Robotics I

March 27, 2018

Exercise 1

Consider the 5-dof spatial robot in Fig. 1, having the third and fifth joints of the prismatic type while the others are revolute.

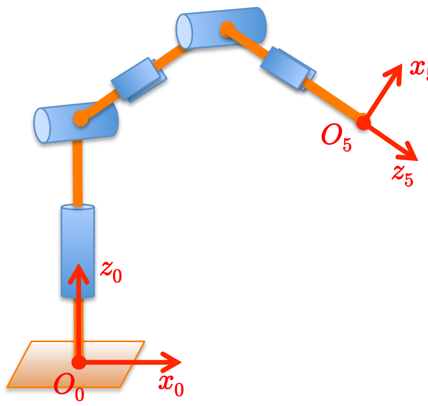


Figure 1: A 5-dof robot, with a RRPRP joint sequence, moving in 3D space.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are *non-negative*. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 5 are already assigned and should not be modified. [Please, make clean drawings and return the completed sheet with your name written on it.]
- Sketch the robot in the configuration $\boldsymbol{q}_a = \begin{pmatrix} 0 & \frac{\pi}{2} & 1 & \frac{\pi}{2} & 1 \end{pmatrix}^T$ [rad, rad, m, rad, m].
- For which value $\boldsymbol{q}_b \in \mathbb{R}^5$ does the robot assume a stretched upward configuration?
- Determine the symbolic expression of the 6×5 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for this robot.
- In the configuration $\boldsymbol{q}_a$, find as many independent *wrench* vectors $\boldsymbol{w} \in \mathbb{R}^6$ (of forces and moments) as possible, with

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix} \neq \boldsymbol{0}, \qquad \boldsymbol{f} \in \mathbb{R}^3, \qquad \boldsymbol{m} \in \mathbb{R}^3,$$

  such that when any of these wrenches is applied to the end-effector, the robot remains in static equilibrium without the need of balancing generalized forces at the joints ($\boldsymbol{\tau} = \boldsymbol{0}$, with some components being forces and some torques).

Exercise 2

A number of statements are reported on the extra sheet #2, regarding singularity issues in the direct kinematics of serial manipulators. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the completed sheet with your name on it.]

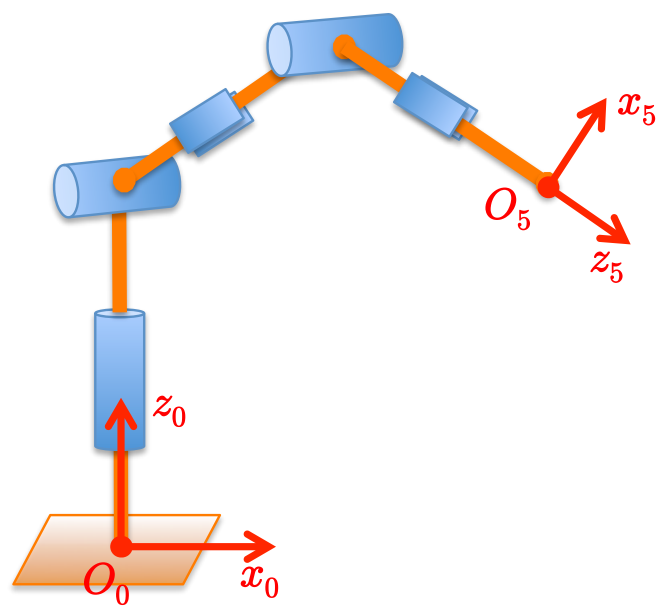
[180 minutes, open books but no computer or smartphone]

5-dof spatial robot – DH frames assignment and table

Name: ____

i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				

all constant DH parameters
should be ≥ 0



Robotics I – Sheet for Exercise 1, March 27, 2018

Robotics I - Sheet for Exercise 2

March 27, 2018

Name: ______________________________

Consider only serial manipulators having $\boldsymbol{q} \in \mathbb{R}^6$, with direct kinematics expressed by homogenous transformation matrices ${}^0\boldsymbol{T}_6(\boldsymbol{q})$, and their 6×6 geometric Jacobians $\boldsymbol{J}(\boldsymbol{q})$. Check if each of the following statements about singularities is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.

