Robotics I

April 11, 2017

Exercise 1

The kinematics of a 3R spatial robot is specified by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	d_i	a_i	θ_i
1	$\pi/2$	L_1	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

Table 1: Table of DH parameters of a 3R spatial robot.

- Given a position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin of the end-effector frame, provide the analytic expression of the solution to the inverse kinematics problem.
- For $L_1 = 1$ [m] and $L_2 = L_3 = 1.5$ [m], determine all inverse kinematics solutions in numerical form associated to the end-effector position $\boldsymbol{p} = \begin{pmatrix} -1 & 1 & 1.5 \end{pmatrix}^T$ [m].

Exercise 2

A robot joint should move in minimum time between an initial value q_a and a final value q_b, with an initial velocity $\dot{q}_a$ and a final velocity $\dot{q}_b$, under the bounds $|\dot{q}| \leq V$ and $|\ddot{q}| \leq A$.

- Provide the analytic expression of the minimum feasible motion time T^* when $\Delta q = q_b - q_a > 0$ and the initial and final velocities are arbitrary in sign and magnitude (but both satisfy the velocity bound, i.e., $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$).
- Using the data $q_a = -90°$, $q_b = 30°$, $\dot{q}_a = 45°/s$, $\dot{q}_b = -45°/s$, $V = 90°/s$, $A = 200°/s^2$, determine the numerical value of the minimum feasible motion time T^* and draw the velocity and acceleration profiles of the joint motion.

[180 minutes, open books but no computer or smartphone]

Solution

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Exercise 1

From the direct kinematics, using Tab. 1, we obtain for the position of the origin of the end-effector frame

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)$$

$$\Rightarrow \quad \boldsymbol{p} = \begin{pmatrix} (L_2\cos q_2 + L_3\cos(q_2+q_3))\cos q_1 \\ (L_2\cos q_2 + L_3\cos(q_2+q_3))\sin q_1 \\ L_1 + L_2\sin q_2 + L_3\sin(q_2+q_3) \end{pmatrix}. \tag{1}$$

The analytic inversion of eq. (1) for $\boldsymbol{p} = \boldsymbol{p}_d = \begin{pmatrix} p_{dx} & p_{dy} & p_{dz} \end{pmatrix}^T$ proceeds as follows. After moving L_1 to the left-hand side of the third equation, squaring and adding the three equations yields the numeric value c_3 (for $\cos q_3$)

$$c_3 = \frac{p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 - L_2^2 - L_3^2}{2L_2L_3}. \tag{2}$$

The desired end-effector position will belong to the robot workspace if and only if $c_3 \in [-1, 1]$. Note that this condition holds no matter if L_2 and L_3 are equal or different. Under such premises, we compute

$$s_3 = \sqrt{1 - c_3^2} \tag{3}$$

and

$$q_3^{\{+\}} = \text{ATAN2}\{s_3, c_3\}, \qquad q_3^{\{-\}} = \text{ATAN2}\{-s_3, c_3\}, \tag{4}$$

yielding by definition two opposite values $q_3^{\{-\}} = -q_3^{\{+\}}$. If $c_3 = \pm 1$, the robot is in a kinematic singularity: the forearm is either stretched or folded, in both cases on the boundary of the workspace. In particular, when $c_3 = 1$, $q_3^{\{+\}}$ and $q_3^{\{-\}}$ are both equal to 0; when $c_3 = -1$, the two solutions will be taken[1] equal to π. Instead, when $c_3 \notin [-1, 1]$, the inverse kinematics algorithm should output a warning message ("desired position is out of workspace") and exit.

