Exercise 1

Consider the 5-dof spatial robot in Fig. 1 having the third and fifth joints of the prismatic type while the others are revolute.

Figure 1: A 5-dof robot, with a RRPRP joint sequence, moving in 3D space.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are non-negative. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 5 are already assigned and should not be modified. [Please, make clean drawings and return the completed sheet with your name written on it.]

- Sketch the robot in the configuration $q_a = \begin{pmatrix} 0 & \frac{\pi}{2} & 1 & \frac{\pi}{2} & 1 \end{pmatrix}^T$ [rad, rad, m, rad, m].

- For which value $q_b \in \mathbb{R}^5$ does the robot assume a stretched upward configuration?

- Determine the symbolic expression of the $6 \times 5$ geometric Jacobian $J(q)$ for this robot.

- In the configuration $q_a$, find as many independent wrench vectors $w \in \mathbb{R}^6$ (of forces and moments) as possible, with

$$w = \begin{pmatrix} f \\ m \end{pmatrix} \neq 0, \quad f \in \mathbb{R}^3, \quad m \in \mathbb{R}^3,$$

such that when any of these wrenches is applied to the end-effector, the robot remains in static equilibrium without the need of balancing generalized forces at the joints ($\tau = 0$, with some components being forces and some torques).

Exercise 2

A number of statements are reported on the extra sheet #2, regarding singularity issues in the direct kinematics of serial manipulators. Check if each statement is True or False, providing also a very short motivation/explanation for your answer. [Return the completed sheet with your name on it.]

[180 minutes, open books but no computer or smartphone]
5-dof spatial robot – DH frames assignment and table

Name: ________________________________

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All constant DH parameters should be $\geq 0$

Robotics I – Sheet for Exercise 1, March 27, 2018
Consider only serial manipulators having \( q \in \mathbb{R}^6 \), with direct kinematics expressed by homogeneous transformation matrices \( {}^0T_6(q) \), and their \( 6 \times 6 \) geometric Jacobians \( J(q) \). Check if each of the following statements about singularities is \textbf{True} or \textbf{False}, and provide a \textit{very short} motivating/explanation sentence.

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.
   \[ \text{True} \quad \text{False} \]

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.
   \[ \text{True} \quad \text{False} \]

3. Close to a singularity of \( J \), some Cartesian directions of motion are not accessible.
   \[ \text{True} \quad \text{False} \]

4. In a singularity, the end-effector angular velocities \( \omega \) are linearly dependent on the linear velocities \( v \).
   \[ \text{True} \quad \text{False} \]

5. In a singular configuration, \( \mathcal{R}\{J^T\} \oplus \mathcal{N}\{J\} \neq \mathbb{R}^6 \).
   \[ \text{True} \quad \text{False} \]

6. The linear part \( J_L(q) \) and the angular part \( J_A(q) \) of the Jacobian cannot lose rank simultaneously.
   \[ \text{True} \quad \text{False} \]

7. The lower is the rank of \( J \), the larger is the loss of mobility of the end-effector.
   \[ \text{True} \quad \text{False} \]

8. All singularities of a manipulator can be found by inspecting the null space \( \mathcal{N}\{J(q)\} \).
   \[ \text{True} \quad \text{False} \]

9. There cannot be singularities of \( J(q) \) outside the joint range of the manipulator.
   \[ \text{True} \quad \text{False} \]

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.
    \[ \text{True} \quad \text{False} \]
Exercise 1

A possible DH frame assignment and the associated table of parameters are reported in Fig. 2 and Tab. 1 respectively. All constant parameters are non-negative, as requested.

Figure 2: A DH frame assignment for the spatial RRPRP robot.

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<tr>
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<td>0</td>
<td>$q_5$</td>
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</table>

Table 1: Parameters associated to the DH frames in Fig. 2
For later use, based on Tab. 1 the five DH homogeneous transformation matrices are:

\[ 0A_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0R_1(q_1) \quad 0p_1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ 1A_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1R_2(q_2) \quad 1p_2) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ 2A_3(q_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (2R_3 \quad 2p_3) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ 3A_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 & 0 \\ \sin q_4 & 0 & -\cos q_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (3R_4(q_4) \quad 3p_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ 4A_5(q_5) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (4R_5 \quad 4p_5) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

A sketch of the robot in the configuration \( q_a = (0, \pi/2, 1, \pi/2, 1) \) is given on the left of Fig. 3 while on the right a stretched upward configuration is shown, corresponding to \( q_b = (0, \pi, 1, \pi, 1) \).

