Exercise 1

The kinematics of a 3R spatial robot is specified by the Denavit-Hartenberg parameters in Tab. 1.

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$a_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>$L_1$</td>
<td>0</td>
<td>$q_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$L_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$L_3$</td>
<td>$q_3$</td>
</tr>
</tbody>
</table>

Table 1: Table of DH parameters of a 3R spatial robot.

- Given a position $p \in \mathbb{R}^3$ of the origin of the end-effector frame, provide the analytic expression of the solution to the inverse kinematics problem.
- For $L_1 = 1$ [m] and $L_2 = L_3 = 1.5$ [m], determine all inverse kinematics solutions in numerical form associated to the end-effector position $p = (-1 \ 1 \ 1.5)^T$ [m].

Exercise 2

A robot joint should move in minimum time between an initial value $q_a$ and a final value $q_b$, with an initial velocity $\dot{q}_a$ and a final velocity $\dot{q}_b$, under the bounds $|\dot{q}| \leq V$ and $|\ddot{q}| \leq A$.

- Provide the analytic expression of the minimum feasible motion time $T^*$ when $\Delta q = q_b - q_a > 0$ and the initial and final velocities are arbitrary in sign and magnitude (but both satisfy the velocity bound, i.e., $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$).
- Using the data $q_a = -90^\circ$, $q_b = 30^\circ$, $\dot{q}_a = 45^\circ/s$, $\dot{q}_b = -45^\circ/s$, $V = 90^\circ/s$, $A = 200^\circ/s^2$, determine the numerical value of the minimum feasible motion time $T^*$ and draw the velocity and acceleration profiles of the joint motion.

[180 minutes, open books but no computer or smartphone]
Solution
April 11, 2017

Exercise 1
From the direct kinematics, using Tab. 1, we obtain for the position of the origin of the end-effector frame

\[
p_H = A_1(q_1) \left( A_2(q_2) \left( A_3(q_3) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right) = \begin{pmatrix} p_1 \\ 0 \end{pmatrix} A_1(q_1)
\]

\[
\Rightarrow p = \begin{pmatrix} L_2 \cos q_2 + L_3 \cos(q_2 + q_3) \cos q_1 \\ L_2 \cos q_2 + L_3 \cos(q_2 + q_3) \sin q_1 \\ L_1 + L_2 \sin q_2 + L_3 \sin(q_2 + q_3) \end{pmatrix}.
\] (1)

The analytic inversion of eq. (1) for \( p = p_d = \begin{pmatrix} p_{dx} \\ p_{dy} \\ p_{dz} \end{pmatrix}^T \) proceeds as follows. After moving \( L_1 \) to the left-hand side of the third equation, squaring and adding the three equations yields the numeric value \( c_3 \) (for \( \cos q_3 \))

\[
c_3 = \frac{p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 - L_2^2 - L_3^2}{2L_2 L_3}.
\] (2)

The desired end-effector position will belong to the robot workspace if and only if \( c_3 \in [-1, 1] \). Note that this condition holds no matter if \( L_2 \) and \( L_3 \) are equal or different. Under such premises, we compute

\[
s_3 = \sqrt{1 - c_3^2}
\]

and

\[
q_3^{(+)} = \text{ATAN2} \{s_3, c_3\}, \quad q_3^{(-)} = \text{ATAN2} \{-s_3, c_3\},
\] (4)

yielding by definition two opposite values \( q_3^{(-)} = -q_3^{(+)} \). If \( c_3 = \pm 1 \), the robot is in a kinematic singularity: the forearm is either stretched or folded, in both cases on the boundary of the workspace. In particular, when \( c_3 = 1 \), \( q_3^{(+)} \) and \( q_3^{(-)} \) are both equal to 0; when \( c_3 = -1 \), the two solutions will be taken equal to \( \pi \). Instead, when \( c_3 \notin [-1, 1] \), the inverse kinematics algorithm should output a warning message (“desired position is out of workspace”) and exit.