When $p_{dx}^2 + p_{dy}^2 > 0$, from the first two equations in (1) we can further compute

$$p_{dx}^2 + p_{dy}^2 = (L_2\cos q_2 + L_3\cos(q_2+q_3))^2 \;\Rightarrow\; \cos q_1 = \frac{p_{dx}}{\pm\sqrt{p_{dx}^2 + p_{dy}^2}}, \quad \sin q_1 = \frac{p_{dy}}{\pm\sqrt{p_{dx}^2 + p_{dy}^2}},$$

and thus

$$q_1^{\{+\}} = \text{ATAN2}\{p_{dy}, p_{dx}\}, \qquad q_1^{\{-\}} = \text{ATAN2}\{-p_{dy}, -p_{dx}\}. \tag{5}$$

These two values belong to $(-\pi, \pi]$ and will always differ by π. Instead, when $p_{dx} = p_{dy} = 0$, the first joint angle q_1 remains undefined and the robot will be in a kinematic singularity (with the end-effector placed along the axis of joint 1). The solution algorithm should output a warning message ("singular case: angle q_1 is undefined"), possibly set a flag ($sing_1 = ON$), but continue.

[1] Remember that we use as conventional range $q \in (-\pi, \pi]$, for all angles q. Thus, if the output of a generic computation is $-\pi$, we always replace it with $+\pi$.

At this stage, we can rewrite a suitable combination of the first two equations in (1) as well as the third equation in the following way:

$$\cos q_1 p_{dx} + \sin q_1 p_{dy} = L_2 \cos q_2 + L_3 \cos(q_2 + q_3) = (L_2 + L_3 \cos q_3) \cos q_2 - L_3 \sin q_3 \sin q_2$$

and

$$p_{dz} - L_1 = L_2 \sin q_2 + L_3 \sin(q_2 + q_3) = L_3 \sin q_3 \cos q_2 + (L_2 + L_3 \cos q_3) \sin q_2.$$

Plugging the (multiple) values found so far for q_1 and q_3, we obtain four similar 2×2 linear systems in the trigonometric unknowns $c_2 = \cos q_2$ and $s_2 = \sin q_2$:

$$\begin{pmatrix} L_2 + L_3 c_3 & -L_3 s_3^{\{+,-\}} \\ L_3 s_3^{\{+,-\}} & L_2 + L_3 c_3 \end{pmatrix} \begin{pmatrix} c_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} \cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy} \\ p_{dz} - L_1 \end{pmatrix} \iff \boldsymbol{A}^{\{+,-\}} \boldsymbol{x} = \boldsymbol{b}^{\{+,-\}}. \tag{6}$$

In (6), we should use (2) and the values from (4) and (5). This gives rise to four possible combinations for the matrix/vector pair $\left(\boldsymbol{A}^{\{+,-\}}, \boldsymbol{b}^{\{+,-\}}\right)$, which will eventually lead to four solutions for q_2 that are in general distinct[2]. These will be labeled as

$$q_2^{\{f,u\}} \quad q_2^{\{f,d\}} \quad q_2^{\{b,u\}} \quad q_2^{\{b,d\}} \qquad \Rightarrow \qquad \boldsymbol{q}^{\{f,u\}} \quad \boldsymbol{q}^{\{f,d\}} \quad \boldsymbol{q}^{\{b,u\}} \quad \boldsymbol{q}^{\{b,d\}}$$

depending on whether the robot is facing (f) of backing (b) the desired position quadrant —due to the choice of q_1, and on whether the elbow is up (u) or down (d) —due to the combined choice of q_1 and q_3. If the (common) determinant of the coefficient matrix is different from zero, i.e., using eq. (2),

$$\det \boldsymbol{A}^{\{+,-\}} = (L_2 + L_3 c_3)^2 + L_3^2 \left(s_3^{\{+,-\}}\right)^2 = L_2^2 + L_3^2 + 2 L_2 L_3 c_3 = p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 > 0,$$

the solution for q_2 of each of the above four cases is uniquely determined from

$$\begin{pmatrix} c_2^{\{\{f,b\},\{u,d\}\}} \\ s_2^{\{\{f,b\},\{u,d\}\}} \end{pmatrix} = \begin{pmatrix} (L_2 + L_3 c_3) \left(\cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy}\right) + L_3 s_3^{\{+,-\}} (p_{dz} - L_1) \\ (L_2 + L_3 c_3) (p_{dz} - L_1) - L_3 s_3^{\{+,-\}} \left(\cos q_1^{\{+,-\}} p_{dx} + \sin q_1^{\{+,-\}} p_{dy}\right) \end{pmatrix},$$

and henceforth

$$q_2^{\{\{f,b\},\{u,d\}\}} = \text{ATAN2} \left\{ s_2^{\{\{f,b\},\{u,d\}\}}, c_2^{\{\{f,b\},\{u,d\}\}} \right\}. \tag{7}$$

