Robotics 1

July 8, 2025

Exercise 1

The articulated robotic structure in Fig. 1 is used for diagnostic and interventional imaging in cardiology and radiology. The robot kinematics is a serial chain of bodies with five revolute joints, followed by two symmetric final ‘branches’. Each branch can be modeled as having two elementary joints (one prismatic and one revolute) that move in perfect coordination with those of the other branch. This structure provides a convenient positioning and orienting flexibility to the imaging device that operates along the segment between the tips of the two branches.

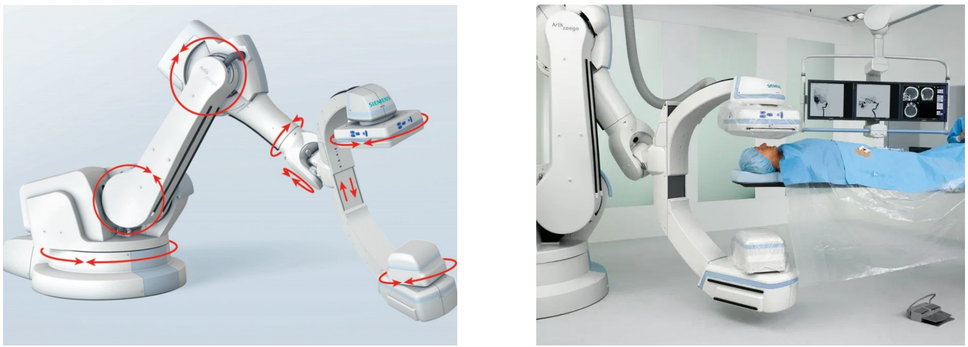

Figure 1: The Siemens Artis Zeego robot: its degrees of freedom (left) and in operative conditions (right)

Consider only one branch of the robot, i.e., a serial kinematic chain with 7 dofs. Assign the robot frames according to the Denavit-Hartenberg convention, so that the constant a_i and d_i parameters are all non-negative. Draw the frames and fill the table of DH parameters on the Extra Sheet.

Let then P be the midpoint of the (variable length) segment joining the tips of the two branches of the complete robot. The motion of which joints will never affect the position of P? If the prismatic joints are held fixed, can we say that the complete robot has a spherical wrist?

Exercise 2

For a 2-dof robot, consider the differential mapping between the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ and the task velocity $\dot{\boldsymbol{r}} \in \mathbb{R}^2$ at a given configuration $\boldsymbol{q}$. Joint velocities are bounded as $|\dot{q}_i| \leq V_i$, for $i = 1, 2$. The task velocity is specified as $\dot{\boldsymbol{r}} = s\dot{\boldsymbol{r}}_d$, with a scaling factor $s \in [0, 1]$ to reduce as little as possible the original task velocity $\dot{\boldsymbol{r}}_d$ in case of an unfeasible joint velocity (i.e., that violates one or both bounds). Draw in the plane $(\dot{q}_1, \dot{q}_2)$ a qualitative picture for each of the following situations.

a) Regular robot configuration and feasible joint velocity. Represent geometrically the two linear equations of the differential map $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{r}}_d$.

b) Regular configuration and unfeasible joint velocity. Find geometrically the largest task scaling factor that recovers feasibility, together with the corresponding joint velocity.

c) Singular configuration, realizable $\dot{\boldsymbol{r}}_d$, but unfeasible joint velocity. Find geometrically the largest task scaling factor that recovers feasibility and the corresponding joint velocity.

d) Singular configuration and unrealizable $\dot{\boldsymbol{r}}_d$. Find geometrically the smallest joint velocity in norm that minimizes the norm of the task velocity error $\|\dot{\boldsymbol{r}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}\|$. When the joint velocity obtained in this way is unfeasible, find geometrically the largest task scaling factor that recovers feasibility and the corresponding joint velocity.

[Exercise 2 continues . . .]

For a planar 2R robot having links of unitary length and velocity bounds $V_1 = 1$ and $V_2 = 1.5$ [rad/s], let $\boldsymbol{r} = \boldsymbol{p} \in \mathbb{R}^2$ be the position of its end-effector. Based on the previous qualitative analysis, provide the solutions for the following two numerical cases, indicating also which of the above situations applies:

- $\boldsymbol{q} = (\pi/3, \pi)$ [rad], $\dot{\boldsymbol{r}}_d = (-\sqrt{3}/2, 0.5)$ [m/s];
- $\boldsymbol{q} = (0, \pi/2)$ [rad], $\dot{\boldsymbol{r}}_d = (1, -1.2)$ [m/s].