When \( p_{dx}^2 + p_{dy}^2 > 0 \), from the first two equations in (1) we can further compute

\[
p_{dx}^2 + p_{dy}^2 = (L_2 \cos q_2 + L_3 \cos(q_2 + q_3))^2 \Rightarrow \cos q_1 = \frac{p_{dx}}{\pm \sqrt{p_{dx}^2 + p_{dy}^2}}, \quad \sin q_1 = \frac{p_{dy}}{\pm \sqrt{p_{dx}^2 + p_{dy}^2}},
\]

and thus

\[
q_1^{(+)} = \text{ATAN2} \{p_{dy}, p_{dx}\}, \quad q_1^{(-)} = \text{ATAN2} \{-p_{dy}, -p_{dx}\}.
\] (5)

These two values belong to \( (-\pi, \pi] \) and will always differ by \( \pi \). Instead, when \( p_{dx} = p_{dy} = 0 \), the first joint angle \( q_1 \) remains undefined and the robot will be in a kinematic singularity (with the end-effector placed along the axis of joint 1). The solution algorithm should output a warning message (“singular case: angle \( q_1 \) is undefined”), possibly set a flag (\( \text{sing}_1 = \text{ON} \)), but continue.

Remember that we use as conventional range \( q \in (-\pi, \pi] \), for all angles \( q \). Thus, if the output of a generic computation is \(-\pi\), we always replace it with \(+\pi\).
At this stage, we can rewrite a suitable combination of the first two equations in [1] as well as the third equation in the following way:

\[
\cos q_1 p_{dx} + \sin q_1 p_{dy} = L_2 \cos q_2 + L_3 \cos(q_2 + q_3) = (L_2 + L_3 \cos q_3) \cos q_2 - L_3 \sin q_3 \sin q_2
\]
and

\[
p_{dz} - L_1 = L_2 \sin q_2 + L_3 \sin(q_2 + q_3) = L_3 \sin q_3 \cos q_2 + (L_2 + L_3 \cos q_3) \sin q_2.
\]

Plugging the (multiple) values found so far for \(q_1\) and \(q_3\), we obtain four similar \(2 \times 2\) linear systems in the trigonometric unknowns \(c_2 = \cos q_2\) and \(s_2 = \sin q_2\):

\[
\begin{pmatrix}
L_2 + L_3 c_3 \\
L_3 s_3
\end{pmatrix}
\begin{pmatrix}
c_2 \\
s_2
\end{pmatrix}
= \begin{pmatrix}
\cos q_1^{(\pm,-)} p_{dx} + \sin q_1^{(\pm,-)} p_{dy} \\
p_{dz} - L_1
\end{pmatrix}
\quad \iff \quad A^{(+,-)} x = b^{(+,-)}.
\]

In [6], we should use the values from [4] and [5]. This gives rise to four possible combinations for the matrix/vector pair \((A^{(+,-)}, b^{(+,-)})\), which will eventually lead to four solutions for \(q_2\) that are in general distinct. These will be labeled as

\[
q_2^{(f,u)} q_2^{(f,d)} q_2^{(b,u)} q_2^{(b,d)} \quad \Rightarrow \quad q^{(f,u)} q^{(f,d)} q^{(b,u)} q^{(b,d)}
\]

depending on whether the robot is facing \((f)\) of backing \((b)\) the desired position quadrant — due to the choice of \(q_1\), and on whether the elbow is up \((u)\) or down \((d)\) — due to the combined choice of \(q_1\) and \(q_3\). If the (common) determinant of the coefficient matrix is different from zero, i.e., using eq. [2],

\[
\det A^{(+,-)} = (L_2 + L_3 c_3)^2 + L_3^2 (s_3^{(\pm,-)})^2 = L_2^2 + 2 L_2 L_3 c_3 + p_{dx}^2 + p_{dy}^2 + (p_{dz} - L_1)^2 > 0,
\]

the solution for \(q_2\) of each of the above four cases is uniquely determined from

\[
\begin{pmatrix}
c_2^{(f,b),(u,d)} \\
s_2^{(f,b),(u,d)}
\end{pmatrix}
= \begin{pmatrix}
(L_2 + L_3 c_3) \\
(L_2 + L_3 c_3)
\end{pmatrix}
\begin{pmatrix}
\cos q_1^{(\pm,-)} p_{dx} + \sin q_1^{(\pm,-)} p_{dy} \\
(p_{dz} - L_1)
\end{pmatrix}
\quad \iff \quad A^{(+,-)} x = b^{(+,-)}.
\]

and henceforth

\[
q_2^{(f,b),(u,d)} = \text{ATAN2} \left( \frac{s_2^{(f,b),(u,d)}}{c_2^{(f,b),(u,d)}} \right) = \text{ATAN2} \left( \frac{s_2^{(f,b),(u,d)}}{c_2^{(f,b),(u,d)}} \right).
\]