Instead, when $p_{dx} = p_{dy} = 0$ and $p_{dy} = L_1$, the robot will be in a *double* kinematic singularity, with the arm folded and the end-effector placed along the axis of joint 1. Note that this situation can only occur in case the robot has $L_2 = L_3$ (otherwise the singular Cartesian point would be out of the robot workspace). The solution algorithm should output a warning message ("singular case: angle q_2 is undefined"), possibly set a second flag ($sing_2 = ON$), and then exit. In this case, only a single value $q_3 = \pi$ for the third joint angle will be defined.

Moving next to the requested numerical case with $L_1 = 1$, $L_2 = 1.5$, and $L_3 = 1.5$ [m], and for the desired position

$$\boldsymbol{p}_d = \begin{pmatrix} -1 \\ 1 \\ 1.5 \end{pmatrix} \text{ [m]},$$

[2]A special case arises when the joint angle q_1 remains undefined (a singularity with flag $sing_1 = ON$). The first component of the known vector $\boldsymbol{b}$ in (6) will vanish ($p_{dx} = p_{dy} = 0$) and only two solutions would be left for q_2. The case in which these two well-defined solutions collapse into a single value is left to the reader's analysis.

we can see that $\boldsymbol{p}_d$ belongs to the robot workspace and that this is not a singular case since

$$c_3 = -0.5 \in [-1, 1], \qquad p_{dx}^2 + p_{dy}^2 = 2 > 0.$$

We note that the desired position is in the second quadrant ($x < 0, y > 0$). Thus, the four inverse kinematics solutions obtained from (4), (5) and (7) are:

$$\begin{aligned}
\boldsymbol{q}^{\{f,u\}} &= \begin{pmatrix} 2.3562 \\ 1.3870 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 1.3870 \\ -2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 135.00^\circ \\ 79.47^\circ \\ -120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{f,d\}} &= \begin{pmatrix} 2.3562 \\ -0.7074 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 2.3562 \\ 2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} 135.00^\circ \\ -40.53^\circ \\ 120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{b,u\}} &= \begin{pmatrix} -0.7854 \\ 1.7546 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ 1.7546 \\ 2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} -45.00^\circ \\ 100.53^\circ \\ 120.00^\circ \end{pmatrix} \\
\boldsymbol{q}^{\{b,d\}} &= \begin{pmatrix} -0.7854 \\ -2.4342 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ -2.4342 \\ -2\pi/3 \end{pmatrix} [\text{rad}] = \begin{pmatrix} -45.00^\circ \\ -139.47^\circ \\ -120.00^\circ \end{pmatrix}.
\end{aligned} \tag{8}$$

As a double-check of correctness, it is always highly recommended to evaluate the direct kinematics with the obtained solutions (8). In return, one should get every time the desired position $\boldsymbol{p}_d$.

Exercise 2

This exercise is a generalization of the minimum-time trajectory planning problem for a single joint under velocity and acceleration bounds, with zero initial and final velocity (rest-to-rest) as boundary conditions.

It is useful to recap first the solution to the rest-to-rest problem. The minimum-time motion is given by a trapezoidal velocity profile (or a bang-coast-bang profile in acceleration), with minimum motion time T^* and symmetric initial and final acceleration/deceleration phases of duration T_s given by

$$T^* = \frac{|\Delta q|}{V} + \frac{V}{A} > 2T_s, \qquad T_s = \frac{V}{A} > 0. \tag{9}$$

This solution is only valid when the distance $|\Delta q|$ to travel (in absolute value) and the limit velocity and acceleration values $V > 0$ and $A > 0$ satisfy the inequality

$$|\Delta q| \geq \frac{V^2}{A}, \tag{10}$$

namely, when the distance is "sufficiently long" with respect to the ratio of the squared velocity limit to the acceleration limit. When the equality holds in (10), the maximum velocity V is reached only at the single instant $T^*/2 = T_s$, when half of the motion has been completed. Instead, when (10) is violated, the minimum-time motion is given by a bang-bang acceleration profile (i.e., with a triangular velocity profile) having only the acceleration/deceleration phases, each of duration

$$T_s = \sqrt{\frac{|\Delta q|}{A}} \qquad \Rightarrow \qquad T^* = 2T_s. \tag{11}$$

The crusing phase with maximum velocity V is not reached in this case. For all the above cases, when $\Delta q < 0$ the optimal velocity and acceleration profiles are simply changed of sign (flipped over the time axis).

