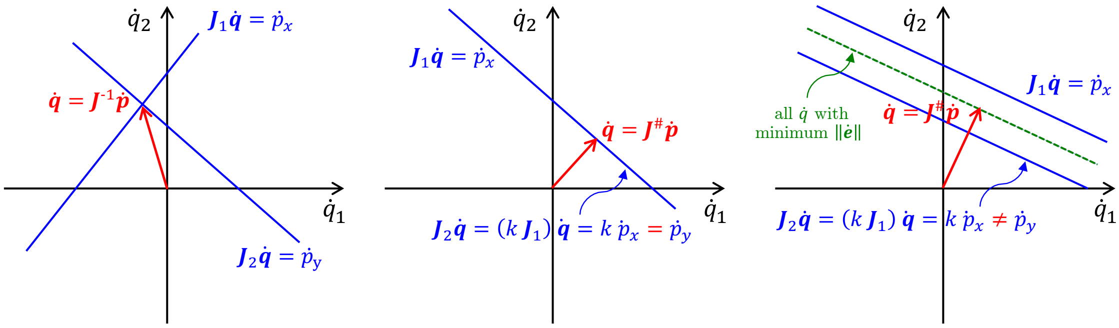

Figure 1: The three possible situations for $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$ in the two-dimensional case

One special case which is not represented in Fig. 1 is when one of the two equations has all zero coefficients —say, $\boldsymbol{J}_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$; as a consequence, the line associated to the second equation cannot be drawn. Then, for case *ii)*, it is necessarily $\dot{p}_y = 0$ and this second equation vanishes (it is simply the identity $0 = 0$), so that the linear system has only one equation in two unknowns and thus infinite solutions; instead, case *iii)* has $\dot{p}_y \neq 0$ and all the solutions to the first equation $\boldsymbol{J}_1\dot{\boldsymbol{q}} = \dot{p}_x$ will have the same error norm $\|\dot{\boldsymbol{e}}\| = |\dot{p}_y| \neq 0$. In all these situations, use of the pseudoinverse provides again the joint velocity with minimum norm.

Let us turn to some numerical examples. For the considered 2R planar robot, one has

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) \qquad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix},$$

with $l_1 = 1$, $l_2 = 0.5$ [m]. The determinant is $\det \boldsymbol{J}(\boldsymbol{q}) = l_1 l_2 \sin q_2$. Consider the end-effector velocity $\dot{\boldsymbol{p}} = (-1, 1)$ [m/s] and choose first the configuration $\boldsymbol{q} = (0, \pi/2)$ [rad], for which

$$\boldsymbol{J} = \begin{pmatrix} -0.5 & -0.5 \\ 1 & 0 \end{pmatrix}$$

is clearly nonsingular. Then, the unique solution is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [rad/s]}.$$

Move next the robot to the singular configuration $\boldsymbol{q} = (\pi/4, 0)$ [rad]. The Jacobian becomes

$$\boldsymbol{J} = \begin{pmatrix} -1.5/\sqrt{2} & -1/\sqrt{2} \\ 1.5/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1.0607 & -0.3536 \\ 1.0607 & 0.3536 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix}$$

which is clearly singular, being $\boldsymbol{J}_2 = k\boldsymbol{J}_1$, with $k = -1$. However, since $\dot{p}_y = 1 = -1 \cdot -1 = k\,\dot{p}_x$, we are in case *ii)* and the pseudoinverse solution will provide no error with respect to the desired end-effector velocity.

Interlude. In the absence of a numerical tool (e.g., Matlab) for computing the pseudoinverse of a singular matrix, which would require in general the SVD decomposition of $\boldsymbol{J}$, one can use direct formulas that exploit the simple structure of the matrix to be pseudoinverted.[1] In fact, it is easy to verify that:

