Exercise 1

Consider the 3R robot in Fig. 1, with the associated Denavit-Hartenberg parameters of Tab. 1. An extra frame is shown on the robot end-effector, representing the typical frame associated to an eye-in-hand camera.

![Robot](image)

Figure 1: A 3R robot

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$a_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>$L_1$</td>
<td>0</td>
<td>$q_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$L_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$L_3$</td>
<td>$q_3$</td>
</tr>
</tbody>
</table>

Table 1: Table of DH parameters

- Draw on the figure the Denavit-Hartenberg frames specified by Tab. 1.
- Derive the explicit expression of the $3 \times 3$ Jacobian $\mathbf{J}(q)$ relating the joint velocity $\dot{\mathbf{q}}$ to the linear velocity $\mathbf{v}$ of the origin of the camera frame expressed in the camera frame as
  \[ \mathbf{v} = \mathbf{J}(q) \dot{\mathbf{q}}. \]

Exercise 2

Let two absolute orientations $^0 \mathbf{R}_i$ (initial) and $^0 \mathbf{R}_f$ (final) be assigned through their minimal representation with the $(Z,X,Y)$ Euler angles:

\[
\begin{pmatrix}
\alpha_i & \beta_i & \gamma_i \\
\frac{\pi}{4} & -\frac{\pi}{2} & 0
\end{pmatrix}
\quad \begin{pmatrix}
\alpha_f & \beta_f & \gamma_f \\
-\frac{\pi}{2} & 0 & \frac{\pi}{2}
\end{pmatrix}.
\]

- Design a rest-to-rest orientation trajectory that joins $^0 \mathbf{R}_i$ to $^0 \mathbf{R}_f$ in time $T = 1.5$ s using the axis-angle method and a cubic polynomial as timing law.
- Provide the expression of the orientation $^0 \mathbf{R}(t)$ at a generic instant $t \in (0, T)$ of the planned motion and the associated angular velocity $^0 \mathbf{\omega}(t)$, both expressed in the absolute reference frame.
- What is the maximum value $\omega_{\text{max}}$ of the norm of the angular velocity $^0 \mathbf{\omega}(t)$ for $t \in [0, T]$?

[180 minutes; open books & software]
Solution
April 26, 2012

Exercise 1

The correct frame assignment is shown in Fig. 2, where the second and third joint as well as the second link are illustrated in transparency for better clarity.

![Figure 2: The DH frames for the 3R robot](image)

For later use, we can see that the constant rotation from the end-effector to the camera frame is given by

\[
^3 R_c = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

Moreover, from the DH table we can build the homogenous transformation matrices \(^0 A_1(q_1), ^1 A_2(q_2), \) and \(^2 A_3(q_3)\) containing the rotation matrices

\[
^0 R_1 = \begin{pmatrix}
\cos q_1 & 0 & \sin q_1 \\
\sin q_1 & 0 & -\cos q_1 \\
0 & 1 & 0
\end{pmatrix},
^1 R_2 = \begin{pmatrix}
\cos q_2 & -\sin q_2 & 0 \\
\sin q_2 & \cos q_2 & 0 \\
0 & 0 & 1
\end{pmatrix},
^2 R_3 = \begin{pmatrix}
\cos q_3 & -\sin q_3 & 0 \\
\sin q_3 & \cos q_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

that will be needed in the following.

The position \(p\) of the origin \(O_3\) of frame 3 can be computed (in homogeneous coordinates) as

\[
\begin{pmatrix}
p \\ 1
\end{pmatrix} = ^0 A_1(q_1)^1 A_2(q_2)^2 A_3(q_3) \begin{pmatrix}
0 \\ 1
\end{pmatrix}
\]

yielding

\[
p = f(q) = \begin{pmatrix}
\cos q_1 (L_2 \cos q_2 + L_3 \cos (q_2 + q_3)) \\
\sin q_1 (L_2 \cos q_2 + L_3 \cos (q_2 + q_3)) \\
L_1 + L_2 \sin q_2 + L_3 \sin (q_2 + q_3)
\end{pmatrix}.
\]
The Jacobian related to the linear velocity $^0v$ ($= ^0v_3$) of the origin of frame 3 and expressed in the base frame is obtained as

$$^0J(q) = \frac{\partial f(q)}{\partial q} = \begin{pmatrix}
-\sin q_1(L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) & -\cos q_1(L_2 \sin q_2 + L_3 \sin(q_2 + q_3)) & -L_3 \cos q_1 \sin(q_2 + q_3) \\
\cos q_1(L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) & -\sin q_1(L_2 \sin q_2 + L_3 \sin(q_2 + q_3)) & -L_3 \sin q_1 \sin(q_2 + q_3) \\
0 & L_2 \cos q_2 + L_3 \cos(q_2 + q_3) & L_3 \cos(q_2 + q_3)
\end{pmatrix}.$$  

