Robotics 1

June 12, 2023

Exercise 1

Consider the ABB CBR 15000 collaborative robot in Fig. 1, with six revolute joints. This robot has offsets at the elbow and at the wrist. More geometric information is available in the accompanying extra sheet.

Figure 1: The ABB CBR 15000 collaborative robot.

Assign the frames according to the standard Denavit-Hartenberg (DH) convention and fill in the corresponding table of parameters. The origin of the first DH frame is placed on the floor and that of the last frame should coincide with the center of the final flange. The assignment has to be consistent with the positive rotations of the joint variables, as specified by the manufacturer (see again the extra sheet). Moreover, none of the linear DH parameters should be negative (specify also their actual numerical value). Provide the values of the joint variables q_i, $i = 1, \ldots, 6$, in the configuration shown in the extra sheet.

Exercise 2

A unitary mass moves along a circular path centered at the origin of the (x, y) plane and having radius $R > 0$. At the initial time $t = 0$, the mass is in $A = (R, 0)$ while at the final time $t = T$ it should be in $B = (-R, 0)$. The timing law is chosen as a cubic rest-to-rest profile. If the norm of the Cartesian acceleration $\|\ddot{\boldsymbol{p}}\|$ is bounded by $A > 0$, what is the minimum feasible time T to execute the desired trajectory? At which time instant(s) is the bound attained? Provide a closed-form solution to the problem in symbolic form, and then evaluate it with the data $R = 1.5$ [m], $A = 3$ [m/s^2]. Sketch the time profile of the norm $\|\ddot{\boldsymbol{p}}(t)\|$ and of the components $\ddot{p}_x(t)$ and $\ddot{p}_y(t)$ of the obtained Cartesian acceleration $\ddot{\boldsymbol{p}}(t)$.

Exercise 3

- For the 4R spatial robot in Fig. 2, compute the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and find all its singular configurations $\boldsymbol{q}_s$, i.e., where rank $\boldsymbol{J}(\boldsymbol{q}_s) < 4$.
- Verify that $\boldsymbol{q}_0 = \boldsymbol{0}$ is NOT a singular configuration. With the robot at $\boldsymbol{q}_0$, show that one of the two following six-dimensional end-effector velocities

$$\boldsymbol{V}_a = \begin{pmatrix} \boldsymbol{v}_a \\ \boldsymbol{\omega}_a \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{V}_b = \begin{pmatrix} \boldsymbol{v}_b \\ \boldsymbol{\omega}_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

  is admissible while the other is not, being $\boldsymbol{v} \in I\!R^3$ the velocity of point $P = O_4$ and $\boldsymbol{\omega} \in I\!R^3$ the angular velocity of the DH reference frame RF_4.
- For the admissible end-effector velocity, determine a joint velocity $\dot{\boldsymbol{q}}_0 \in I\!R^4$ that realizes it, i.e., such that $\boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \boldsymbol{V}_i$, for either $i = a$ or $i = b$.

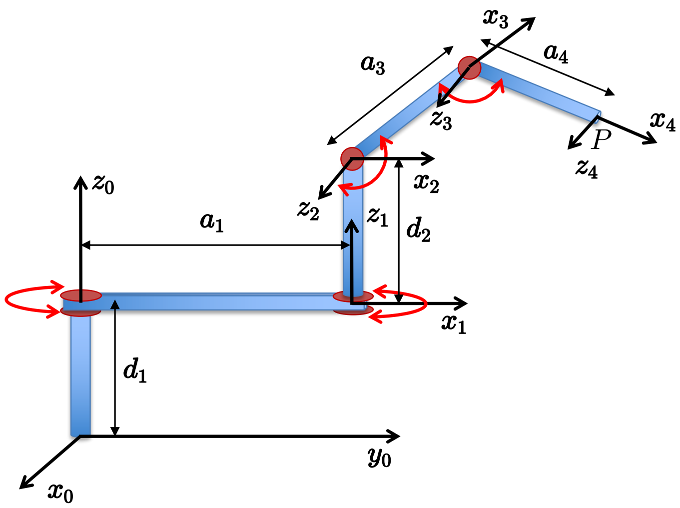

Figure 2: A 4R spatial robot, with DH frames and non-zero linear parameters shown.