Exercise 3

A planar RP robot with base at the origin should move its end-effector along a circular path centered at $C_0 = (4, 4)$, from point $P_1 = (4, 3)$ to point $P_2 = (3, 4)$ [m]. The desired motion is rest-to-rest and is executed in a total time $T = 2$ s, with a bang-bang acceleration profile as timing law. Determine:

- the parametric expression of the path $\boldsymbol{p}(s)$, for $s \in [0, L]$, where L is the Cartesian path length, together with its first and second derivatives $\boldsymbol{p}'(s) = d\boldsymbol{p}/ds$ and $\boldsymbol{p}''(s) = d^2\boldsymbol{p}/ds^2$;
- the expression of the corresponding parametrized joint path $\boldsymbol{q}(s) = (q_1(s), q_2(s))$, for $s \in [0, L]$;
- the remaining parameters and the expression of the timing law $s = s(t)$, for $t \in [0, T]$, together with its speed $\dot{s}(t)$ and scalar acceleration $\ddot{s}(t)$;
- the joint trajectory $\boldsymbol{q}(t) = (q_1(t), q_2(t))$ corresponding to the desired Cartesian trajectory;
- the maximum value reached by the absolute velocity $|\dot{q}_1(t)|$ of the first joint in the interval $[0, T]$, and the value of $\dot{q}_2(t)$ at the same instant;
- the maximum attained values of the norms of the end-effector velocity and acceleration, respectively $\|\dot{\boldsymbol{p}}(t)\|$ and $\|\ddot{\boldsymbol{p}}(t)\|$, for $t \in [0, T]$.

Illustrate your solution by sketching the plots of the most relevant quantities in the problem.

[4 hours; open books]

Solution

July 8, 2025

Exercise 1

A DH frame assignment satisfying the requests is shown in Fig. 2. The DH parameters are in Tab. 1.

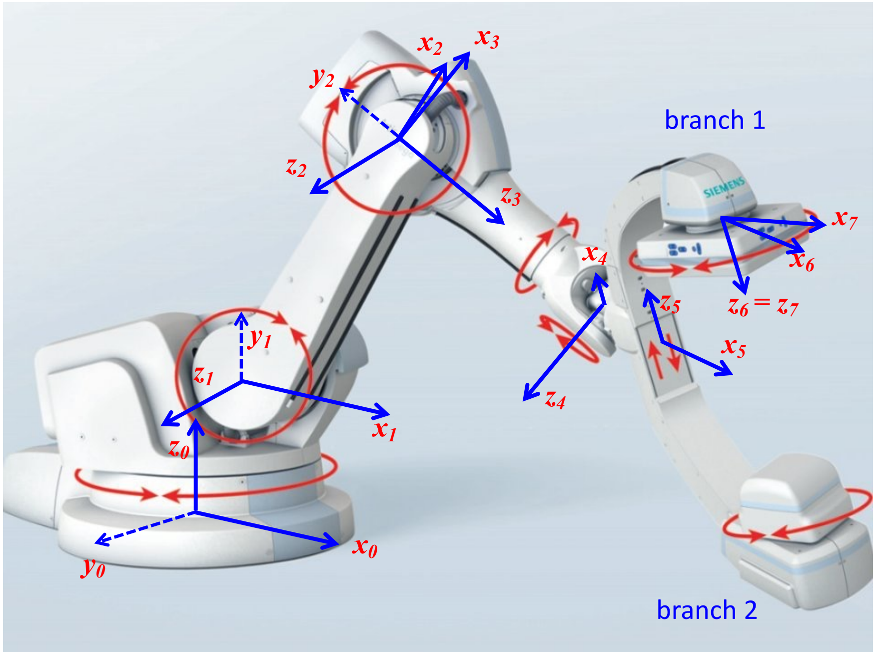

Figure 2: Assignment of DH frames for one of the two branches of the Siemens Artis Zeego robot

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	$\pi/2$	0	0	q_3
4	$-\pi/2$	0	$d_4 > 0$	q_4
5	$-\pi/2$	$a_5 > 0$	0	q_5
6	π	$a_6 > 0$	q_6	0
7	0	0	0	q_7

Table 1: DH parameters corresponding to the frame assignment in Fig. 2

For the complete robot, the motion of the midpoint P between the tips of the two branches is never affected by the prismatic joint(s), since $\dot{q}_6$ in the two branches move equally and in opposite directions, nor by the last rotational joint(s), since $\dot{q}_7$ rotate in both branches by the same amount and around a common direction passing through point P.

When fixing the prismatic joints, the complete robotic structure will have five revolute joints in common and two final branches, each with a revolute joint. While the fourth and fifth joint axes of the serial sub-structure intersect at a point, namely in the origin O_4 of RF_4, the common axis of the last revolute joints, i.e., the line joining the tips of the two branches, will never pass through this point (a consequence of having $a_5 \neq 0$ in Tab. 1). Thus, even in the case of reduced DoFs, this robot has no spherical wrist.

