Robotics 1

January 12, 2026

Exercise 1

Consider the robot sketched in Fig. 1 with two revolute and two prismatic joints, including the revolute joint of its actuated end-effector tool that rolls on a vertical wall. An absolute base frame RF_B is placed on the ground.

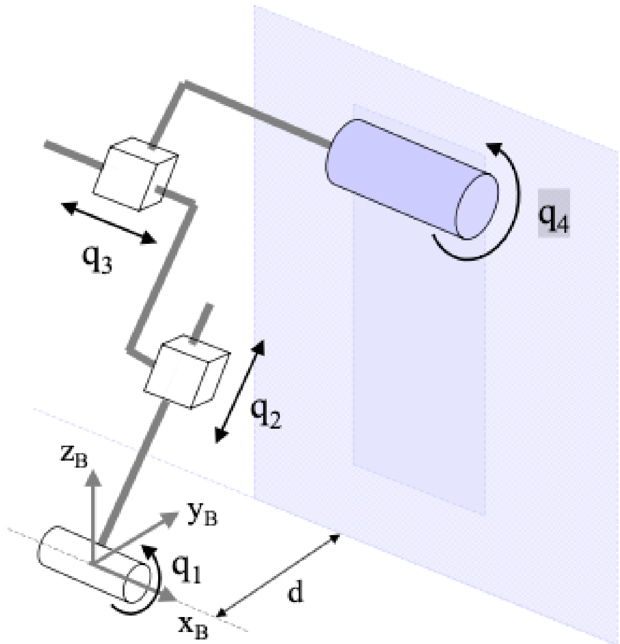

Figure 1: A 4-dof robot used for painting walls

- Assign a set of frames according to the standard Denavit-Hartenberg (DH) convention and determine the corresponding table of parameters. Place the origin of the last DH frame at the center of the rolling tool. Draw the frames directly on the figure.
- Choose a minimal set of parameters that characterizes the pose of the end-effector tool with respect to the base frame RF_B and compute the associated direct kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$.
- For a given $\boldsymbol{r}_d$, determine a closed-form expression for the inverse kinematics $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{r}_d)$.
- Compute the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ of the origin O_4 of the last DH frame. Determine, if any, the configurations at which the 3×4 Jacobian matrix in $\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ looses its full rank.

Exercise 2

Consider the RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ defined by the rotations around the YZX sequence of fixed axes and provide the symbolic expression of the corresponding rotation matrix $\boldsymbol{R}(\boldsymbol{\phi})$. Solve then the inverse representation problem for a given rotation matrix $\boldsymbol{R}_d$ with elements r_{ij}, and determine the cases when there are more than two inverse solutions $\boldsymbol{\phi}$. Finally, compute the map $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity vector $\boldsymbol{\omega} \in \mathbb{R}^3$ and find the singularities of the matrix $\boldsymbol{T}(\boldsymbol{\phi})$.

Exercise 3

For the velocity transformation $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$, where $\boldsymbol{J}(\boldsymbol{q})$ is the 2×2 analytic Jacobian of a 2R planar robot with link lengths $l_1 = 1$, $l_2 = 0.5$ [m] and $\boldsymbol{p} \in \mathbb{R}^2$ is its end-effector position, build three case studies of the pair $(\boldsymbol{q}, \dot{\boldsymbol{p}})$ for which: *i)* the solution $\dot{\boldsymbol{q}}$ is unique; *ii)* there are infinite solutions $\dot{\boldsymbol{q}}$ and you choose the one with minimum norm $\|\dot{\boldsymbol{q}}\|$; *iii)* there is no solution $\dot{\boldsymbol{q}}$ and you choose the joint velocity with minimum norm that minimizes also the error norm $\|\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}}\|$. Sketch graphically the associated situations in the joint velocity plane $(\dot{q}_1, \dot{q}_2)$ and provide the numerical values of $\dot{\boldsymbol{q}}$ for the three case studies.