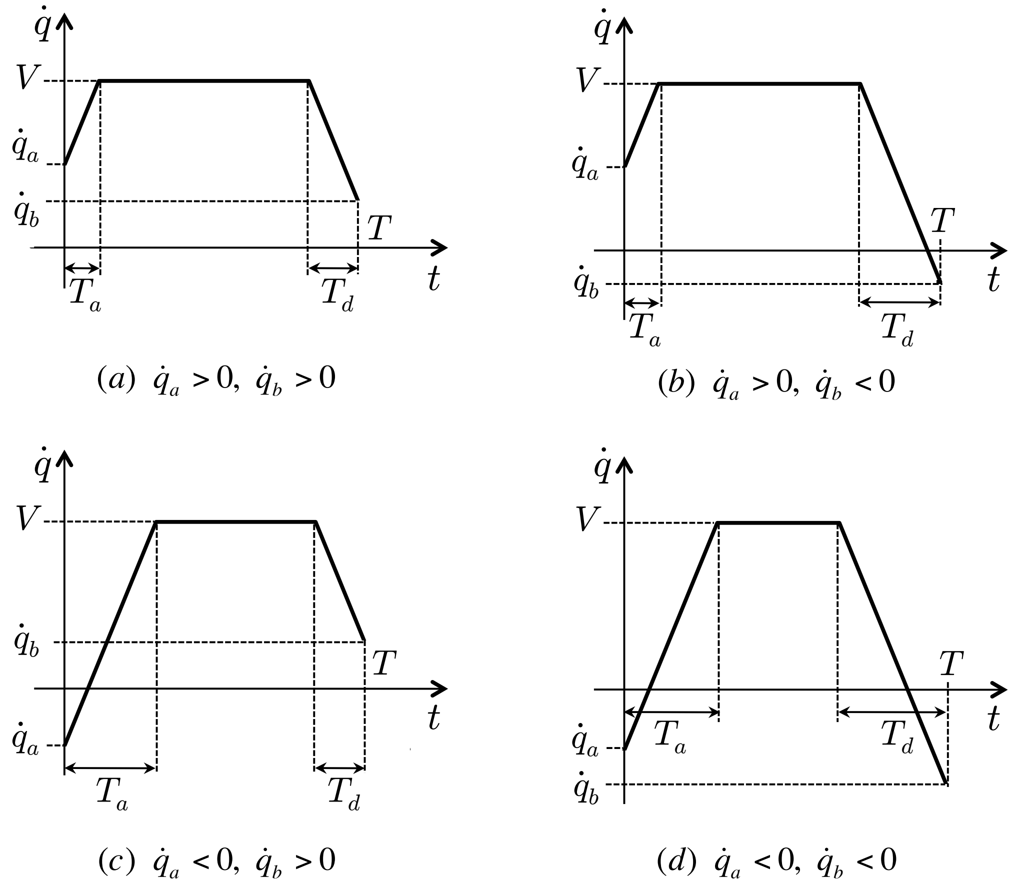

Figure 1: Qualitative asymmetric velocity profiles of the trapezoidal type for the four combinations of signs of the initial and final velocity $\dot{q}_a$ and $\dot{q}_b$. It is assumed that $\Delta q > 0$, and that this distance is sufficiently long so as to have a non-vanishing cruising interval at maximum velocity $\dot{q} = V$.