[210 minutes, open books]

Solution

June 12, 2023

Exercise 1

An assignment of DH frames for the ABB robot consistent with the positive joint rotations specified by the manufacturer is shown in Fig. 3. The associated parameters are reported in Tab. 1. The numerical values of the linear DH parameters, expressed in [mm], are taken from the side view of the robot (see the extra sheet). The angular values of the joint variables correspond to the configuration shown in Fig. 3. Figure 4 shows the same frame assignment drawn on the side view picture of the robot.[1]

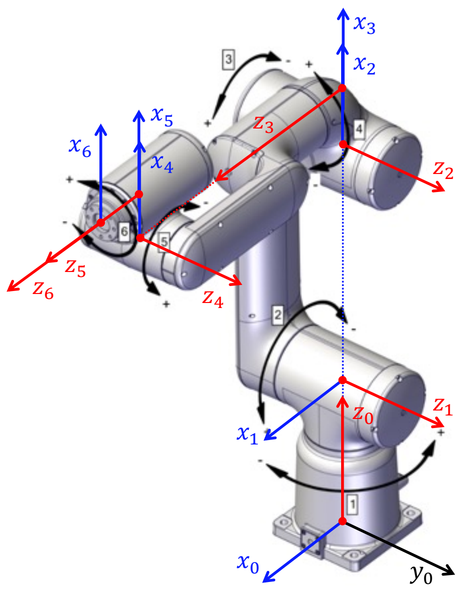

Figure 3: Assignment of DH frames for the ABB CBR 15000 robot.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 = 265$	$q_1 = 0$
2	0	$a_2 = 444$	0	$q_2 = -\pi/2$
3	$-\pi/2$	$a_3 = 110$	0	$q_3 = 0$
4	$\pi/2$	0	$d_4 = 470$	$q_4 = 0$
5	$-\pi/2$	$a_5 = 80$	0	$q_5 = 0$
6	0	0	$d_6 = 101$	$q_6 = 0$

Table 1: Table of DH parameters for the frame assignment in Fig. 3. Lengths are expressed in [mm]. The values of the joint variables (in blue) correspond to the configuration shown in Fig. 3.

[1] When compared to Fig. 3, this view is seen from the opposite side of the robot: the axes $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ and $\boldsymbol{z}_4$, which are not shown, are entering the page.

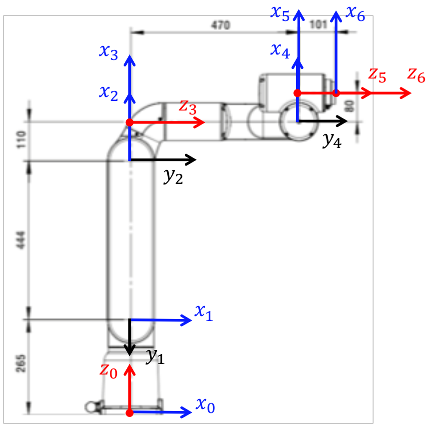

Figure 4: Another view of the DH frames assigned in Fig. 3.

Exercise 2

The given circular path from A to B in the plane (x, y) can be parametrized by

$$\boldsymbol{p} = \boldsymbol{p}(s) = R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \qquad \text{with } s \in [0, \Delta], \quad \Delta = \pi > 0.$$

The first and second spatial derivatives of $\boldsymbol{p}(s)$ are

$$\boldsymbol{p}' = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix}, \qquad \boldsymbol{p}'' = -R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}.$$

Further, the rest-to-rest cubic timing law is

$$s = s(t) = \Delta \left(3\tau^2 - 2\tau^3\right), \qquad \text{with } t \in [0, T], \quad \tau = \frac{t}{T} \in [0, 1],$$

where the total motion time T is to be determined. The first and second time derivatives of $s(t)$ are

$$\dot{s} = \frac{6\Delta}{T}\left(\tau - \tau^2\right), \qquad \ddot{s} = \frac{6\Delta}{T^2}\left(1 - 2\tau\right).$$

Accordingly, the Cartesian velocity and acceleration of the unitary mass will be

$$\dot{\boldsymbol{p}} = \boldsymbol{p}'\dot{s} = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \dot{s},$$

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \boldsymbol{p}'\ddot{s} + \boldsymbol{p}''\dot{s}^2 = R \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \ddot{s} - R \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \dot{s}^2 = R \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} -\dot{s}^2 \\ \ddot{s} \end{pmatrix}.$$

Therefore, the norm of the acceleration is computed as

$$\|\ddot{\boldsymbol{p}}\| = \sqrt{\ddot{\boldsymbol{p}}^T\ddot{\boldsymbol{p}}} = R\sqrt{\dot{s}^4 + \ddot{s}^2} = \frac{6R\Delta}{T^2}\sqrt{36\,\Delta^2\left(\tau(1-\tau)\right)^4 + (1-2\tau)^2} = \frac{6R\Delta}{T^2}\sqrt{\alpha(\tau)}. \tag{1}$$

For given Δ and R, this norm is only a function of the total motion time T, which has to be minimized while satisfying the bound $\|\ddot{\boldsymbol{p}}\| \le A$. Thus, we proceed with the analysis of the functional behavior of the acceleration norm.