Exercise 4

Consider a trapezoidal speed profile for a joint that has to move in minimum time from q_i to q_f, under the bounds $|\dot{q}| \leq V$ and $|\ddot{q}| \leq A$. Draw the position, velocity, and acceleration profiles and compute the relevant parameters for the data $q_i = \pi$, $q_f = \pi/4$ [rad] and the bounds $V = 3$ rad/s and $A = 5$ rad/s^2. Assume now that the same displacement should occur under the same bounds in a motion time $\bar{T}$ that is twice as long as the minimum feasible time T, by using a trapezoidal speed profile that has the same duration T_r of the acceleration/deceleration phases of the minimum time motion. Compute the associated velocity $\bar{V}$ and acceleration $\bar{A}$ of the new trajectory and draw the corresponding position, velocity, and acceleration profiles.

[270 minutes (4,5 hours); open books]

Solution

January 12, 2026

Exercise 1

The 4-dof robot in Fig. 1 can be considered a (spatial) RPPR manipulator. A correct assignment of DH frames is shown in Fig. 2. The corresponding parameters are given in Tab. 1, where the signs of the constant parameters a_3 and d_4 is specified, together with those of the joint variables q_i, $i = 1, \ldots, 4$, in the configuration shown.

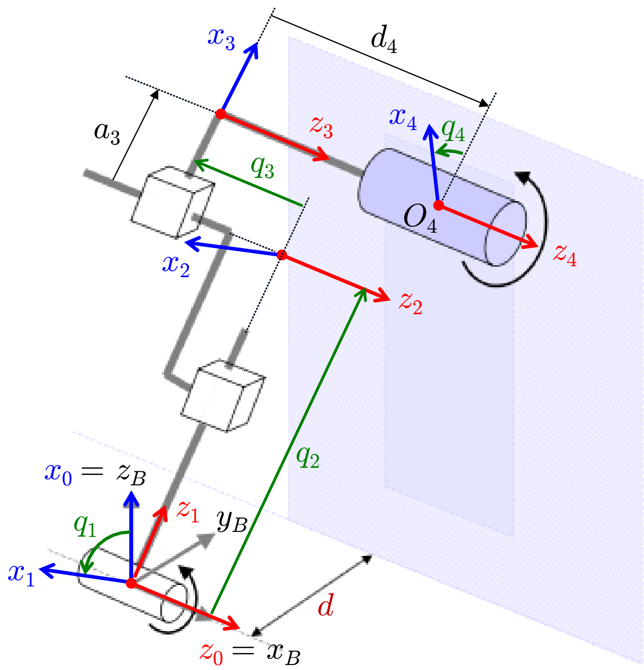

Figure 2: DH frames for the RPPR robot in Fig. 1

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	$q_1 > 0$
2	$-\pi/2$	0	$q_2 > 0$	0
3	0	$a_3 > 0$	$q_3 < 0$	$-\pi/2$
4	0	0	$d_4 > 0$	$q_4 > 0$

Table 1: DH parameters for the frame assignment in Fig. 2

The pose of the rolling end-effector tool is completely specified by the vector $\boldsymbol{r} \in \mathbb{R}^4$ containing the coordinates of the position ${}^B\boldsymbol{p}_4 = ({}^Bp_{4x}, {}^Bp_{4y}, {}^Bp_{4z})$ of the origin O_4 of the last DH frame with respect to the base frame RF_B and the angle φ between y_B (the axis of the base frame lying on the horizontal plane in the direction to the wall) and x_4. By simple inspection, it is

$$\boldsymbol{r} = \begin{pmatrix} {}^{B}p_{4x} \\ {}^{B}p_{4y} \\ {}^{B}p_{4z} \\ \varphi \end{pmatrix} = \begin{pmatrix} q_3 + d_4 \\ (q_2 + a_3)\cos q_1 \\ (q_2 + a_3)\sin q_1 \\ q_1 + q_4 \end{pmatrix}. \tag{1}$$