Exercise 2

The robot differential kinematics at a given configuration $\boldsymbol{q}$ is a *linear* mapping from $\dot{\boldsymbol{q}}$ to $\dot{\boldsymbol{r}}$, represented by the (analytic) Jacobian $\boldsymbol{J}(\boldsymbol{q})$. In the present case, when evaluated at $\boldsymbol{q}$, this is a 2×2 constant matrix $\boldsymbol{J}$ that we can write by its scalar elements J_{ij} or by its 1×2 rows J_i as

$$\boldsymbol{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix},$$

so that

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{r}} \quad \Rightarrow \quad \begin{array}{l} J_1\dot{\boldsymbol{q}} = \dot{r}_1 \\ J_2\dot{\boldsymbol{q}} = \dot{r}_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} J_{11}\dot{q}_1 + J_{12}\dot{q}_2 = \dot{r}_1 \\ J_{21}\dot{q}_1 + J_{22}\dot{q}_2 = \dot{r}_2. \end{array} \tag{1}$$

For a given $\dot{\boldsymbol{r}} = (\dot{r}_1, \dot{r}_2)$, the two linear equations in the unknowns $\dot{q}_1$ and $\dot{q}_2$ in (1) can be represented geometrically by two lines in the plane $(\dot{q}_1, \dot{q}_2)$. The solution of these equations, if it exists, is at the intersection of these two lines. If this intersection corresponds to a point $\dot{\boldsymbol{q}}$ that lies inside the rectangular box defined by the joint velocity bounds,

$$-V_1 \leq \dot{q}_1 \leq V_1 \qquad \text{and} \qquad -V_2 \leq \dot{q}_2 \leq V_2,$$

then the solution will be *feasible*. The feasible region is shaded in green in Figs. 3–4.

Further, the homogeneous version $J_i\dot{\boldsymbol{q}} = 0$ of each of the equations in (1) is a line passing through the origin; thus, $J_i\dot{\boldsymbol{q}} = \dot{r}_i$, for any $\dot{r}_i \neq 0$, is represented by a line parallel to this line. When the task velocity is written in the *scaled* form $\dot{\boldsymbol{r}} = s\dot{\boldsymbol{r}}_d$, namely as a desired vector $\dot{\boldsymbol{r}}_d = (\dot{r}_{d1}, \dot{r}_{d2})$ possibly scaled by a constant $s \in [0, 1]$, each of the corresponding linear equations in (1) will be represented by a line parallel to the line $J_i\dot{\boldsymbol{q}} = \dot{r}_{di}$. The smaller is the scalar $s \geq 0$, the closer to the origin will this line be; as a consequence, for the scaled equations, also the intersection of the two lines, if it exists, will be closer to the origin. Thus, by suitable task scaling, we can turn an unfeasible joint velocity solution into a feasible one.

It is well known that a solution $\dot{\boldsymbol{q}}$ to the linear system $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{r}}$ exists if and only if $\dot{\boldsymbol{r}} \in \mathcal{R}(\boldsymbol{J})$, i.e., the vector $\dot{\boldsymbol{r}}$ can be expressed as a combination of the columns of $\boldsymbol{J}$. We say in this case that $\dot{\boldsymbol{r}}$ is *realizable* (or *admissible*). In particular, when the square matrix $\boldsymbol{J}$ is nonsingular, the solution is unique and is given by $\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{r}}$, for any vector $\dot{\boldsymbol{r}}$. Note that, due to the linearity of the mapping, scaling the task velocity transfers directly the same scaling to the joint velocity solution:

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{r}}_d \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_s = \boldsymbol{J}^{-1}s\dot{\boldsymbol{r}}_d = s\dot{\boldsymbol{q}}.$$

Figure 3 shows the two situations a) and b) of the problem for a regular (nonsingular) Jacobian. On the left, we have a realizable solution $\dot{\boldsymbol{q}}$ which is also feasible; on the right, the original task velocity $\dot{\boldsymbol{r}}_d$ generates an unfeasible solution $\dot{\boldsymbol{q}}$ (shown in red), which is converted to a feasible $\dot{\boldsymbol{q}}_s$ (shown in green) by a suitable positive task scaling factor $s < 1$. Note that the two lines representing the two equations move in a parallel and coordinated way when changing s (at the scaled solution $\dot{\boldsymbol{q}}_s$, these are the shaded blue lines). Their intersection will always lie on the line connecting the origin to the original unscaled solution $\dot{\boldsymbol{q}}$. Moreover, the obtained feasible solution uses the largest

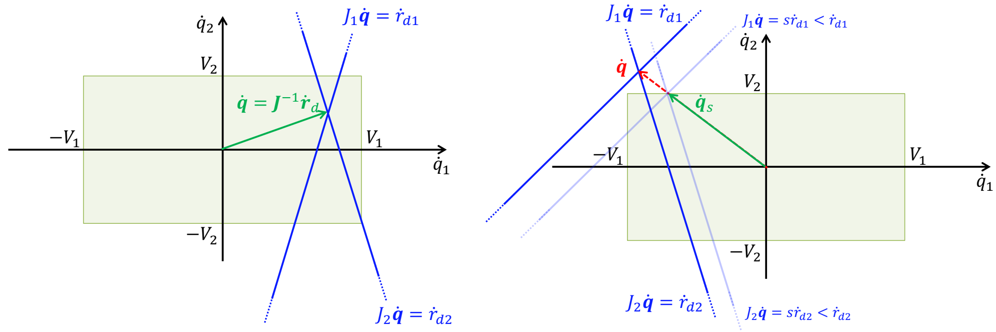

Figure 3: *Left:* Situation a) Regular case, feasible solution $\dot{\boldsymbol{q}}$. *Right:* Situation b) Regular case, unfeasible solution $\dot{\boldsymbol{q}}$ converted to a feasible $\dot{\boldsymbol{q}}_s$ by task scaling