The maximum of the norm (1) occurs when the argument $\alpha(\tau)$ of the square root has its maximum. This occurs either at the boundaries of the closed interval $[0, 1]$ of definition for τ or when the time derivative of $\alpha(\tau)$ vanishes. At the boundaries, we have

$$\alpha(0) = \alpha(1) = 1 \qquad \Rightarrow \qquad \|\ddot{\boldsymbol{p}}(t=0)\| = \|\ddot{\boldsymbol{p}}(t=T)\| = \frac{6R\Delta}{T^2} = \frac{6R\pi}{T^2}.$$

On the other hand, by zeroing the time derivative

$$\frac{d\alpha}{d\tau} = 144\,\Delta^2\,(\tau(1-\tau))^3\,(1-2\tau) - 4(1-2\tau) = \left(144\,\Delta^2\tau^3(1-\tau)^3 - 4\right)(1-2\tau) = 0, \tag{2}$$

we see that a first root is at $\tau = 0.5$ (i.e., $t = T/2$), in correspondence to which the norm takes the value

$$\|\ddot{\boldsymbol{p}}(t = T/2)\| = \frac{6R\,\Delta}{T^2}\cdot\frac{3\Delta}{8} = \frac{6R\,\pi}{T^2}\cdot\frac{3\pi}{8} > \frac{6R\,\pi}{T^2}.$$

Note that the acceleration norm at $t = T/2$ is larger than at the boundaries ($t = 0$ and $t = T$) because the path length to travel (as parametrized by the angle $\Delta = \pi$) is sufficiently long[2]. Next, when deleting the factor $(1-2\tau) \neq 0$ from (2), any other root $\tau = \tau^* \in [0,1]$ should satisfy

$$\tau^3(1-\tau)^3 = \frac{1}{36\Delta^2}. \tag{3}$$

However, by substituting (3) in the expression (1) of $\|\ddot{\boldsymbol{p}}\|$ and simplifying, it is easy to see that

$$\|\ddot{\boldsymbol{p}}(\tau = \tau^*)\| = \frac{6R\,\Delta}{T^2}\sqrt{\tau^*(1-\tau^*) + (1-2\tau^*)^2} = \frac{6R\,\Delta}{T^2}\sqrt{3\tau^{*2} - 3\tau^* + 1} \leq \frac{6R\,\Delta}{T^2},$$

where the last inequality holds for any $\tau^* \in [0,1]$. Thus, also in the stationary points of $\alpha(\tau)$ at the instants $\tau = \tau^*$, the acceleration norm is not larger than at $\tau = 0.5$.

In summary, we have shown that, for the given $\Delta = \pi$, the maximum acceleration norm occurs at $t = T/2$ and its value is

$$\max_{t\in[0,T]} \|\ddot{\boldsymbol{p}}(t)\| = \|\ddot{\boldsymbol{p}}(t = T/2)\| = \frac{9R\,\pi^2}{4T^2}.$$

Thus, the minimum feasible time T is found by equating the maximum acceleration norm to its bound A:

$$\frac{9R\,\pi^2}{4T^2} = A \qquad \Rightarrow \qquad T = \frac{3\pi}{2}\sqrt{\frac{R}{A}}.$$

Finally, substituting the numerical data $R = 1.5$ [m] and $A = 3$ [m/s^2], we obtain $T = 3.3322$ [s]. Figures 5–6 show the time evolution of $\|\ddot{\boldsymbol{p}}(t)\|$ and of its Cartesian components $\ddot{p}_x(t)$ and $\ddot{p}_y(t)$ for the obtained optimal solution.