This result can be obtained also from the more lengthy computation

$${}^{B}\boldsymbol{T}_4(\boldsymbol{q}) = {}^{B}\boldsymbol{T}_0 \, {}^{0}\boldsymbol{A}_1(q_1) \, {}^{1}\boldsymbol{A}_1(q_2) \, {}^{2}\boldsymbol{A}_3(q_2) \, {}^{3}\boldsymbol{A}_4(q_4),$$

where

$${}^{B}\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and ${}^{i-1}\boldsymbol{A}_i(q_i)$, $i = 1, \ldots, 4$, are the DH homogeneous transformation matrices associated to Tab. 1.

For a given value $\boldsymbol{r}_d = (r_{1d}, r_{2d}, r_{3d}, r_{4d})$ of $\boldsymbol{r}$, the inverse kinematics is solved by

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\,\{r_{3d}, r_{2d}\} \\ \sqrt{r_{2d}^2 + r_{3d}^2} - a_3 \\ r_{1d} - d_4 \\ r_{4d} - q_1 \end{pmatrix}, \tag{2}$$

where q_1 in the last component should be replaced by the value of the first component. Note also that, being the wall at a distance $d > 0$ from the origin of the base frame along the y_B direction, the given data $\boldsymbol{r}_d$ should have $r_{2d} = d > 0$ for feasibility.

The velocity $\boldsymbol{v} \in \mathbb{R}^3$ of the origin O_4 is obtained differentiating with respect to time the first three components of (1)

$$\begin{aligned} \boldsymbol{v} = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix} &= \begin{pmatrix} \dot{q}_3 \\ -(q_2 + a_3)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 + a_3)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ -(q_2 + a_3)\sin q_1 & \cos q_1 & 0 & 0 \\ (q_2 + a_3)\cos q_1 & \sin q_1 & 0 & 0 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \end{aligned} \tag{3}$$

with the 3×4 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ associated to the task. This matrix has always full rank $\rho = 3$, except when $q_2 + a_3 = 0$. Such a singularity will never occur, as the end-effector tool should roll on the vertical wall placed at a distance $r_2 = r_{2d} = d > 0$, and thus $q_2 + a_3 > 0$ —see also the second equation in (2).

Exercise 2

The elementary rotation matrices involved with the given RPY-type sequence $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ are:

$$\boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad \boldsymbol{R}_z(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{R}_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}.$$

Since RPY-type rotations are defined around fixes axes, the final orientation for the YZX sequence is obtained by the product of the elementary rotation matrices in the reverse order of definition:

$$\begin{aligned}\boldsymbol{R}_{YZX}(\boldsymbol{\phi}) &= \boldsymbol{R}_x(\gamma)\boldsymbol{R}_z(\beta)\boldsymbol{R}_y(\alpha)\\ &= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\beta & \sin\alpha\cos\beta \\ \cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma-\cos\alpha\sin\gamma \\ \cos\alpha\sin\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\beta\sin\gamma & \cos\alpha\cos\gamma+\sin\alpha\sin\beta\sin\gamma \end{pmatrix}.\end{aligned}$$

For the inverse problem, only the first row and the second column of $\boldsymbol{R}_{YZX}(\boldsymbol{\phi})$ matter. Given a rotation matrix $\boldsymbol{R}_d = \{r_{ij}\}$, it is

$$\beta = \text{ATAN2}\Big\{-r_{12}, \pm\sqrt{r_{11}^2+r_{13}^2}\Big\}. \tag{4}$$

Provided that $r_{11}^2+r_{13}^2 \neq 0$ (regular case), for each sign in (4) one has the pair

$$\alpha = \text{ATAN2}\Big\{\frac{r_{13}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\Big\} \qquad \gamma = \text{ATAN2}\Big\{\frac{r_{32}}{\cos\beta}, \frac{r_{22}}{\cos\beta}\Big\}.$$

In the singular case, $\cos\beta = 0$ so that β is either $\pi/2$ ($\sin\beta = 1$) or $-\pi/2$ ($\sin\beta = -1$), depending on the (opposite) sign of r_{12}. Then, only the difference $\alpha-\gamma$ or, respectively, the sum $\alpha+\gamma$ is defined (an infinite number of solutions for $\boldsymbol{\phi}$).