- for a n-dimensional (column or row) vector $\boldsymbol{a} \neq \boldsymbol{0}$, the pseudoinverse is $\boldsymbol{a}^{\#} = \boldsymbol{a}^T/\|\boldsymbol{a}\|^2$;
- for a $2 \times n$ matrix $\boldsymbol{A}$ with a row of zeros (or an $n \times 2$ matrix $\boldsymbol{B}$ with a column of zeros), the pseudoinverse is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{0} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{a}^{\#} & \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{0} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{B}^{\#} = \begin{pmatrix} \boldsymbol{b}^{\#} \\ \boldsymbol{0} \end{pmatrix},$$

  and similarly when the zeros are in the first row (or column);
- for a singular (i.e., not full rank) $2 \times n$ matrix $\boldsymbol{J}$,

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ k\boldsymbol{J}_1 \end{pmatrix} \qquad k \neq 0,$$

  the pseudoinverse is computed from the previous results using the factorization

$$\boldsymbol{J} = \boldsymbol{B}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ k \end{pmatrix},$$

  yielding

$$\boldsymbol{J}^{\#} = \boldsymbol{A}^{\#}\boldsymbol{B}^{\#} = \begin{pmatrix} \boldsymbol{J}_1^{\#} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{b}^{\#} \\ \boldsymbol{0} \end{pmatrix}.$$

With the above in mind, one can compute

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}} = \begin{pmatrix} -0.4243 & 0.4243 \\ -0.1414 & 0.1414 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8485 \\ 0.2828 \end{pmatrix} \text{ [rad/s]}.$$

It is easy to verify that this joint velocity is a correct solution, returning the desired end-effector velocity ($\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$).

Finally, suppose that the robot is in a different singular configuration, e.g., in $\boldsymbol{q} = (0, 0)$, namely with both links stretched in the x_0-direction. Then, since

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 \\ 1.5 & 0.5 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin \mathcal{R}(\boldsymbol{J}) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

we are in case *iii)*. The joint velocity computed using the previous pseudoinverse formulas

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}} = \begin{pmatrix} 0 & 0.6 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \end{pmatrix} \text{ [rad/s]},$$

will return only the part of the desired end-effector velocity that lies in the range of $\boldsymbol{J}$,

$$\dot{\boldsymbol{p}}^{\perp} = \boldsymbol{J}\dot{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \dot{\boldsymbol{p}},$$

[1] These formulas have been presented also during the lectures in the classroom.

namely the second component only. The end-effector velocity error has $\|\dot{\boldsymbol{e}}\| = 1$, which is the smallest possible norm for any $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. The joint velocity computed with the pseudoinverse has $\|\dot{\boldsymbol{q}}\| = \sqrt{0.6^2 + 0.2^2} = \sqrt{0.4} = 0.6325$ rad/s, which is the smallest norm for all joint velocities that achieve the minimum value for the norm of $\dot{\boldsymbol{e}}$.

Exercise 3

The DH parameters in Tab. 1 correspond to an RRP robot. Thus, the geometric Jacobian in

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

takes the form

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{0,3} & \boldsymbol{z}_1 \times \boldsymbol{p}_{1,3} & \boldsymbol{z}_2 \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}, \tag{1}$$

where

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 \qquad \boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0,$$

$$\begin{pmatrix} \boldsymbol{p}_{0,3} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right) \qquad \begin{pmatrix} \boldsymbol{p}_{0,1} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \qquad \boldsymbol{p}_{1,3} = \boldsymbol{p}_{0,3} - \boldsymbol{p}_{0,1}.$$

Remember that the 3×3 linear part of the geometric Jacobian (1) can be equivalently obtained by analytic differentiation of the direct kinematics as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_{0,3}}{\partial \boldsymbol{q}}, \tag{2}$$

which may be easier to compute by hand.

With the DH parameters, we build the DH homogeneous transformation matrices

$${}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1 & \boldsymbol{p}_{0,1} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & L\cos q_1 \\ \sin q_1 & \cos q_1 & 0 & L\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2 & \boldsymbol{p}_{1,2} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2\boldsymbol{A}_2(q_3) = \begin{pmatrix} {}^0\boldsymbol{R}_1 & \boldsymbol{p}_{2,3} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and thus

$$\boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{z}_2 = \begin{pmatrix} \sin(q_1 + q_2) \\ -\cos(q_1 + q_2) \\ 0 \end{pmatrix}$$

and

$$
\boldsymbol{p}_{0,3} = \begin{pmatrix} L\cos q_1 + q_3\sin(q_1+q_2) \\ L\sin q_1 - q_3\cos(q_1+q_2) \\ 0 \end{pmatrix} \qquad \boldsymbol{p}_{0,1} = \begin{pmatrix} L\cos q_1 \\ L\sin q_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{p}_{1,3} = \begin{pmatrix} q_3\sin(q_1+q_2) \\ -q_3\cos(q_1+q_2) \\ 0 \end{pmatrix}
$$