possible scaling factor s or, equivalently, corresponds to the smallest proportional reduction of the original task velocity $\dot{\boldsymbol{r}}_d$.

The same applies also to the case in which matrix $\boldsymbol{J}$ is singular but $\dot{\boldsymbol{r}}$ is still realizable. For the 2×2 case, this implies that the two columns of the Jacobian $\boldsymbol{J}$ are linearly dependent and vector $\dot{\boldsymbol{r}}$ is proportional to (any of) these columns. Indeed, in this case also the two rows of $\boldsymbol{J}$ will be linearly dependent, i.e., we can write their proportionality as $J_2 = \alpha J_1$, for a non-zero factor $\alpha \in \mathbb{R}$. As a result, the same proportionality holds between the two components of a realizable $\dot{\boldsymbol{r}}$:

$$J_1\dot{\boldsymbol{q}} = \dot{r}_{d1} \qquad \Rightarrow \qquad J_2\dot{\boldsymbol{q}} = \alpha J_1\dot{\boldsymbol{q}} = \alpha\,\dot{r}_{d1} = \dot{r}_{d2}.$$

Geometrically, this means that the two equations in (1) are represented by the same line in the plane $(\dot{q}_1, \dot{q}_2)$. In other terms, we have only one equation in two unknowns and the linear system has an infinite number of solutions —all points along this line. The solution with minimum norm is the closest to the origin (thus, more likely to be also feasible with respect to the joint velocity bounds) and is generated by the pseudoinverse of the Jacobian as $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}$. This solution is feasible when the line intersects the green box *and* the velocity with minimum norm on this line belongs to the feasible region. Figure 4, left, represents the situation c) when the obtained solution is still unfeasible: a task scaling recovers feasibility also in this case, since we have

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}_d \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_s = \boldsymbol{J}^{\#}s\dot{\boldsymbol{r}}_d = s\dot{\boldsymbol{q}}.$$

The last situation d) is when the task velocity $\dot{\boldsymbol{r}}_d$ is not realizable —and necessarily the Jacobian is singular. We have then:

$$\begin{array}{l} J_1\dot{\boldsymbol{q}} = \dot{r}_{d1} \\ J_2\dot{\boldsymbol{q}} = \alpha J_1\dot{\boldsymbol{q}} = \alpha\dot{r}_{d1} \neq \dot{r}_{d2} \end{array} \qquad \Rightarrow \qquad \text{no solution } \dot{\boldsymbol{q}}.$$

The two lines representing these equations are now parallel and there is no intersection for finite $\dot{\boldsymbol{q}}$. Nonetheless, we can look for a joint velocity $\dot{\boldsymbol{q}}$ that minimizes the norm of the task velocity error, i.e., tries to satisfy at best both equations in (1) in a least squares sense. The pseudoinverse solution has the following known property:

$$\begin{array}{l} \min\limits_{\dot{\boldsymbol{q}}\in S} H = \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2 \\ \qquad S = \{\dot{\boldsymbol{q}} \in \mathbb{R}^n : \|\dot{\boldsymbol{r}}_d - \boldsymbol{J}\dot{\boldsymbol{q}}\| \text{ is minimum}\} \end{array} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}_d.$$

This case is represented in Fig. 4, right. There are infinite joint velocities $\dot{\boldsymbol{q}}$ minimizing the error norm $\|\dot{\boldsymbol{r}}_d - \boldsymbol{J}\dot{\boldsymbol{q}}\|$: these are all the points on the (shaded red) line equidistant from the two blue