As for the map $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ between the time derivative $\dot{\boldsymbol{\phi}}$ and the angular velocity vector $\boldsymbol{\omega} \in \mathbb{R}^3$, the algebraic construction provides the sum of three contributions

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot\gamma} + \boldsymbol{\omega}_{\dot\beta} + \boldsymbol{\omega}_{\dot\alpha} = \begin{pmatrix}1\\0\\0\end{pmatrix}\dot\gamma + \boldsymbol{R}_x(\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix}\dot\beta + \boldsymbol{R}_x(\gamma)\boldsymbol{R}_z(\beta)\begin{pmatrix}0\\1\\0\end{pmatrix}\dot\alpha,$$

which is reorganized in matrix form as

$$\boldsymbol{\omega} = \begin{pmatrix} -\sin\beta & 0 & 1 \\ \cos\beta\cos\gamma & -\sin\gamma & 0 \\ \cos\beta\sin\gamma & \cos\gamma & 0 \end{pmatrix}\begin{pmatrix}\dot\alpha\\ \dot\beta\\ \dot\gamma\end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

The determinant of the transformation matrix is $\det \boldsymbol{T}(\boldsymbol{\phi}) = \cos\beta$, so that this matrix is singular at $\beta = \pm\pi/2$ (exactly when the inverse problem of representing orientation by the sequence YZX of RPY-type angles has infinite solutions). When in a singularity, angular velocities of the form $\boldsymbol{\omega} = k\,(0, \cos\gamma, \sin\gamma)$, with $k \neq 0$, cannot be generated by any $\dot{\boldsymbol{\phi}}$.

Exercise 3

At a given configuration $\boldsymbol{q}$ and for an assigned $\dot{\boldsymbol{p}}$, the linear system of equations $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$ in the unknown $\dot{\boldsymbol{q}}$, with $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q})$ being a square matrix:

i) has the unique solution $\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}}$ if $\det\boldsymbol{J} \neq 0$;

ii) if $\det\boldsymbol{J} = 0$ and $\dot{\boldsymbol{p}} \in \mathcal{R}(\boldsymbol{J})$, has infinite solutions and $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}}$ is the solution with minimum norm, being $\boldsymbol{J}^{\#}$ the (unique) pseudoinverse of $\boldsymbol{J}$;

iii) if $\det\boldsymbol{J} = 0$ and $\dot{\boldsymbol{p}} \notin \mathcal{R}(\boldsymbol{J})$, has no solutions and $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}}$ is the joint velocity with minimum norm among all those that minimize the norm of the velocity error $\dot{\boldsymbol{e}} = \boldsymbol{J}\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}}$.

For a 2×2 matrix $\boldsymbol{J}$, let

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \end{pmatrix} \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}.$$

Then, the two linear equations

$$\boldsymbol{J}_1\dot{\boldsymbol{q}} = J_{11}\dot{q}_1 + J_{12}\dot{q}_2 = \dot{p}_x \qquad \boldsymbol{J}_2\dot{\boldsymbol{q}} = J_{21}\dot{q}_1 + J_{22}\dot{q}_2 = \dot{p}_y$$

can be represented in the plane $(\dot{q}_1, \dot{q}_2)$ as two lines that may or may not intersect. The above three cases *i)–iii)* are shown in Fig. 3. When $\det \boldsymbol{J} = 0$, the two rows are linearly dependent, i.e., $\boldsymbol{J}_2 = k\boldsymbol{J}_1$ for some $k \neq 0$; moreover, if $\dot{p}_y \neq k\dot{p}_x$ the two equations are inconsistent and there is no exact solution for both. The dashed line of the right picture in Fig. 3 is equidistant from the other two and represents the set of joint velocities $\dot{\boldsymbol{q}}$ yielding the minimum norm of the velocity error.