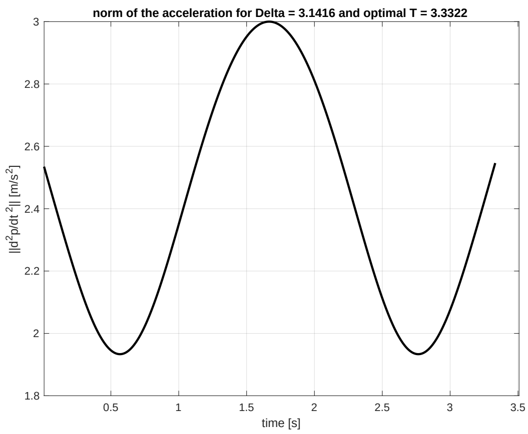

Figure 5: Norm of the optimal Cartesian acceleration $\ddot{\boldsymbol{p}}(t)$ for the given data.

[2]The crossover point is at $\Delta = 8/3 \approx 2.6666$. For smaller values, the path would be too short for the peak velocity $\dot{s}$ at $t = T/2$ to become dominant and the maximum norm would then occur at the boundaries of the time interval, where $\ddot{s}$ is maximum.

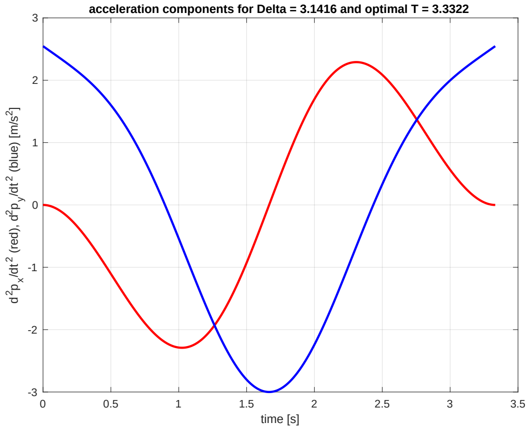

Figure 6: Cartesian components $\ddot{p}_x(t)$ (in red) and $\ddot{p}_y(t)$ (in blue) of the optimal acceleration $\ddot{\boldsymbol{p}}(t)$.

Exercise 3

The 6×4 geometric Jacobian of the 4R spatial robot in Fig. 2 is defined as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0 \times \boldsymbol{p}_{04} & \boldsymbol{z}_1 \times \boldsymbol{p}_{14} & \boldsymbol{z}_2 \times \boldsymbol{p}_{24} & \boldsymbol{z}_3 \times \boldsymbol{p}_{34} \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \boldsymbol{z}_3 \end{pmatrix}, \tag{4}$$

where $\boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and the other elements are determined through the computation of the direct kinematics of the robot. In alternative, the 3×4 linear (upper) block of the Jacobian in (4) can also be computed, perhaps more directly, as

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}, \qquad \text{with } \boldsymbol{p} = \boldsymbol{p}_{04}(\boldsymbol{q}), \tag{5}$$

i.e., using only the positional part of the direct kinematics for the origin O_4 of the DH frame RF_4.

Table 2 reports the DH parameters associated to the frames shown in Fig. 2 for the 4R spatial robot.

i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	q_1
2	$\pi/2$	0	d_2	q_2
3	0	a_3	0	q_3
4	0	a_4	0	q_4

Table 2: Table of DH parameters corresponding to the frames in Fig. 2.

From this, we compute (e.g., with the MATLAB code for the standard DH direct kinematics available in the course material)

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_{01}(q_1) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_{12} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^2\boldsymbol{R}_3(q_3) & {}^2\boldsymbol{p}_{23}(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_{34}(q_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Based on these homogeneous transformation matrices, one obtains

$$
\boldsymbol{p} = \begin{pmatrix} a_1c_1 + c_{12}\left(a_3c_3 + a_4c_{34}\right) \\ a_1s_1 + s_{12}\left(a_3c_3 + a_4c_{34}\right) \\ d_1 + d_2 + a_3s_3 + a_4s_{34} \end{pmatrix} \tag{6}
$$

and

$$
\begin{aligned}
\boldsymbol{z}_1 &= {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
\boldsymbol{z}_2 &= {}^0\boldsymbol{R}_2(q_1,q_2)\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0\right) = \begin{pmatrix} s_{12} \\ -c_{12} \\ 0 \end{pmatrix}, \\
\boldsymbol{z}_3 &= {}^0\boldsymbol{R}_3(q_1,q_2,q_3)\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1(q_1)\left({}^1\boldsymbol{R}_2(q_2)\left({}^2\boldsymbol{R}_3(q_3)\boldsymbol{z}_0\right)\right) = \begin{pmatrix} s_{12} \\ -c_{12} \\ 0 \end{pmatrix}.
\end{aligned} \tag{7}
$$