Instead, when \(p_{dx} = p_{dy} = 0\) and \(p_{dz} = L_1\), the robot will be in a \emph{double} kinematic singularity, with the arm folded and the end-effector placed along the axis of joint 1. Note that this situation can only occur in case the robot has \(L_2 = L_3\) (otherwise the singular Cartesian point would be out of the robot workspace). The solution algorithm should output a warning message (“singular case: angle \(q_2\) is undefined”), possibly set a second flag \((\text{sing} q_2 = \text{ON})\), and then exit. In this case, only a single value \(q_3 = \pi\) for the third joint angle will be defined.

Moving next to the requested numerical case with \(L_1 = 1\), \(L_2 = 1.5\), and \(L_3 = 1.5\ [\text{m}]\), and for the desired position

\[
p_d = \begin{pmatrix}
-1 \\
1 \\
1.5
\end{pmatrix}
\quad [\text{m}],
\]

2A special case arises when the joint angle \(q_1\) remains undefined (a singularity with flag \(\text{sing} q_1 = \text{ON}\)). The first component of the known vector \(b\) in \([6]\) will vanish \((p_{dx} = p_{dy} = 0)\) and only two solutions would be left for \(q_2\). The case in which these two well-defined solutions collapse into a single value is left to the reader’s analysis.
we can see that $\mathbf{p}_d$ belongs to the robot workspace and that this is not a singular case since

$$c_3 = -0.5 \in [-1, 1], \quad p_{dx}^2 + p_{dy}^2 = 2 > 0.$$  

We note that the desired position is in the second quadrant ($x < 0, y > 0$). Thus, the four inverse kinematics solutions obtained from (4), (5) and (7) are:

$$q^{(f,u)} = \begin{pmatrix} 2.3562 \\ 1.3870 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 1.3870 \\ -2\pi/3 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 135.00^\circ \\ 79.47^\circ \\ -120.00^\circ \end{pmatrix};$$

$$q^{(f,d)} = \begin{pmatrix} 2.3562 \\ -0.7074 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} 3\pi/4 \\ 2\pi/3 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 135.00^\circ \\ 120.00^\circ \end{pmatrix};$$

$$q^{(b,u)} = \begin{pmatrix} -0.7854 \\ 1.7546 \\ 2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ 2\pi/3 \end{pmatrix} \text{[rad]} = \begin{pmatrix} -45.00^\circ \\ 120.00^\circ \end{pmatrix};$$

$$q^{(b,d)} = \begin{pmatrix} -0.7854 \\ -2.4342 \\ -2.0944 \end{pmatrix} = \begin{pmatrix} -\pi/4 \\ -2\pi/3 \end{pmatrix} \text{[rad]} = \begin{pmatrix} -45.00^\circ \\ -120.00^\circ \end{pmatrix}.$$  

(8)

As a double-check of correctness, it is always highly recommended to evaluate the direct kinematics with the obtained solutions (8). In return, one should get every time the desired position $\mathbf{p}_d$.

**Exercise 2**

This exercise is a generalization of the minimum-time trajectory planning problem for a single joint under velocity and acceleration bounds, with zero initial and final velocity (rest-to-rest) as boundary conditions.