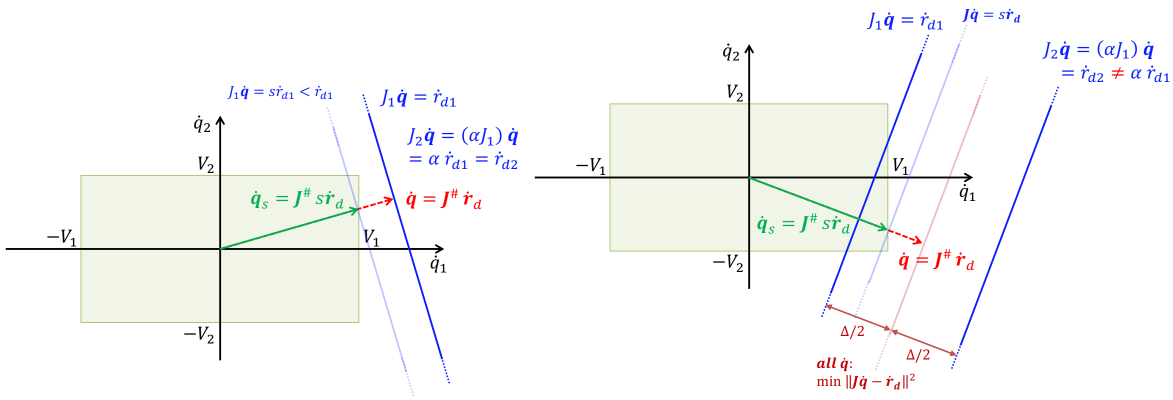

Figure 4: *Left:* Situation c) Singular case, with realizable task velocity $\dot{\boldsymbol{r}}_d$ but unfeasible solution $\dot{\boldsymbol{q}}$, converted to a feasible $\dot{\boldsymbol{q}}_s$ by task scaling. *Right:* Situation d) Singular case, with unrealizable task velocity $\dot{\boldsymbol{r}}_d$; a feasible minimum norm error solution $\dot{\boldsymbol{q}}_s$ is obtained after scaling.

lines representing the singular system of equations. The pseudoinverse solution is again the point on this line with the smallest norm, i.e., the closest to the origin. In the present case, this solution is still unfeasible, so a task scaling is implemented as before to obtain $\dot{\boldsymbol{q}}_s$.

Note that for the two singular situations c) and d), we could find a feasible joint velocity that requires a smaller relaxation of the (realizable or not) original task velocity. Geometrically, in the two-dimensional case of Fig. 4, it is easy to see that this joint velocity is at a corner of the feasible region: the upper right corner $\dot{\boldsymbol{q}} = (V_1, V_2)$ for situation c) and the lower left corner $\dot{\boldsymbol{q}} = (V_1, -V_2)$ for situation d). However, such solutions cannot be obtained by a task scaling procedure with the use of the pseudoinverse, but requires a better distribution of the effort among the robot joints.[1]

Finally, for the two numerical case studies, we have the following results.

- At $\boldsymbol{q} = (\pi/3, \pi)$, the Jacobian of the considered planar 2R robot is singular, but the desired task velocity is realizable. We are then in situation c):

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0.8660 \\ 0 & -0.5 \end{pmatrix} \qquad \dot{\boldsymbol{r}}_d = \begin{pmatrix} -\sqrt{3}/2 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -0.8660 \\ 0.5 \end{pmatrix} \in \mathcal{R}(\boldsymbol{J}).$$

  The pseudoinverse solution (obtained using `pinv` in MATLAB) is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 & 0 \\ 0.8660 & -0.5 \end{pmatrix} \dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

  which is feasible with respect to the joint velocity bounds $V_1 = 1$ and $V_2 = 1.5$ [rad/s]. No scaling is needed in this case.

- At $\boldsymbol{q} = (0, \pi/2)$, the Jacobian is nonsingular. However, the unique joint velocity solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\dot{\boldsymbol{r}}_d = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1.2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1.2 \end{pmatrix} = \begin{pmatrix} -1.2 \\ 0.2 \end{pmatrix}$$

  has the first component out of bound, so that we are in situation b). The optimal task scaling factor is computed in general as

$$s^* = \min\left\{1, \frac{V_1}{|\dot{q}_1|}, \frac{V_2}{|\dot{q}_2|}\right\} = \frac{V_1}{|\dot{q}_1|} = \frac{1}{1.2} = 0.8\bar{3},$$

[1] This topic is addressed in the more advanced Robotics 2 course when dealing with kinematic redundancy, and leads to the so-called Saturation in the Null Space (SNS) optimization algorithm.

and the feasible scaled joint velocity is

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}^{-1} s^* \dot{\boldsymbol{r}}_d = \boldsymbol{J}^{-1} \begin{pmatrix} 0.8\bar{3} \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0.1\bar{6} \end{pmatrix},$$

with the first component at its (negative) limit value.

Exercise 3

It is immediate to see that the circular arc from P_1 to P_2 has a radius $R = \|C_0 - P_1\| = 1$ m and a length $L = (2\pi R)/4 = \pi/2$. Thus, the desired Cartesian path is expressed in parametric form (with s being the arc length) as

$$\boldsymbol{p}(s) = \begin{pmatrix} p_x(s) \\ p_y(s) \end{pmatrix} = C_0 - R \begin{pmatrix} \sin(s/R) \\ \cos(s/R) \end{pmatrix} \qquad s \in [0, L].$$

The arc is traced clockwise, being this the shortest way to go, from $\boldsymbol{p}(0) = P_1$ to $\boldsymbol{p}(L) = P_2$. Accordingly, the first and second spatial derivatives are

$$\boldsymbol{p}'(s) = \frac{d\boldsymbol{p}}{ds} = \begin{pmatrix} -\cos(s/R) \\ \sin(s/R) \end{pmatrix} \qquad \boldsymbol{p}''(s) = \frac{d^2\boldsymbol{p}}{ds^2} = \frac{1}{R} \begin{pmatrix} \sin(s/R) \\ \cos(s/R) \end{pmatrix},$$

from which $\|\boldsymbol{p}'(s)\| = 1$ and $\|\boldsymbol{p}''(s)\| = 1/R$, both constant for all s.