   True ☐ **False** ☐

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.

   True ☐ **False** ☐

3. Close to a singularity of $\boldsymbol{J}$, some Cartesian directions of motion are not accessible.

   True ☐ **False** ☐

4. In a singularity, the end-effector angular velocities $\boldsymbol{\omega}$ are linearly dependent on the linear velocities $\boldsymbol{v}$.

   True ☐ **False** ☐

5. In a singular configuration, $\mathcal{R}\{\boldsymbol{J}^T\} \oplus \mathcal{N}\{\boldsymbol{J}\} \neq \mathbb{R}^6$.

   True ☐ **False** ☐

6. The linear part $\boldsymbol{J}_L(\boldsymbol{q})$ and the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the Jacobian cannot lose rank simultaneously.

   True ☐ **False** ☐

7. The lower is the rank of $\boldsymbol{J}$, the larger is the loss of mobility of the end-effector.

   True ☐ **False** ☐

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q})\}$.

   True ☐ **False** ☐

9. There cannot be singularities of $\boldsymbol{J}(\boldsymbol{q})$ outside the joint range of the manipulator.

   True ☐ **False** ☐

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.

    True ☐ **False** ☐

Solution

March 27, 2018

Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1, respectively. All constant parameters are non-negative, as requested.

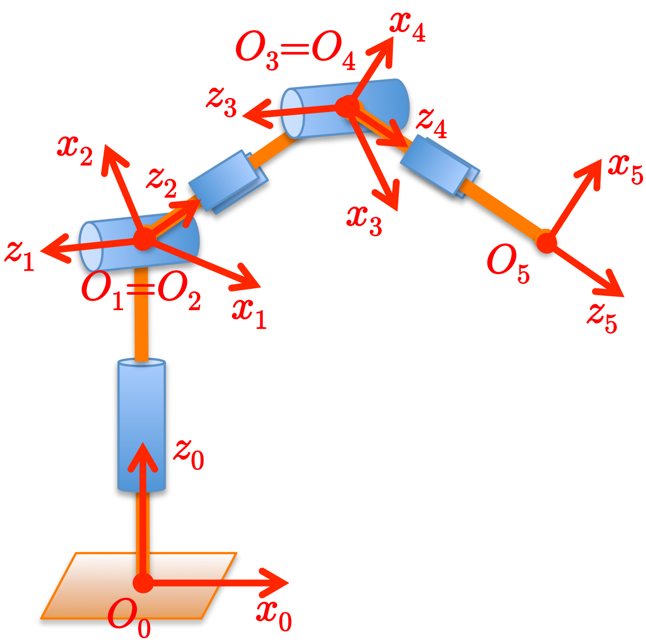


Figure 2: A DH frame assignment for the spatial RRPRP robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$\pi/2$	0	q_3	π
4	$\pi/2$	0	0	q_4
5	0	0	q_5	0

Table 1: Parameters associated to the DH frames in Fig. 2.

For later use, based on Tab. 1, the five DH homogeneous transformation matrices are:

$$
\begin{aligned}
{}^0\boldsymbol{A}_1(q_1) &= \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^1\boldsymbol{A}_2(q_2) &= \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^2\boldsymbol{A}_3(q_3) &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^2\boldsymbol{R}_3 & {}^2\boldsymbol{p}_3(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
{}^3\boldsymbol{A}_4(q_4) &= \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4 \\ \boldsymbol{0}^T & 1 \end{pmatrix}, \\
{}^4\boldsymbol{A}_5(q_5) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^4\boldsymbol{R}_5 & {}^4\boldsymbol{p}_5(q_5) \\ \boldsymbol{0}^T & 1 \end{pmatrix}.
\end{aligned} \tag{2}
$$

A sketch of the robot in the configuration $\boldsymbol{q}_a = (0, \pi/2, 1, \pi/2, 1)$ is given on the left of Fig. 3, while on the right a stretched upward configuration is shown, corresponding to $\boldsymbol{q}_b = (0, \pi, 1, \pi, 1)$.