It is useful to recap first the solution to the rest-to-rest problem. The minimum-time motion is given by a trapezoidal velocity profile (or a bang-coast-bang profile in acceleration), with minimum motion time $T^*$ and symmetric initial and final acceleration/deceleration phases of duration $T_s$ given by

$$T^* = \frac{\left|\Delta q\right|}{V} + \frac{V}{A} > 2T_s, \quad T_s = \frac{V}{A} > 0. \quad (9)$$

This solution is only valid when the distance $\left|\Delta q\right|$ to travel (in absolute value) and the limit velocity and acceleration values $V > 0$ and $A > 0$ satisfy the inequality

$$\left|\Delta q\right| \geq \frac{V^2}{A}, \quad (10)$$

namely, when the distance is “sufficiently long” with respect to the ratio of the squared velocity limit to the acceleration limit. When the equality holds in (10), the maximum velocity $V$ is reached only at the single instant $T^*/2 = T_s$, when half of the motion has been completed. Instead, when (10) is violated, the minimum-time motion is given by a bang-bang acceleration profile (i.e., with a triangular velocity profile) having only the acceleration/deceleration phases, each of duration

$$T_s = \sqrt{\frac{\left|\Delta q\right|}{A}} \quad \Rightarrow \quad T^* = 2T_s. \quad (11)$$
The cruising phase with maximum velocity $V$ is not reached in this case. For all the above cases, when $\Delta q < 0$ the optimal velocity and acceleration profiles are simply changed of sign (flipped over the time axis).

Consider now the problem of moving in minimum time the joint by a distance $\Delta q = q_b - q_a > 0$, but with generic non-zero boundary conditions $\dot{q}(0) = \dot{q}_a$ and $\dot{q}(T) = \dot{q}_b$ on the initial and final velocity. The requirement that $|\dot{q}_a| \leq V$ and $|\dot{q}_b| \leq V$ is obviously mandatory in order to have a feasible solution. With reference to the qualitative trapezoidal velocity profiles sketched in Fig. 1, we see that non-zero initial and final velocities may help in reducing the motion time or work against it. In particular, when both $\dot{q}_a$ and $\dot{q}_b$ are positive (case (a)) it is clear that less time will be needed to ramp up from $\dot{q}_a > 0$ to $V$, rather than from 0 to $V$. The same is true for slowing down from $V$ to $\dot{q}_b > 0$, rather than down to 0. On the contrary, when both $\dot{q}_a$ and $\dot{q}_b$ are negative (case (d)), an extra time will be spent for reversing motion from $\dot{q}_a < 0$ to 0 (in this time interval, the joint will continue to move in the opposite direction to the desired one, until it stops), when finally a positive velocity can be achieved, and, similarly, another extra time will be spent toward the end of the trajectory for bringing the velocity from 0 to $\dot{q}_b < 0$ (also in this second interval, the joint will move in the opposite direction to the desired one). Cases (b) and (c) in Fig. 1 are intermediate situations between (a) and (d), and can be analyzed in a similar way.

Figure 1: Qualitative asymmetric velocity profiles of the trapezoidal type for the four combinations of signs of the initial and final velocity $\dot{q}_a$ and $\dot{q}_b$. It is assumed that $\Delta q > 0$, and that this distance is sufficiently long so as to have a non-vanishing cruising interval at maximum velocity $\dot{q} = V$. 
As a result:

- in general, the acceleration/deceleration phases will have different durations $T_a \geq 0$ and $T_d \geq 0$ (rather than the single $T_s \geq 0$ of the rest-to-rest case);
- the original required distance to travel $\Delta q > 0$ will become in practice longer, since we need to counterbalance the negative displacements introduced during those intervals where the velocity is negative;
- since we need to minimize the total motion time, intervals with negative velocity should be traversed in the least possible time, thus with maximum (positive or negative) acceleration $\ddot{q} = \pm A$.

With the above general considerations in mind, we perform now quantitative calculations. In the (positive) acceleration and (negative) deceleration phases, we have

$$T_a = \frac{V - \dot{q}_a}{A}, \quad T_d = \frac{V - \dot{q}_b}{A}. \quad (12)$$

We note that both these time intervals will be shorter than $T_s = V/A$ for a positive boundary velocity and longer than $T_s$ for a negative one. The area (with sign) underlying the velocity profile should provide, over the total motion time $T > 0$, the required distance $\Delta q > 0$. We compute this area as the sum of three contributions, using the trapezoidal rule for the two intervals where the velocity is changing linearly over time:

$$T_a \cdot \dot{q}_a + \frac{V + \dot{q}_a}{2} + (T - T_a - T_d) \cdot V + T_d \cdot \frac{V + \dot{q}_b}{2} = \Delta q. \quad (13)$$