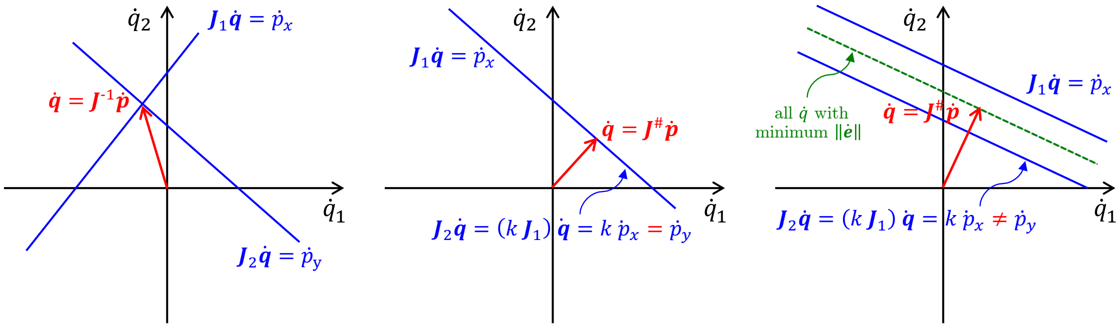

Figure 3: The three possible situations for $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$ in the two-dimensional case

One special case which is not represented in Fig. 3 is when one of the two equations has all zero coefficients —say, $\boldsymbol{J}_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$; as a consequence, the line associated to the second equation cannot be drawn. Then, for case *ii)*, it is necessarily $\dot{p}_y = 0$ and this second equation vanishes (it is simply the identity $0 = 0$), so that the linear system has only one equation in two unknowns and thus infinite solutions; instead, case *iii)* has $\dot{p}_y \neq 0$ and all the solutions to the first equation $\boldsymbol{J}_1\dot{\boldsymbol{q}} = \dot{p}_x$ will have the same error norm $\|\dot{\boldsymbol{e}}\| = |\dot{p}_y| \neq 0$. In all these situations, use of the pseudoinverse provides again the joint velocity with minimum norm.

Let us turn to some numerical examples. For the considered 2R planar robot, one has

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}) \qquad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1s_1 + l_2s_{12}) & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{pmatrix},$$

with $l_1 = 1$, $l_2 = 0.5$ [m]. The determinant is $\det \boldsymbol{J}(\boldsymbol{q}) = l_1 l_2 \sin q_2$. Consider the end-effector velocity $\dot{\boldsymbol{p}} = (-1, 1)$ [m/s] and choose first the configuration $\boldsymbol{q} = (0, \pi/2)$ [rad], for which

$$\boldsymbol{J} = \begin{pmatrix} -0.5 & -0.5 \\ 1 & 0 \end{pmatrix}$$

is clearly nonsingular. Then, the unique solution is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{p}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [rad/s]}.$$

Move next the robot to the singular configuration $\boldsymbol{q} = (\pi/4, 0)$ [rad]. The Jacobian becomes

$$\boldsymbol{J} = \begin{pmatrix} -1.5/\sqrt{2} & -1/\sqrt{2} \\ 1.5/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1.0607 & -0.3536 \\ 1.0607 & 0.3536 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix}$$

which is clearly singular, being $\boldsymbol{J}_2 = k\boldsymbol{J}_1$, with $k = -1$. However, since $\dot{p}_y = 1 = -1 \cdot -1 = k\,\dot{p}_x$, we are in case *ii)* and the pseudoinverse solution will provide no error with respect to the desired end-effector velocity.