Using (2) for the linear part, we obtain the expression of the 6×3 geometric Jacobian in (1) as

$$
\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} q_3\cos(q_1+q_2) - L\sin q_1 & q_3\cos(q_1+q_2) & \sin(q_1+q_2) \\ q_3\sin(q_1+q_2) + L\cos q_1 & q_3\sin(q_1+q_2) & -\cos(q_1+q_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3}
$$

The three rows that are identically zero in $\boldsymbol{J}(\boldsymbol{q})$ reveal that this robot is planar, i.e., it moves in the plane (x_0, y_0), being $v_z = \omega_x = \omega_y = 0$ for any possible joint motion.

To analyze the rank of $\boldsymbol{J}(\boldsymbol{q})$, we can simply eliminate the zero rows from (3) and obtain the 3×3 reduced matrix

$$
\boldsymbol{J}_r(\boldsymbol{q}) = \begin{pmatrix} q_3\cos(q_1+q_2) - L\sin q_1 & q_3\cos(q_1+q_2) & \sin(q_1+q_2) \\ q_3\sin(q_1+q_2) + L\cos q_1 & q_3\sin(q_1+q_2) & -\cos(q_1+q_2) \\ 1 & 1 & 0 \end{pmatrix},
$$

whose determinant is $\det \boldsymbol{J}_r(\boldsymbol{q}) = L\sin q_2$. Therefore, all singular configurations $\boldsymbol{q}_s$ of $\boldsymbol{J}(\boldsymbol{q})$ are characterized by having $\sin q_2 = 0$, i.e., $q_2 = \{0, \pi\}$. Taking for instance $q_2 = 0$, we get

$$
\boldsymbol{J}_s(q_1, q_3) = \boldsymbol{J}_r(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} q_3\cos q_1 - L\sin q_1 & q_3\cos q_1 & \sin q_1 \\ q_3\sin q_1 + L\cos q_1 & q_3\sin q_1 & -\cos q_1 \\ 1 & 1 & 0 \end{pmatrix}.
$$

Being the rank of $\boldsymbol{J}_s$ always equal to 2, its null space is spanned by one basis vector as

$$
\mathcal{N}\{\boldsymbol{J}_s\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ L \end{pmatrix} \right\}.
$$

In a dual fashion, the range space of $\boldsymbol{J}_s^T$ is covered by two basis vectors as

$$
\mathcal{R}\{\boldsymbol{J}_s^T\} = \text{span}\left\{ \begin{pmatrix} \cos q_1 - L\sin q_1 \\ \cos q_1 \\ \sin q_1 \end{pmatrix}, \begin{pmatrix} \sin q_1 + L\cos q_1 \\ \sin q_1 \\ -\cos q_1 \end{pmatrix} \right\}.
$$

These two vectors have been obtained by simply setting $q_3 = 0$ in the first two columns of $\boldsymbol{J}_s^T$. It is easy to see that both basis vectors of $\mathcal{R}\{\boldsymbol{J}_s^T\}$ are orthogonal to $\mathcal{N}\{\boldsymbol{J}_s\}$ and that the three vectors chosen as bases for the two subspaces are linearly independent. In fact,

$$
\det \begin{pmatrix} 1 & \cos q_1 - L\sin q_1 & \sin q_1 + L\cos q_1 \\ -1 & \cos q_1 & \sin q_1 \\ L & \sin q_1 & -\cos q_1 \end{pmatrix} = -(2 + L^2) \neq 0,
$$

confirming that the two subspaces are in direct sum

$$\mathcal{N}\{\boldsymbol{J}_s\} \oplus \mathcal{R}\{\boldsymbol{J}_s^T\} = \mathbb{R}^3.$$