By differentiation of the positional direct kinematics in (6), one has

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -a_1s_1 - s_{12}\left(a_3c_3 + a_4c_{34}\right) & -s_{12}\left(a_3c_3 + a_4c_{34}\right) & -c_{12}\left(a_3s_3 + a_4s_{34}\right) & -a_4c_{12}s_{34} \\ a_1c_1 + c_{12}\left(a_3c_3 + a_4c_{34}\right) & c_{12}\left(a_3c_3 + a_4c_{34}\right) & -s_{12}\left(a_3s_3 + a_4s_{34}\right) & -a_4s_{12}s_{34} \\ 0 & 0 & a_3c_3 + a_4c_{34} & a_4c_{34} \end{pmatrix},
$$

while from the unit vectors in (7) it follows

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & s_{12} & s_{12} \\ 0 & 0 & -c_{12} & -c_{12} \\ 1 & 1 & 0 & 0 \end{pmatrix}.
$$

The obtained geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is expressed in frame RF_0 (or, ${}^0\boldsymbol{J}(\boldsymbol{q})$). Attempting to determine the singularities of this matrix by computing symbolically the determinant of the 4×4 matrix $\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})$

and setting it to zero may be too cumbersome for a standard symbolic manipulation program (such as MATLAB or Mathematica). On the other hand, because of the structure of the first two joints (having parallel axes), for singularity analysis it is definitely more convenient to express the Jacobian in the rotated frame RF_2:

$$
\begin{aligned}
{}^2\boldsymbol{J}(\boldsymbol{q}) &= {}^0\bar{\boldsymbol{R}}_2^T(q_1,q_2)\,{}^0\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{R}_2^T(q_1,q_2) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_2^T(q_1,q_2) \end{pmatrix} \begin{pmatrix} {}^0\boldsymbol{J}_L(\boldsymbol{q}) \\ {}^0\boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \\
&= \begin{pmatrix} a_1 s_2 & 0 & -(a_3 s_3 + a_4 s_{34}) & -a_4 s_{34} \\ 0 & 0 & a_3 c_3 + a_4 c_{34} & a_4 c_{34} \\ -(a_1 c_2 + a_3 c_3 + a_4 c_{34}) & -(a_3 c_3 + a_4 c_{34}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\end{aligned} \tag{8}
$$

Further, a combination of the columns of ${}^2\boldsymbol{J}(\boldsymbol{q})$ in (8) simplifies even more the analysis:

$$
{}^2\bar{\boldsymbol{J}}(\boldsymbol{q}) = {}^2\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{T} = {}^2\boldsymbol{J}(\boldsymbol{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} a_1 s_2 & 0 & -a_3 s_3 & -a_4 s_{34} \\ 0 & 0 & a_3 c_3 & a_4 c_{34} \\ -a_1 c_2 & -(a_3 c_3 + a_4 c_{34}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}
$$

Indeed, it is

$$
\operatorname{rank}\,{}^0\boldsymbol{J}(\boldsymbol{q}) = \operatorname{rank}\,{}^2\boldsymbol{J}(\boldsymbol{q}) = \operatorname{rank}\,{}^2\bar{\boldsymbol{J}}(\boldsymbol{q}).
$$

At this stage, we could evaluate $\det\left({}^2\bar{\boldsymbol{J}}^T(\boldsymbol{q})\,{}^2\bar{\boldsymbol{J}}(\boldsymbol{q})\right) = 0$ and find its solutions. However, we pursue here an alternative method which is even simpler. The rank of the 6×4 matrix in (9) will drop below 4 (singularity) if and only if all its 4×4 minors will simultaneously vanish. Since the fourth row is zero, there are only five minors that matter. Denote by ${}^2\bar{\boldsymbol{J}}_{-\{i,4\}}$ the 4×4 matrix obtained by deleting row 4 and row $i \neq 4$ from ${}^2\bar{\boldsymbol{J}}$. We impose then the following equalities:

$$
\begin{aligned}
\det\,{}^2\bar{\boldsymbol{J}}_{-\{1,4\}} &= -a_1 a_3 c_2 c_3 &&= 0 \\
\det\,{}^2\bar{\boldsymbol{J}}_{-\{2,4\}} &= a_1 a_3 c_2 s_3 &&= 0 \\
\det\,{}^2\bar{\boldsymbol{J}}_{-\{3,4\}} &= -a_1 a_3 s_2 c_3 &&= 0 \\
\det\,{}^2\bar{\boldsymbol{J}}_{-\{5,4\}} &= a_1 a_3 s_2 c_3 \left(a_3 c_3 + a_4 c_{34}\right) &&= 0 \\
\det\,{}^2\bar{\boldsymbol{J}}_{-\{6,4\}} &= -a_1 a_3 a_4 c_2 s_4 &&= 0.
\end{aligned} \tag{10}
$$

It is easy to see[3] that the system of five nonlinear equations (10) has a solution if and only if

$$
q_2 = \pm\frac{\pi}{2} \qquad \textbf{and} \qquad q_3 = \pm\frac{\pi}{2}
$$

(while q_4 does not matter), which characterize then all the singularities of the geometric Jacobian.