Interlude. In the absence of a numerical tool (e.g., Matlab) for computing the pseudoinverse of a singular matrix, which would require in general the SVD decomposition of $\boldsymbol{J}$, one can use direct formulas that exploit the simple structure of the matrix to be pseudoinverted.[1] In fact, it is easy to verify that:

- for a n-dimensional (column or row) vector $\boldsymbol{a} \neq \boldsymbol{0}$, the pseudoinverse is $\boldsymbol{a}^{\#} = \boldsymbol{a}^T/\|\boldsymbol{a}\|^2$;
- for a $2 \times n$ matrix $\boldsymbol{A}$ with a row of zeros (or an $n \times 2$ matrix $\boldsymbol{B}$ with a column of zeros), the pseudoinverse is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{0} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{a}^{\#} & \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{0} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{B}^{\#} = \begin{pmatrix} \boldsymbol{b}^{\#} \\ \boldsymbol{0} \end{pmatrix},$$

  and similarly when the zeros are in the first row (or column);
- for a singular (i.e., not full rank) $2 \times n$ matrix $\boldsymbol{J}$,

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ k\boldsymbol{J}_1 \end{pmatrix} \qquad k \neq 0,$$

  the pseudoinverse is computed from the previous results using the factorization

$$\boldsymbol{J} = \boldsymbol{B}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ k \end{pmatrix},$$

  yielding

$$\boldsymbol{J}^{\#} = \boldsymbol{A}^{\#}\boldsymbol{B}^{\#} = \begin{pmatrix} \boldsymbol{J}_1^{\#} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{b}^{\#} \\ \boldsymbol{0} \end{pmatrix}.$$

With the above in mind, one can compute

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}} = \begin{pmatrix} -0.4243 & 0.4243 \\ -0.1414 & 0.1414 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8485 \\ 0.2828 \end{pmatrix} \text{ [rad/s]}.$$

It is easy to verify that this joint velocity is a correct solution, returning the desired end-effector velocity ($\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$).

Finally, suppose that the robot is in a different singular configuration, e.g., in $\boldsymbol{q} = (0, 0)$, namely with both links stretched in the x_0-direction. Then, since

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 \\ 1.5 & 0.5 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin \mathcal{R}(\boldsymbol{J}) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

we are in case *iii)*. The joint velocity computed using the previous pseudoinverse formulas

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{p}} = \begin{pmatrix} 0 & 0.6 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \end{pmatrix} \text{ [rad/s]},$$

will return only the part of the desired end-effector velocity that lies in the range of $\boldsymbol{J}$,

$$\dot{\boldsymbol{p}}^{\perp} = \boldsymbol{J}\dot{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \dot{\boldsymbol{p}},$$

[1] These formulas have been presented also during the lectures in the classroom.

namely the second component only. The end-effector velocity error has $\|\dot{\boldsymbol{e}}\| = 1$, which is the smallest possible norm for any $\dot{\boldsymbol{q}} \in \mathbb{R}^2$. The joint velocity computed with the pseudoinverse has $\|\dot{\boldsymbol{q}}\| = \sqrt{0.6^2 + 0.2^2} = \sqrt{0.4} = 0.6325$ rad/s, which is the smallest norm for all joint velocities that achieve the minimum value for the norm of $\dot{\boldsymbol{e}}$.

Exercise 4

The minimum time trajectory has a complete trapezoidal velocity profile (equivalently, a bang-coast-bang acceleration). In fact, a cruising phase at maximum speed is reached since

$$|\Delta q| = |q_f - q_i| = \frac{3\pi}{4} = 2.3562 > 1.8 = \frac{9}{5} = \frac{V^2}{A}.$$

The two relevant parameters of this profile (beside the required displacement Δq and the maximum bounds V and A) are the rise time T_r, i.e., the duration of the (maximum) acceleration/deceleration phases, and the total (minimum) time, respectively

$$T_r = \frac{V}{A} = 0.6 \text{ s} \qquad T = \frac{A\,|\Delta q| + V^2}{AV} = 1.3854 \text{ s}.$$

Thus, the cruising phase at maximum speed V lasts for $T - 2T_r = 0.1854$ s. The position, velocity, and acceleration profiles are sketched in Fig. 4.