Figure 2 provides a graphical sketch of this RRP planar robot in a generic configuration, with the joint variables $\boldsymbol{q}$ defined according to Tab. 1, as well as in a singular configuration with $q_2 = 0$. The pictures show also:

a) the instantaneous joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that produces no linear/angular motion of the end-effector, i.e., such that $\dot{\boldsymbol{q}} \in \mathcal{N}\{\boldsymbol{J}_s\}$;

b) a force $\boldsymbol{f} \in \mathbb{R}^2$ and a moment $m_z \in \mathbb{R}$ applied to the end-effector that need no torque $\boldsymbol{\tau} \in \mathbb{R}^3$ for keeping static balance; they produce zero joint torques in $\mathcal{R}\{\boldsymbol{J}_s^T\}$ or, equivalently, the vector $\boldsymbol{F} = (f_x, f_y, m_z) \in \mathbb{R}^3$ belongs to $\mathcal{N}\{\boldsymbol{J}_s^T\}$.

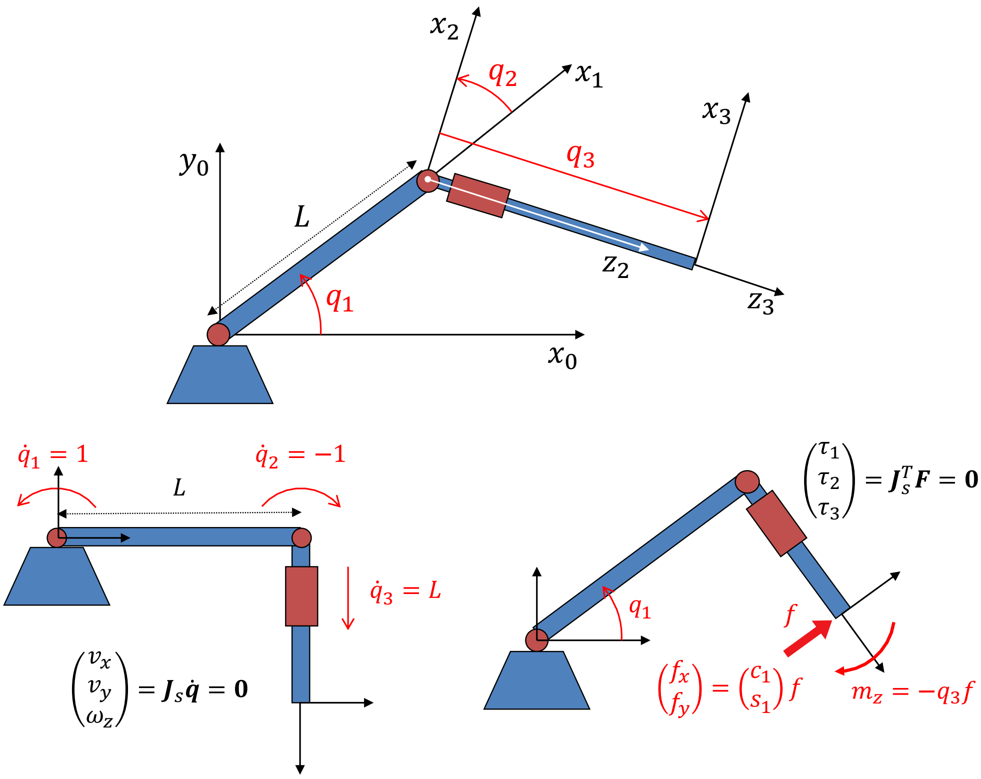

Figure 2: The considered RRP planar robot in a regular (top) and in two singular configurations with $q_2 = 0$ (bottom), where some physical interpretations are also provided

Exercise 4

The minimum time trajectory has a complete trapezoidal velocity profile (equivalently, a bang-coast-bang acceleration). In fact, a cruising phase at maximum speed is reached since

$$|\Delta q| = |q_f - q_i| = \frac{3\pi}{4} = 2.3562 > 1.8 = \frac{9}{5} = \frac{V^2}{A}.$$

The two relevant parameters of this profile (beside the required displacement Δq and the maximum bounds V and A) are the rise time T_r, i.e., the duration of the (maximum) acceleration/deceleration phases, and the total (minimum) time, respectively

$$T_r = \frac{V}{A} = 0.6 \text{ s} \qquad T = \frac{A\,|\Delta q| + V^2}{AV} = 1.3854 \text{ s}.$$

Thus, the cruising phase at maximum speed V lasts for $T - 2T_r = 0.1854$ s. The position, velocity, and acceleration profiles are sketched in Fig. 3.