The joint path $\boldsymbol{q}(s)$ corresponding to the Cartesian path $\boldsymbol{p}(s)$ is found by inverse kinematics for the planar RP robot. Since the direct kinematics is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix},$$

one obtains

$$\boldsymbol{q}(s) = \begin{pmatrix} q_1(s) \\ q_2(s) \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\left\{p_y(s), p_x(s)\right\} \\ \sqrt{p_x^2(s) + p_y^2(s))} \end{pmatrix} \qquad \text{for } s \in [0, L], \tag{2}$$

where the positive sign has been chosen for the prismatic joint. If needed, one can obtain also the first and second spatial derivatives of the joint path, evaluating in sequence (2), then

$$\boldsymbol{q}'(s) = \boldsymbol{J}^{-1}(\boldsymbol{q}(s))\, \boldsymbol{p}'(s), \tag{3}$$

with

$$\boldsymbol{J}^{-1}(\boldsymbol{q}(s)) = \begin{pmatrix} -q_2(s)\sin q_1(s) & \cos q_1(s) \\ q_2(s)\cos q_1(s) & \sin q_1(s) \end{pmatrix}^{-1} = \frac{1}{q_2(s)} \begin{pmatrix} -\sin q_1(s) & \cos q_1(s) \\ q_2(s)\cos q_1(s) & q_2(s)\sin q_1(s) \end{pmatrix},$$

and finally

$$\boldsymbol{q}''(s) = \boldsymbol{J}^{-1}(\boldsymbol{q}(s)) \left(\boldsymbol{p}''(s) - \boldsymbol{J}'(\boldsymbol{q}(s))\, \boldsymbol{q}'(s)\right), \tag{4}$$

where

$$\boldsymbol{J}'(\boldsymbol{q}(s)) = \frac{d\boldsymbol{J}}{ds} = \begin{pmatrix} -q_2'(s)\sin q_1(s) - q_2(s)\, q_1'(s)\cos q_1(s) & -q_1'(s)\sin q_1(s) \\ q_2'(s)\cos q_1(s) - q_2(s)\, q_1'(s)\sin q_1(s) & q_1'(s)\cos q_1(s) \end{pmatrix}.$$

The Cartesian path and the corresponding joint path are shown in Fig. 5. Note the symmetric behavior of q_2 and skew-symmetric one of q_1 with respect to the path midpoint: this is due to the

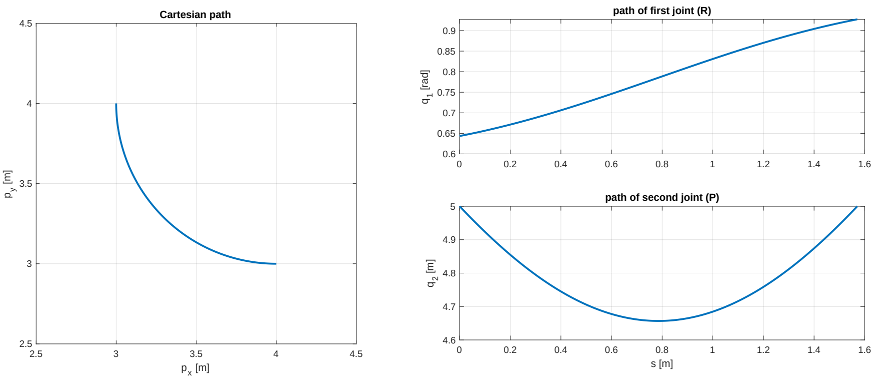

Figure 5: The Cartesian path $\boldsymbol{p}(s)$, an arc of a circle with center in C_0 and radius $R = 1$ m, and the corresponding joint path $\boldsymbol{q}(s) = (q_1(s), q_2(s))$ for the planar RP robot

special placement of the center C_0 of the circle (on the diagonal of the first quadrant) and of the two path points P_1 and P_2, (placed symmetrically with respect to the line passing through the origin and C_0).