In order to compute the linear part \( J_L(q) \) of the geometric Jacobian \( J(q) \) for this robot, it is convenient to compute first the end-effector position \( p \) and then to proceed by symbolic differentiation. For efficiency, we compute this vector (in homogeneous coordinates) using the recursive formula:

\[ p(q) = \begin{pmatrix} p(q) \\ 1 \end{pmatrix} = 0A_1(q_1) \begin{pmatrix} 1A_2(q_2) \begin{pmatrix} 2A_3(q_3) \begin{pmatrix} 3A_4(q_4) \begin{pmatrix} 4A_5(q_5) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix}. \]

Therefore, resorting to the usual compact notation, we obtain

\[ J_L(q) = \frac{\partial p(q)}{\partial q} = \begin{pmatrix} s_1(q_5s_{24} - q_3s_{22}) & c_1(q_4s_{23} - q_5s_{24}) & c_1s_2 & c_1c_2 & -c_1c_2 & -s_1s_{24} \\ -c_1(q_3s_{24} - q_1s_{23}) & s_1(q_4s_{23} - q_5s_{24}) & s_1s_2 & c_1s_2 & -c_1s_2 & -s_1s_{24} \\ 0 & q_3s_{22} - q_5s_{24} & -c_2 & -q_5s_{24} & c_2 & -s_1s_{24} \end{pmatrix}. \]

For the angular part \( J_A(q) \) of the geometric Jacobian, taking into account that the third and fifth
joints are prismatic, we have
\[
J_A(q) = \begin{pmatrix}
0z_0 & 0z_1 & 0 & 0z_3 & 0
\end{pmatrix}
= \begin{pmatrix}
z_0 & 0R_1(q_1)z_1 & 0 & 0R_1(q_1)R_2(q_2)z_2R_3(q_3)z_3 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & s_1 & 0 & s_1 & 0 \\
0 & -c_1 & 0 & -c_1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
being \(z_i = (0, 0, 1)^T = z_0\) for any \(i\).

The complete Jacobian is then
\[
J(q) = \begin{pmatrix}
J_L(q) \\
J_A(q)
\end{pmatrix}.
\]

In the assigned configuration \(q_a = \begin{pmatrix} 0 & \pi/2 & 1 & \pi/2 & 1 \end{pmatrix}^T\) the transpose of this Jacobian matrix takes the value
\[
J^T(q_a) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \text{rank} J^T(q_a) = 4.
It is easy to see that the null space of $J^T(q_a)$ is spanned, e.g., by the two wrenches

$$w_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

The wrench $w_1$ corresponds to a pure moment with $m_x \neq 0$, while $w_2$ is associated to a force $f_y \neq 0$, combined with a moment $m_z \neq 0$. The generalized forces in the joint space needed for balancing any wrench generated by $w_1$ and $w_2$ are indeed

$$\tau = J^T(q_a) (\alpha_1 w_1 + \alpha_2 w_2) = 0, \quad \forall \alpha_1, \alpha_2.$$

**Exercise 2**

1. In a singular configuration, there may be an infinite number of inverse kinematics solutions.
   **True.** The number of solutions changes from the generic case, decreasing or going to infinity.

2. In a singularity, the manipulator can access instantaneously any nearby joint configuration.
   **True.** There is no mobility loss in the joint space commanding motion without inversion of $J$.

3. Close to a singularity of $J$, some Cartesian directions of motion are not accessible.
   **False.** This is true in a singular configuration, not close to it (though motion effort may increase).

4. In a singularity, the end-effector angular velocities $\omega$ are linearly dependent on the linear velocities $v$.
   **False.** Not necessarily. It depends on the geometric relation between subspaces $\mathcal{R}\{J_L\}$ and $\mathcal{R}\{J_A\}$.

5. In a singular configuration, $\mathcal{R}\{J_L^T\} \oplus \mathcal{N}\{J\} \neq \mathbb{R}^6$.
   **False.** The direct sum of these two subspaces covers always the entire joint space.

6. The linear part $J_L(q)$ and the angular part $J_A(q)$ of the Jacobian cannot lose rank simultaneously.
   **False.** Both ranks of $J_L$ and $J_A$ can be $< 3$ (when both are full rank, it may still be rank $J < 6$).

7. The lower is the rank of $J$, the larger is the loss of mobility of the end-effector.
   **True.** For instance, two 6-dim independent Cartesian directions are inaccessible when rank $J = 4$.

8. All singularities of a manipulator can be found by inspecting the null space $\mathcal{N}\{J(q)\}$.
   **True.** $J$ is singular iff its null space is $\neq 0$ —the condition can be used in the search of singularities.

9. There cannot be singularities of $J(q)$ outside the joint range of the manipulator.
   **False.** Singularities are found without considering the joint range. Those outside are then discarded.

10. Cyclic motions in the Cartesian space always correspond to cyclic motions in the joint space.
    **False.** Crossing a singular configuration on a feasible Cartesian cycle can destroy joint-space cyclicity.