Consider now the problem of moving in minimum time the joint by a distance $\Delta q = q_b - q_a > 0$, but with generic non-zero boundary conditions $\dot{q}(0) = \dot{q}_a$ and $\dot{q}(T) = \dot{q}_b$ on the initial and final velocity. The requirement that $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$ is obviously mandatory in order to have a feasible solution. With reference to the qualitative trapezoidal velocity profiles sketched in Fig. 1, we see that non-zero initial and final velocities may help in reducing the motion time or work against it. In particular, when both $\dot{q}_a$ and $\dot{q}_b$ are positive (case (a)) it is clear that less time will be needed to ramp up from $\dot{q}_a > 0$ to V, rather than from 0 to V. The same is true for slowing down from V to $\dot{q}_b > 0$, rather than down to 0. On the contrary, when both $\dot{q}_a$ and $\dot{q}_b$ are negative (case (d)), an extra time will be spent for reversing motion from $\dot{q}_a < 0$ to 0 (in this time interval, the joint will continue to move in the opposite direction to the desired one, until it stops), when finally a positive velocity can be achieved, and, similarly, another extra time will be spent toward the end of the trajectory for bringing the velocity from 0 to $\dot{q}_b < 0$ (also in this second interval, the joint will move in the opposite direction to the desired one). Cases (b) and (c) in Fig. 1 are intermediate situations between (a) and (d), and can be analyzed in a similar way.

As a result:

- in general, the acceleration/deceleration phases will have different durations $T_a \geq 0$ and $T_d \geq 0$ (rather than the single $T_s \geq 0$ of the rest-to-rest case);
- the original required distance to travel $\Delta q > 0$ will become in practice longer, since we need to counterbalance the negative displacements introduced during those intervals where the velocity is negative;
- since we need to minimize the total motion time, intervals with negative velocity should be traversed in the least possible time, thus with maximum (positive or negative) acceleration $\ddot{q} = \pm A$.

With the above general considerations in mind, we perform now quantitative calculations. In the (positive) acceleration and (negative) deceleration phases, we have

$$T_a = \frac{V - \dot{q}_a}{A}, \qquad T_d = \frac{V - \dot{q}_b}{A}. \tag{12}$$

We note that both these time intervals will be shorter than $T_s = V/A$ for a positive boundary velocity and longer than T_s for a negative one. The area (with sign) underlying the velocity profile should provide, over the total motion time $T > 0$, the required distance $\Delta q > 0$. We compute this area as the sum of three contributions, using the trapezoidal rule for the two intervals where the velocity is changing linearly over time:

$$T_a \cdot \frac{\dot{q}_a + V}{2} + (T - T_a - T_d) \cdot V + T_d \cdot \frac{V + \dot{q}_b}{2} = \Delta q. \tag{13}$$

Substituting (12) in (13) and rearranging terms gives

$$\frac{(V + \dot{q}_a)(V - \dot{q}_a)}{2A} + \left(T - \frac{2V}{A} + \frac{\dot{q}_a + \dot{q}_b}{A}\right) \cdot V + \frac{(V + \dot{q}_b)(V - \dot{q}_b)}{2A} = \Delta q. \tag{14}$$

Solving for the motion time T, we obtain finally the optimal value

$$T^* = \frac{\Delta q}{V} + \frac{(V - \dot{q}_a)^2 + (V - \dot{q}_b)^2}{2AV}. \tag{15}$$

This is the generalization (for $\Delta q > 0$) of the minimum motion time formula (9) of the rest-to-rest case (which we recover by setting $\dot{q}_a = \dot{q}_b = 0$). This solution is only valid when the distance to travel $\Delta q > 0$, the velocity and acceleration limit values $V > 0$ and $A > 0$, and the boundary velocities $\dot{q}_a$ and $\dot{q}_b$ satisfy the inequality

$$\Delta q \geq \frac{2V^2 - \left(\dot{q}_a^2 + \dot{q}_b^2\right)}{2A} \; (\geq 0), \tag{16}$$

which is again the generalization of condition (10). This inequality is obtained by imposing that the sum of the first and third term in the left-hand side of (14), i.e, the space traveled during the acceleration and deceleration phases, does not exceed Δq (a cruising phase at maximum speed $V > 0$ would no longer be necessary).