The timing law $s(t)$ for $T = 2$ s has the profile shown in Fig. 6, where also the speed $\dot{s}(t)$ and the scalar acceleration $\ddot{s}(t)$ are reported. The analytic expression of the timing law is

$$s(t) = \begin{cases} \frac{1}{2} A\, t^2 & t \in [0, T/2] \\ \frac{L}{2} + \frac{AT}{2}\,(t - T/2) - \frac{1}{2}\, A\,(t - T/2)^2 & t \in [T/2, T], \end{cases} \tag{5}$$

and thus

$$\dot{s}(t) = \begin{cases} A\, t & t \in [0, T/2] \\ \frac{AT}{2} - A\,(t - T/2) & t \in [T/2, T] \end{cases} \qquad \ddot{s}(t) = \begin{cases} A & t \in [0, T/2] \\ -A & t \in [T/2, T]. \end{cases}$$

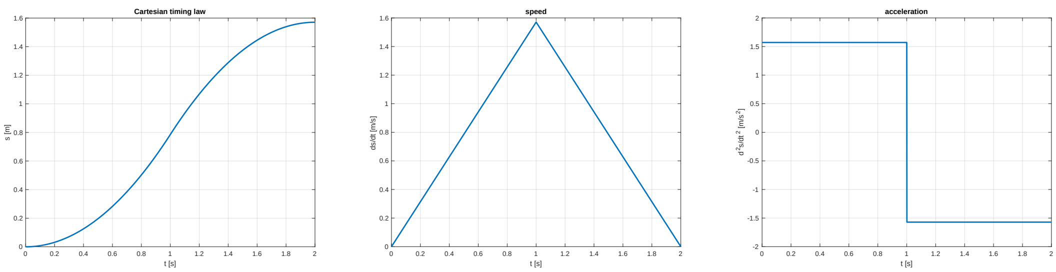

Figure 6: The Cartesian timing law $s(t)$, with trapezoidal speed $\dot{s}(t)$ and bang-bang acceleration $\ddot{s}(t)$

The only missing parameter is the maximum (absolute) value A of the bang-bang acceleration, which is obtained as follows. The triangular linear speed $\dot{s}(t)$ reaches its peak V at $t = T/2$,

$$V = \dot{s}(T/2) = A\,\frac{T}{2},$$

and the area below this profiles gives the total displacement $L = \pi/2$. Thus,

$$L = V\,\frac{T}{2} = A\left(\frac{T}{2}\right)^2 \qquad \Rightarrow \qquad A = \frac{4L}{T^2} = \frac{\pi}{2} = 1.5708 \text{ m/s}^2.$$

As a result, the joint trajectory $\boldsymbol{q}(t) = \boldsymbol{q}(s(t)) = (q_1(s(t)), q_2(s(t)))$ for $t \in [0, T]$ is obtained by combining the joint path $\boldsymbol{q}(s)$ in (2) with the timing law $s(t)$ in (5). Figure 7 compares the spatial and the time evolution of the joints, the latter being clearly modulated by the timing law $s(t)$.

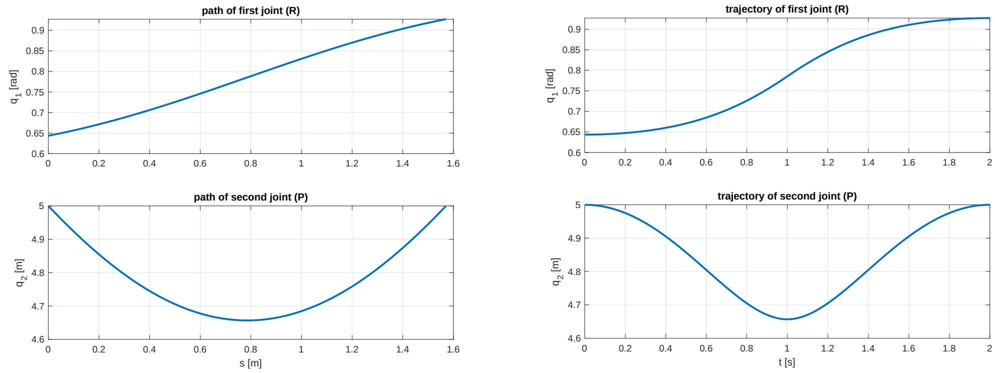

Figure 7: Comparison between the two components of the joint path $\boldsymbol{q}(s)$ and of the joint trajectory $\boldsymbol{q}(t)$

Similarly, the joint velocity and joint acceleration are obtained as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{q}'(s(t))\,\dot{s}(t) \qquad \ddot{\boldsymbol{q}}(t) = \boldsymbol{q}'(s(t))\,\ddot{s}(t) + \boldsymbol{q}''(s(t))\,\dot{s}^2(t),$$

where one uses the evaluation of (3) and (4) for $s = s(t)$ —see Fig. 8.

The maximum velocity of the first joint is reached at the trajectory midtime $T/2 = 1$ s: $\dot{q}_1(T/2) = q_1'(L/2) \cdot V = 0.2147\,(\pi/2) = 0.3373$ [rad/s]. At this instant, we have instead $\dot{q}_2(T/2) = 0$, since the prismatic joint reverses its direction of motion (the joint is here in its most retracted position).