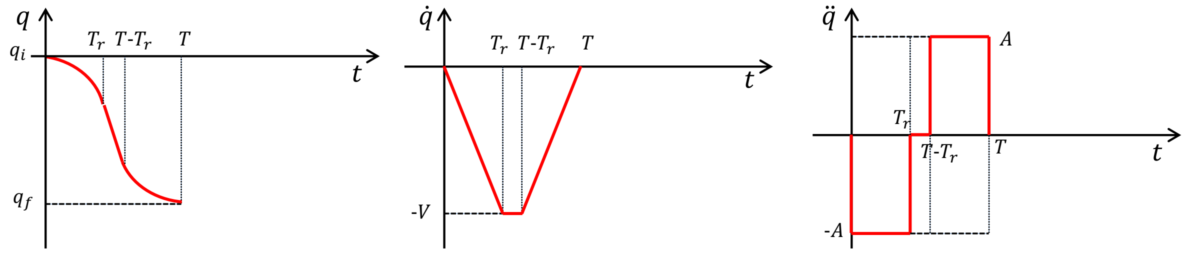

Figure 3: Time profiles of the minimum time rest-to-rest joint trajectory

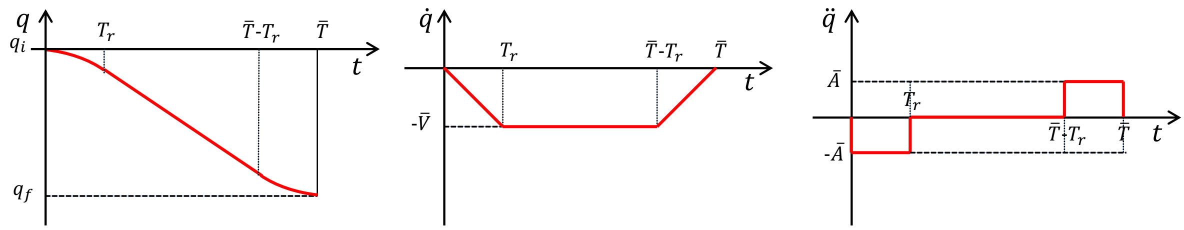

Figure 4: Time profiles of the rescaled rest-to-rest joint trajectory

The new requested trajectory has to accomplish the same joint displacement with the same type of trapezoidal velocity profile, but should last $\bar{T} = 2T = 2.7708$ s and have the same original duration $\bar{T}_r = T_r = 0.6$ s for the acceleration/deceleration phases. Therefore, we can solve for the new cruise velocity $\bar{V}$ and acceleration $\bar{A}$ as follows. Since the area below the new velocity profile should be equal to the original displacement, one obtains

$$\bar{V}(\bar{T} - \bar{T}_r) = |\Delta q| \qquad \Rightarrow \qquad \bar{V} = \frac{|\Delta q|}{\bar{T} - \bar{T}_r} = 1.0854 \text{ rad/s}.$$

Thus, the acceleration needed to reach $\bar{V}$ in a time $\bar{T}_r$ is

$$\bar{A} = \frac{\bar{V}}{\bar{T}_r} = 1.8090 \text{ rad/s}^2.$$

Both values $\bar{V}$ and $\bar{A}$ are reduced with respect to the original V and A. The new position, velocity, and acceleration profiles are sketched in Fig. 4.

Note finally that the operation performed is *not* a uniform time scaling of the original trajectory by the factor $k = 2 = \bar{T}/T$. The latter would have brought to the following new parameters:

$$T_{r,s} = kT_r = 1.2 \text{ s} \qquad V_s = \frac{V}{k} = 1.5 \text{ rad/s} \qquad A_s = \frac{A}{k^2} = 1.25 \text{ rad/s}^2.$$

* * * * *