It is interesting to note that, for a given triple Δq, V, and A, the inequality (16) would be easier to enforce as soon as $\dot{q}_a \neq 0$ and/or $\dot{q}_b \neq 0$, independently from their signs. The physical reason, however, is slightly different for a positive or negative boundary velocity, say of $\dot{q}_a$. When $\dot{q}_a > 0$, less time is needed in order to reach the maximum velocity $V > 0$; thus, it is more likely that the same problem data will imply a cruising velocity phase. Instead, when $\dot{q}_a < 0$, a negative displacement will occur in the initial phase, which needs to be recovered; thus, it is more likely that a cruising phase at maximum velocity V will be needed later.

Finally, we point out that:

- when inequality (16) is violated, or for special values of $\dot{q}_a$ or $\dot{q}_b$ (e.g., $\dot{q}_a = V$), a number of sub-cases arise; their complete analysis is out of the present scope and is left as an exercise for the reader;
- for $\Delta q < 0$, it is easy to show that the formulas corresponding to (12), (15), and (16) are

$$T_a = \frac{V + \dot{q}_a}{A}, \qquad T_d = \frac{V + \dot{q}_b}{A}, \qquad T^* = \frac{|\Delta q|}{V} + \frac{(V + \dot{q}_a)^2 + (V + \dot{q}_b)^2}{2AV},$$

$$|\Delta q| \geq \frac{2V^2 - \left(\dot{q}_a^2 + \dot{q}_b^2\right)}{2A}.$$

  Indeed, the velocity profiles in Fig. 1 will use the value $-V$ as cruising velocity.

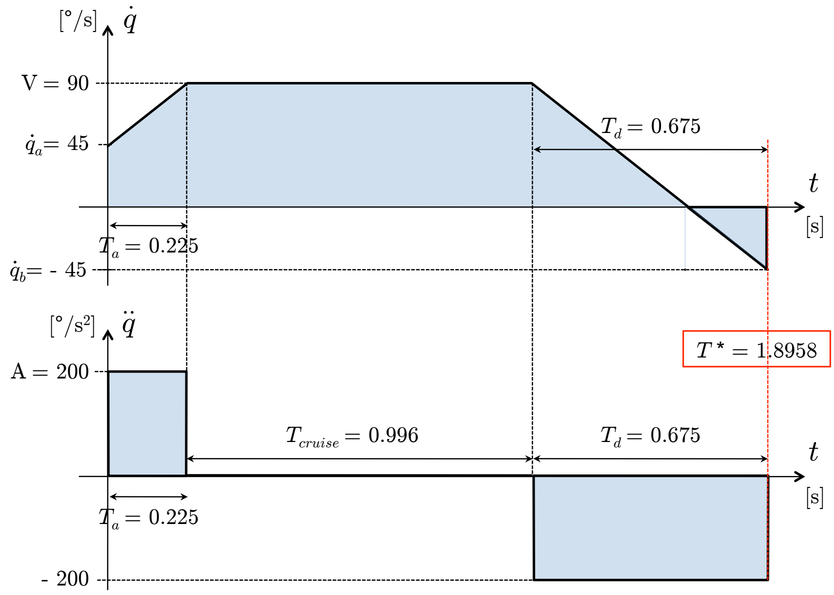

Figure 2: Time-optimal velocity and acceleration profiles for the numerical problem in Exercise 2.

Moving to the given numerical problem, from $\Delta q = q_b - q_a = 30° - (-90°) = 120° > 0$, $\dot{q}_a = 45°/\text{s}$, $\dot{q}_b = -45°/\text{s}$, $V = 90°/\text{s}$, and $A = 200°/\text{s}^2$, we evaluate first the inequality (16) and verify that

$$120 > \frac{2 \cdot 90^2 - \left(45^2 + (-45)^2\right)}{2 \cdot 200} = 30.375,$$

so that the general formula (15) applies. This yields

$$T^* = 1.8958 \text{ [s]},$$

while from (12) we obtain

$$T_a = 0.225 \text{ [s]}, \qquad T_d = 0.675 \text{ [s]},$$

with an interval of duration $T_{cruise} = T^* - T_a - T_d = 0.9958$ [s] in which the joint is cruising at $V = 90°/\text{s}$. The associated time-optimal velocity and acceleration profiles are reported in Fig. 2.

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