In order to compute the linear part $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for this robot, it is convenient to compute first the end-effector position $\boldsymbol{p}$ and then to proceed by symbolic differentiation. For efficiency, we compute this vector (in homogeneous coordinates) using the recursive formula:

$$
\begin{aligned}
\boldsymbol{p}_h(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{p}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left({}^1\boldsymbol{A}_1(q_2) \left({}^2\boldsymbol{A}_3(q_3) \left({}^3\boldsymbol{A}_4(q_4) \left({}^4\boldsymbol{A}_5(q_5) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right) \right) \right) \right) \\
&= \begin{pmatrix} \cos q_1 \left(q_3 \sin q_2 - q_5 \sin(q_2 + q_4) \right) \\ \sin q_1 \left(q_3 \sin q_2 - q_5 \sin(q_2 + q_4) \right) \\ d_1 - q_3 \cos q_2 + q_5 \cos(q_2 + q_4) \\ 1 \end{pmatrix}.
\end{aligned}
$$

Therefore, resorting to the usual compact notation, we obtain

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} s_1 \left(q_5 s_{24} - q_3 s_2 \right) & c_1 \left(q_3 c_2 - q_5 c_{24} \right) & c_1 s_2 & -q_5 c_1 c_{24} & -c_1 s_{24} \\ -c_1 \left(q_5 s_{24} - q_3 s_2 \right) & s_1 \left(q_3 c_2 - q_5 c_{24} \right) & s_1 s_2 & -q_5 s_1 c_{24} & -s_1 s_{24} \\ 0 & q_3 s_2 - q_5 s_{24} & -c_2 & -q_5 s_{24} & c_{24} \end{pmatrix}.
$$

For the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian, taking into account that the third and fifth

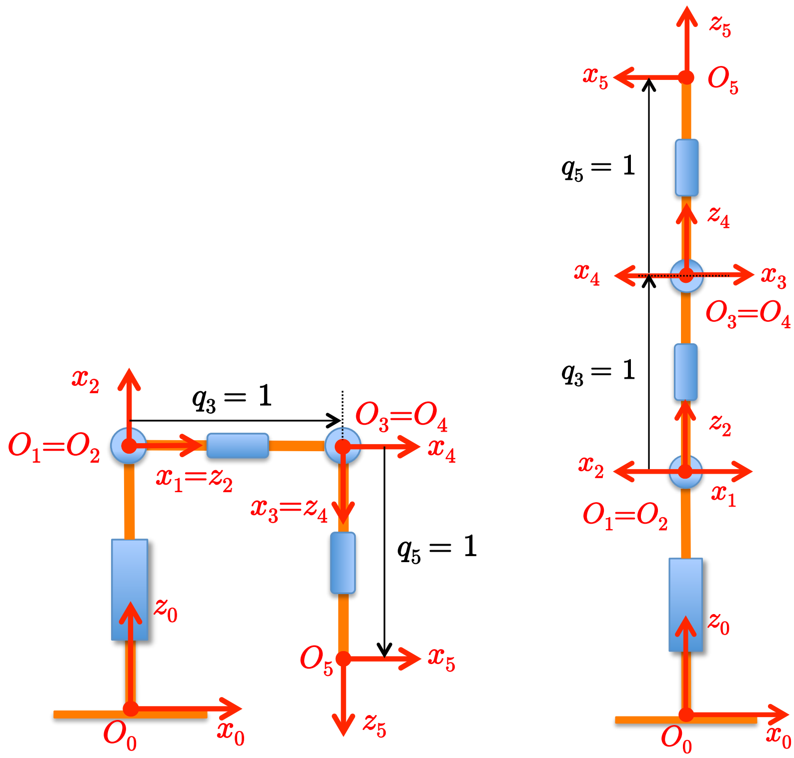


Figure 3: A side view of the RRPRP robot in the configuration $\boldsymbol{q}_a = (0, \pi/2, 1, \pi/2, 1)$ and in a stretched upward configuration with $\boldsymbol{q}_b = (0, \pi, 1, \pi, 1)$.