The requested Jacobian $^cJ(q)$ that relates $\dot{q}$ to $^c\dot{v}$ ($= ^c\dot{v}_3$) is obtained by applying suitable rotation matrices:

$$^cJ(q) = ^0R_c^T(q)^0J(q) = ^3R_c^T(q_3) \left( ^1R_c^T(q_2) \left( ^0R_c^T(q_1) \left( ^0J(q) \right) \right) \right) .$$

The following is a symbolic Matlab script performing intermediate and final computations.

```matlab
clear all
clc
syms L1 L2 L3 q1 q2 q3 alfa d a theta pi real
% DH parameters
alfa1=pi/2; alfa2=0; alfa3=0; d1=L1; d2=0; d3=0; a1=0; a2=L2; a3=L3;
% DH homogeneous matrix
A=[cos(theta) -sin(theta)*cos(alfa) sin(theta)*sin(alfa) a*cos(theta);
    sin(theta) cos(theta)*cos(alfa) -cos(theta)*sin(alfa) a*sin(theta);
    0 sin(alfa) cos(alfa) d;
    0 0 0 1] ;
% evaluations
A1=subs(A, {alfa,d,a,theta}, {alfa1,d1,a1,q1})
A2=subs(A, {alfa,d,a,theta}, {alfa2,d2,a2,q2})
A3=subs(A, {alfa,d,a,theta}, {alfa3,d3,a3,q3})
R1=A1(1:3,1:3); R2=A2(1:3,1:3); R3=A3(1:3,1:3);
% camera frame
Rc=[0 0 1; 0 1 0; -1 0 0];
% position of O3
phom=A1*(A2*(A3*[0 0 0 1]'));
p=simplify(phom(1:3))
% Jacobian in frame 0
q=[q1 q2 q3]';
J=jacobian(p,q)
% Jacobian in frame 1,2,3
J1=simplify(R1'*J)
J2=simplify(R2'*J1)
J3=simplify(R3'*J2)
% final Jacobian in camera frame
Jc=simplify(Rc'*J3)
% end
```
Exercise 2

The rotation matrix associated to the \((\alpha, \beta, \gamma)\) angles in the \((Z, X, Y)\) Euler representation, i.e., for a sequence of rotations around the axes \(Z, X', (\text{moved}), \text{and} \ Y'' (\text{moved})\), is obtained from the elementary rotation matrices

\[
R_Z(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
R_X(\beta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \beta & -\sin \beta \\
0 & \sin \beta & \cos \beta
\end{pmatrix},
R_Y(\gamma) = \begin{pmatrix}
\cos \gamma & 0 & \sin \gamma \\
0 & 1 & 0 \\
-\sin \gamma & 0 & \cos \gamma
\end{pmatrix},
\]

as

\[
R_{ZXY}(\alpha, \beta, \gamma) = R_Z(\alpha)R_X(\beta)R_Y(\gamma),
\]
or

\[
R_{ZXY}(\alpha, \beta, \gamma) = \begin{pmatrix}
\cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma & -\sin \alpha \cos \gamma - \sin \alpha \sin \beta \cos \gamma & \cos \alpha \sin \gamma + \sin \alpha \sin \beta \cos \gamma \\
\sin \alpha \cos \gamma + \cos \alpha \sin \beta \sin \gamma & \cos \alpha \sin \gamma - \cos \alpha \sin \beta \cos \gamma & \sin \beta \cos \gamma \\
-\cos \beta \sin \gamma & \sin \beta & \cos \beta \cos \gamma
\end{pmatrix}.
\]

Thus, we can compute the rotation matrices associated to the given \((\alpha_i, \beta_i, \gamma_i)\)

\[
{0}R_i = R_{ZXY}(\alpha_i, \beta_i, \gamma_i) = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
0 & -1 & 0
\end{pmatrix},
\]
and to \((\alpha_f, \beta_f, \gamma_f)\)

\[
{0}R_f = R_{ZXY}(\alpha_f, \beta_f, \gamma_f) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
-1 & 0 & 0
\end{pmatrix}.
\]

The relative rotation between the initial and final orientation is thus

\[
R_{if} = {0}R_i^{-T} {0}R_f = \begin{pmatrix}
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
1 & 0 & 0 \\
0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{pmatrix}.
\]

Note that this rotation matrix is defined with respect to the initial orientation \(0R_i\) (or \(R_{if} = iR_{if}\)).

We extract then the angle \(\theta_{if}\) and the invariant axis \(r\) (a unit vector) from the elements \(R_{ij}\) of the rotation matrix \(R_{if}\):

\[
\theta_{if} = \text{ATAN2} \left\{ \sqrt{(R_{21} - R_{12})^2 + (R_{31} - R_{13})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1 \right\} = 2.5936 \ [\text{rad}]
\]
(or, in degrees, \(\theta_{if} = 148.6^\circ\)). Being \(\sin \theta_{if} \neq 0\), we have