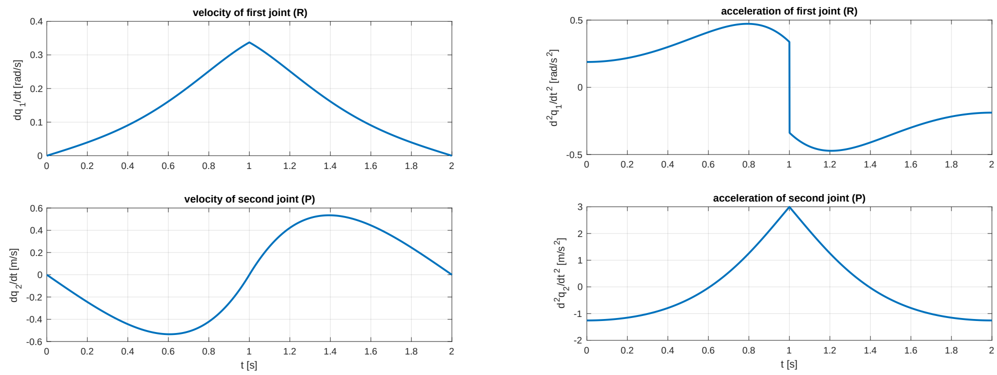

Figure 8: The two components of the joint velocity $\dot{\boldsymbol{q}}(t)$ and joint acceleration $\ddot{\boldsymbol{q}}(t)$

Finally, the Cartesian velocity and acceleration and their norms are respectively

$$\dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{ds}\frac{ds}{dt} = \boldsymbol{p}'\,\dot{s} \qquad \Rightarrow \qquad \|\dot{\boldsymbol{p}}\| = \|\boldsymbol{p}'\|\,|\dot{s}| = |\dot{s}|,$$

and

$$\ddot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{ds}\frac{d^2s}{dt^2} + \frac{d^2\boldsymbol{p}}{ds^2}\left(\frac{ds}{dt}\right)^2 = \boldsymbol{p}'\,\ddot{s} + \boldsymbol{p}''\,\dot{s}^2 = \begin{pmatrix} -\cos(s/R) & \sin(s/R) \\ \sin(s/R) & \cos(s/R) \end{pmatrix}\begin{pmatrix} \ddot{s} \\ \dot{s}^2/R \end{pmatrix} = \bar{\boldsymbol{R}}\begin{pmatrix} \ddot{s} \\ \dot{s}^2/R \end{pmatrix}$$

$$\Rightarrow \qquad \|\ddot{\boldsymbol{p}}\| = \sqrt{\ddot{\boldsymbol{p}}^T\ddot{\boldsymbol{p}}} = \sqrt{\|\boldsymbol{p}'\|^2\,\ddot{s}^2 + \|\boldsymbol{p}''\|^2\,(\dot{s}^2)^2} = \sqrt{\ddot{s}^2 + (\dot{s}^2/R)^2},$$

being $\bar{\boldsymbol{R}}$ an orthonormal matrix ($\bar{\boldsymbol{R}}^T \bar{\boldsymbol{R}} = \boldsymbol{I}$). Figure 9 shows the time evolution of these norms. Their maximum values are

$$\max_{t\in[0,T]} \|\dot{\boldsymbol{p}}(t)\| = \max_{t\in[0,T]} |\dot{s}(t)| = \dot{s}(T/2) = V = \frac{\pi}{2} = 1.5708 \text{ m/s}$$

and

$$\begin{aligned}\max_{t\in[0,T]} \|\ddot{\boldsymbol{p}}(t)\| &= \max_{t\in[0,T]} \sqrt{\ddot{s}^2(t) + (\dot{s}^2(t)/R)^2} = \sqrt{\ddot{s}^2(T/2) + (\dot{s}^2(T/2)/R)^2} \\ &= \sqrt{A^2 + (V^2)/R)^2} = \frac{\pi}{2}\sqrt{1 + (\pi/2)^2} = 2.9250 \text{ m/s}^2.\end{aligned}$$

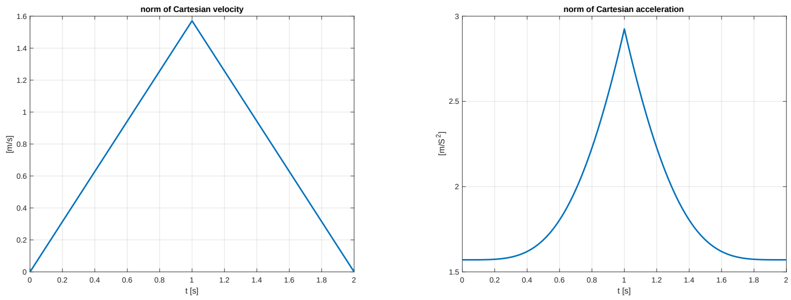

Figure 9: Time evolution of the norms of the end-effector velocity and acceleration, i.e., $\|\dot{\boldsymbol{p}}(t)\|$ and $\|\ddot{\boldsymbol{p}}(t)\|$

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