[3] This result is obtained by simple inspection of the equations. It can also be found by using MATLAB, with the two instructions:

```
eqn = [det(J_14) == 0; det(J_24) == 0; det(J_34) == 0; det(J_54) == 0; det(J_64) == 0];
q_sing = solve(eqn, [q_2 q_3], 'Real', true)
```

.

For $\boldsymbol{q}_0 = \boldsymbol{0}$, the geometric Jacobian

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 + a_3 + a_4 & a_3 + a_4 & 0 & 0 \\ 0 & 0 & a_3 + a_4 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is clearly nonsingular (rank $\boldsymbol{J}_0 = 4$, for non-vanishing a_1 and a_3). From the structure of the matrix $\boldsymbol{J}_0$ it follows that a component $\omega_x \neq 0$ can never be generated: thus, having $\omega_{b,x} = 1$, $\boldsymbol{V}_b$ is not admissible in this configuration,. On the other hand, $\boldsymbol{V}_a = \begin{pmatrix} 0 & 3 & -3 & 0 & 0 & 1 \end{pmatrix}^T$ is admissible since

$$\text{rank} \begin{pmatrix} \boldsymbol{J}_0 & \boldsymbol{V}_a \end{pmatrix} = \text{rank}\, \boldsymbol{J}_0 = 4 \qquad \Rightarrow \qquad \boldsymbol{V}_a \in \mathcal{R}\{\boldsymbol{J}_0\}.$$

A joint velocity that realizes $\boldsymbol{V}_a$ can be obtained by pseudoinversion of $\boldsymbol{J}_0$:

$$\begin{aligned} \dot{\boldsymbol{q}}_0 &= \boldsymbol{J}_0^{\#}\, \boldsymbol{V}_a = \left(\boldsymbol{J}_0^T \boldsymbol{J}_0\right)^{-1} \boldsymbol{J}_0^T\, \boldsymbol{V}_a \\ &= \begin{pmatrix} 0 & \dfrac{1}{a_1} & 0 & 0 & 0 & -\dfrac{a_3 + a_4}{a_1} \\ 0 & -\dfrac{1}{a_1} & 0 & 0 & 0 & \dfrac{a_1 + a_3 + a_4}{a_1} \\ 0 & 0 & \dfrac{1}{a_3} & 0 & \dfrac{a_4}{a_3} & 0 \\ 0 & 0 & -\dfrac{1}{a_3} & 0 & -\dfrac{a_3 + a_4}{a_3} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{a_3 + a_4 - 3}{a_1} \\ \dfrac{a_1 + a_3 + a_4 - 3}{a_1} \\ -\dfrac{3}{a_3} \\ \dfrac{3}{a_3} \end{pmatrix}. \end{aligned}$$

It can be immediately check that $\boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \boldsymbol{V}_a$. Moreover, being $\boldsymbol{J}_0$ full column rank, the joint velocity $\dot{\boldsymbol{q}}_0$ is the unique solution.

Finally, note that one can also compute $\dot{\boldsymbol{q}}_b = \boldsymbol{J}_0^{\#}\, \boldsymbol{V}_b = \begin{pmatrix} -\frac{a_3+a_4}{a_1} & \frac{a_1+a_3+a_4}{a_1} & \frac{1}{a_3} & -\frac{1}{a_3} \end{pmatrix}^T$, but this joint velocity will not return the desired end-effector velocity, being $\dot{\boldsymbol{e}}_b = \boldsymbol{V}_b - \boldsymbol{J}_0\, \dot{\boldsymbol{q}}_b = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T$. We immediately see that the velocity error is restricted to the inadmissible component ω_x. Moreover, thanks to the property of the pseudoinverse, $\dot{\boldsymbol{q}}_b$ is the approximate solution that minimizes the norm of the task velocity error $\dot{\boldsymbol{e}}_b$ among all possible joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^4$.

* * * * *