\[
r = \frac{1}{2 \sin \theta_{if}} \begin{pmatrix}
R_{12} - R_{21} \\
R_{13} - R_{31} \\
R_{21} - R_{12}
\end{pmatrix} = \frac{1}{1.042} \begin{pmatrix}
-\sqrt{2}/2 \\
-\sqrt{2}/2 \\
1 - (\sqrt{2}/2)
\end{pmatrix} = \begin{pmatrix}
-0.6786 \\
-0.6786 \\
0.2811
\end{pmatrix}.
\]
Indeed, this orientation is relative to the initial one \( \theta \) such that \( \theta(0) = 0 \) and \( \theta(T) = \theta_{ij} \), and its time derivative

\[
\dot{\theta}(t) = \frac{\theta_{ij}}{T} \left( \theta(t) \cos \left( \frac{3T}{T} \right) - 2 \left( \frac{t}{T} \right)^3 \right)
\]

is such that \( \theta(0) = 0 \) and \( \dot{\theta}(T) = \theta_{ij} \), and its time derivative

\[
\ddot{\theta}(t) = \frac{\theta_{ij}}{T^2} \left( \frac{6}{T} - 6 \left( \frac{t}{T} \right)^2 \right),
\]

satisfies \( \ddot{\theta}(0) = \ddot{\theta}(T) = 0 \) as required. The maximum rotation speed is attained at \( t = T/2 \):

\[
\dot{\theta} \left( \frac{T}{2} \right) = \frac{3\theta_{ij}}{2T} > 0 \quad \Rightarrow \quad \text{(for } T = 1.5 \text{)} \quad \hat{\theta}_{\max} = \dot{\theta}(0.75) = 2.5936 \text{ [rad/s].}
\]

Using the obtained \( r \), the orientation at a generic instant \( t \in [0,T] \) is

\[
R(r, \theta(t)) = \begin{pmatrix}
0.5395 \cos \theta(t) + 0.4605 & 0.4605(1 - \cos \theta(t)) - 0.2811 \sin \theta(t) & -0.1907(1 - \cos \theta(t)) - 0.6786 \sin \theta(t) \\
0.4605(1 - \cos \theta(t)) + 0.2811 \sin \theta(t) & 0.5395 \cos \theta(t) + 0.4605 & -0.1907(1 - \cos \theta(t)) + 0.6786 \sin \theta(t) \\
-0.1907(1 - \cos \theta(t)) + 0.6786 \sin \theta(t) & -0.1907(1 - \cos \theta(t)) - 0.6786 \sin \theta(t) & 0.9210 \cos \theta(t) + 0.07901
\end{pmatrix}.
\]

Indeed, this orientation is relative to the initial one \( R_i \), or \( R(r, \theta(t)) = R^i_r (r, \theta(t)) \). For check, it is easy to see that at \( t = 0 \) (\( \theta(0) = 0 \)) it is \( R(r,0) = I \). Similarly, at \( t = T \) (\( \theta(T) = \theta_{ij} \)) it is \( R(r, \theta_{ij}) = R_{ij} \). The absolute orientation is simply obtained as

\[
R(r(\theta(t)) = 0 R_i R^i_r (r, \theta(t)) = R^0 r_{ij} (r, \theta(t)) = R^0 (r, \theta(t)) = \begin{pmatrix}
0.2466 \cos \theta(t) - 0.4798 \sin \theta(t) & 0.4605(1 - \cos \theta(t)) + 0.2811 \sin \theta(t) & -0.1907 - 0.5164 \cos \theta(t) - 0.4798 \sin \theta(t) \\
0.5164 \cos \theta(t) + 0.4798 \sin \theta(t) + 0.1907 & 0.1907(1 - \cos \theta(t)) - 0.6786 \sin \theta(t) & 0.7861 \cos \theta(t) - 0.4798 \sin \theta(t) - 0.07901 \\
-0.4605(1 - \cos \theta(t)) - 0.2811 \sin \theta(t) & -0.5395 \cos \theta(t) - 0.4605 & 0.1907(1 - \cos \theta(t)) - 0.6786 \sin \theta(t)
\end{pmatrix}.
\]

Finally, the angular velocity associated to the planned motion expressed in the frame \( R_i \) is

\[
\dot{i} \omega(t) = \dot{i} r \dot{\theta}(t) = \begin{pmatrix}
-0.6786 \\
-0.6786 \\
0.2811
\end{pmatrix} \dot{\theta}(t),
\]

and in the absolute frame

\[
0 \omega(t) = 0 R_i \dot{i} \omega(t) = 0 R_i \dot{i} r \dot{\theta}(t) = 0 r \dot{\theta}(t) = \begin{pmatrix}
-0.6786 \\
-0.2811 \\
0.6786
\end{pmatrix} \dot{\theta}(t).
\]
Its maximum value in norm (invariant with respect to the frame of definition) is simply

$$\max_{t \in [0, T]} \| \dot{0}\omega(t) \| = \max_{t \in [0, T]} \| \dot{\omega}(t) \| = \| \dot{\rho} \| \cdot \max_{t \in [0, T]} |\dot{\theta}(t)| = 1 \cdot \dot{\theta}_{max} = 2.5936 \text{ [rad/s]}. $$

A symbolic/numeric Matlab script supporting the computations of Exercise 2 is available.

* * * * *