joints are prismatic, we have

$$
\begin{aligned}
\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{z}_3 & \boldsymbol{0} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\,{}^2\boldsymbol{R}_3(q_3)\,{}^3\boldsymbol{z}_3 & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} 0 & s_1 & 0 & s_1 & 0 \\ 0 & -c_1 & 0 & -c_1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T = \boldsymbol{z}_0$ for any i.

The complete Jacobian is then

$$
\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}.
$$

In the assigned configuration $\boldsymbol{q}_a = \begin{pmatrix} 0 & \frac{\pi}{2} & 1 & \frac{\pi}{2} & 1 \end{pmatrix}^T$ the transpose of this Jacobian matrix takes the value

$$
\boldsymbol{J}^T(\boldsymbol{q}_a) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank} \boldsymbol{J}^T(\boldsymbol{q}_a) = 4.
$$

It is easy to see that the null space of $\boldsymbol{J}^T(q_a)$ is spanned, e.g., by the two wrenches

$$\boldsymbol{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The wrench $\boldsymbol{w}_1$ corresponds to a pure moment with $m_x \neq 0$, while $\boldsymbol{w}_2$ is associated to a force $f_y \neq 0$, combined with a moment $m_z \neq 0$. The generalized forces in the joint space needed for balancing any wrench generated by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are indeed

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}_a)\left(\alpha_1 \boldsymbol{w}_1 + \alpha_2 \boldsymbol{w}_2\right) = \boldsymbol{0}, \qquad \forall \alpha_1, \alpha_2.$$

Exercise 2

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.

   True. The number of solutions changes from the generic case, decreasing or going to infinity.

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.

   True. There is no mobility loss in the joint space commanding motion without inversion of $\boldsymbol{J}$.

3. Close to a singularity of $\boldsymbol{J}$, some Cartesian directions of motion are not accessible.

   False. This is true in a singular configuration, not close to it (though motion effort may increase).

4. In a singularity, the end-effector angular velocities $\boldsymbol{\omega}$ are linearly dependent on the linear velocities $\boldsymbol{v}$.

   False. Not necessarily. It depends on the geometric relation between subspaces $\mathcal{R}\{\boldsymbol{J}_L\}$ and $\mathcal{R}\{\boldsymbol{J}_A\}$.

5. In a singular configuration, $\mathcal{R}\{\boldsymbol{J}^T\} \oplus \mathcal{N}\{\boldsymbol{J}\} \neq \mathbb{R}^6$.

   False. The direct sum of these two subspaces covers always the entire joint space.

6. The linear part $\boldsymbol{J}_L(\boldsymbol{q})$ and the angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the Jacobian cannot lose rank simultaneously.

   False. Both ranks of $\boldsymbol{J}_L$ and $\boldsymbol{J}_A$ can be < 3 (when both are full rank, it may still be rank $\boldsymbol{J} < 6$).

7. The lower is the rank of $\boldsymbol{J}$, the larger is the loss of mobility of the end-effector.

   True. For instance, two 6-dim independent Cartesian directions are inaccessible when rank $\boldsymbol{J} = 4$.

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q})\}$.

   True. $\boldsymbol{J}$ is singular iff its null space is $\neq \boldsymbol{0}$ —the condition can be used in the search of singularities.

9. There cannot be singularities of $\boldsymbol{J}(\boldsymbol{q})$ outside the joint range of the manipulator.

   False. Singularities are found without considering the joint range. Those outside are then discarded.

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.

    False. Crossing a singular configuration on a feasible Cartesian cycle can destroy joint-space cyclicity.

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