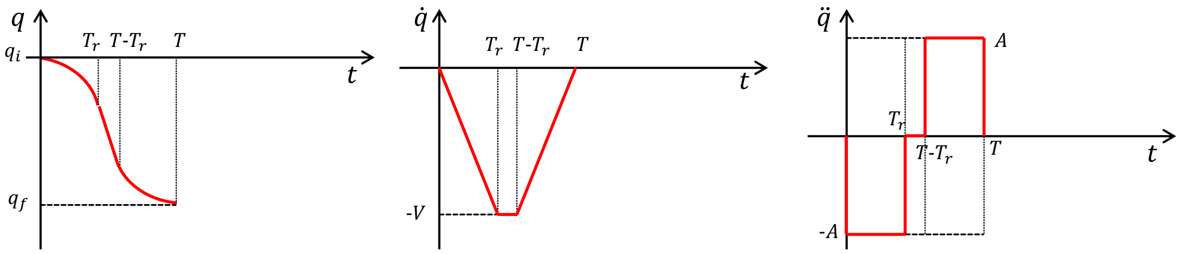

Figure 4: Time profiles of the minimum time rest-to-rest joint trajectory

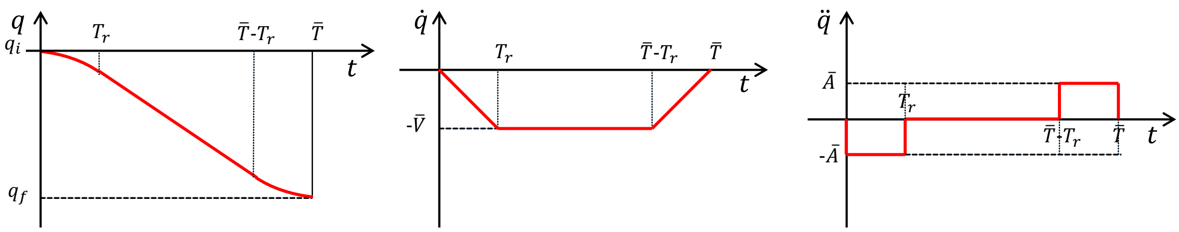

Figure 5: Time profiles of the rescaled rest-to-rest joint trajectory

The new requested trajectory has to accomplish the same joint displacement with the same type of trapezoidal velocity profile, but should last $\bar{T} = 2T = 2.7708$ s and have the same original duration $\bar{T}_r = T_r = 0.6$ s for the acceleration/deceleration phases. Therefore, we can solve for the new cruise velocity $\bar{V}$ and acceleration $\bar{A}$ as follows. Since the area below the new velocity profile should be equal to the original displacement, one obtains

$$\bar{V}(\bar{T} - \bar{T}_r) = |\Delta q| \qquad \Rightarrow \qquad \bar{V} = \frac{|\Delta q|}{\bar{T} - \bar{T}_r} = 1.0854 \text{ rad/s}.$$

Thus, the acceleration needed to reach $\bar{V}$ in a time $\bar{T}_r$ is

$$\bar{A} = \frac{\bar{V}}{\bar{T}_r} = 1.8090 \text{ rad/s}^2.$$

Both values $\bar{V}$ and $\bar{A}$ are reduced with respect to the original V and A. The new position, velocity, and acceleration profiles are sketched in Fig. 5.

Note finally that the operation performed is *not* a uniform time scaling of the original trajectory by the factor $k = 2 = \bar{T}/T$. The latter would have brought to the following new parameters:

$$T_{r,s} = kT_r = 1.2 \text{ s} \qquad V_s = \frac{V}{k} = 1.5 \text{ rad/s} \qquad A_s = \frac{A}{k^2} = 1.25 \text{ rad/s}^2.$$

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