Substituting (12) in (13) and rearranging terms gives

$$\frac{(V + \dot{q}_a)(V - \dot{q}_a)}{2A} + \left( T - \frac{2V}{A} + \dot{q}_a + \dot{q}_b \right) \cdot V + \frac{(V + \dot{q}_b)(V - \dot{q}_b)}{2A} = \Delta q. \quad (14)$$

Solving for the motion time $T$, we obtain finally the optimal value

$$T^* = \frac{\Delta q}{V} + \frac{(V - \dot{q}_a)^2 + (V - \dot{q}_b)^2}{2AV}. \quad (15)$$

This is the generalization (for $\Delta q > 0$) of the minimum motion time formula (9) of the rest-to-rest case (which we recover by setting $\dot{q}_a = \dot{q}_b = 0$). This solution is only valid when the distance to travel $\Delta q > 0$, the velocity and acceleration limit values $V > 0$ and $A > 0$, and the boundary velocities $\dot{q}_a$ and $\dot{q}_b$ satisfy the inequality

$$\Delta q \geq \frac{2V^2 - (\dot{q}_a^2 + \dot{q}_b^2)}{2A} \ (\geq 0), \quad (16)$$

which is again the generalization of condition (10). This inequality is obtained by imposing that the sum of the first and third term in the left-hand side of (14), i.e., the space traveled during the acceleration and deceleration phases, does not exceed $\Delta q$ (a cruising phase at maximum speed $V > 0$ would no longer be necessary).

It is interesting to note that, for a given triple $\Delta q, V$, and $A$, the inequality (16) would be easier to enforce as soon as $\dot{q}_a \neq 0$ and/or $\dot{q}_b \neq 0$, independently from their signs. The physical reason, however, is slightly different for a positive or negative boundary velocity, say of $\dot{q}_a$. When $\dot{q}_a > 0$, less time is needed in order to reach the maximum velocity $V > 0$; thus, it is more likely that
the same problem data will imply a cruising velocity phase. Instead, when \( \dot{q}_a < 0 \), a negative displacement will occur in the initial phase, which needs to be recovered; thus, it is more likely that a cruising phase at maximum velocity \( V \) will be needed later.

Finally, we point out that:
- when inequality (16) is violated, or for special values of \( \dot{q}_a \) or \( \dot{q}_b \) (e.g., \( \dot{q}_a = V \)), a number of sub-cases arise; their complete analysis is out of the present scope and is left as an exercise for the reader;
- for \( \Delta q < 0 \), it is easy to show that the formulas corresponding to (12), (15), and (16) are

\[
T_a = \frac{V + \dot{q}_a}{A}, \quad T_d = \frac{V + \dot{q}_b}{A}, \quad T^* = \frac{|\Delta q|}{V} + \frac{(V + \dot{q}_a)^2 + (V + \dot{q}_b)^2}{2AV},
\]

\[
|\Delta q| \geq \frac{2V^2 - \left(\dot{q}_a^2 + \dot{q}_b^2\right)}{2A}.
\]

Indeed, the velocity profiles in Fig. 1 will use the value \(-V\) as cruising velocity.

Moving to the given numerical problem, from \( \Delta q = q_b - q_a = 30^\circ - (-90^\circ) = 120^\circ > 0 \), \( \dot{q}_a = 45^\circ/s \), \( \dot{q}_b = -45^\circ/s \), \( V = 90^\circ/s \), and \( A = 200^\circ/s^2 \), we evaluate first the inequality (16) and verify that

\[
120 > \frac{2 \cdot 90^2 - \left(45^2 + (-45)^2\right)}{2 \cdot 200} = 30.375,
\]

so that the general formula (15) applies. This yields

\[
T^* = 1.8958 \text{ [s]},
\]

while from (12) we obtain

\[
T_a = 0.225 \text{ [s]}, \quad T_d = 0.675 \text{ [s]},
\]

with an interval of duration \( T_{\text{cruise}} = T^* - T_a - T_d = 0.9958 \text{ [s]} \) in which the joint is cruising at \( V = 90^\circ/s \). The associated time-optimal velocity and acceleration profiles are reported in Fig. 2.

\* \* \* \* \*