# Robotics I

**April 26, 2012**

### Exercise 1

Consider the 3R robot in Fig. 1, with the associated Denavit-Hartenberg parameters of Tab. 1. An extra frame is shown on the robot end-effector, representing the typical frame associated to an eye-in-hand camera.

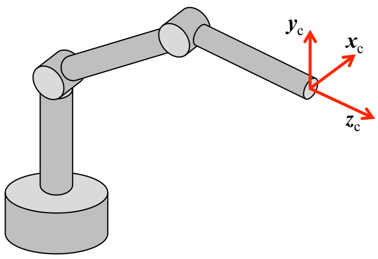


Figure 1: A 3R robot

| $i$ | $\alpha_i$ | $d_i$ | $a_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $\pi/2$ | $L_1$ | 0 | $q_1$ |
| 2 | 0 | 0 | $L_2$ | $q_2$ |
| 3 | 0 | 0 | $L_3$ | $q_3$ |

Table 1: Table of DH parameters

- Draw on the figure the Denavit-Hartenberg frames specified by Tab. 1.
- Derive the explicit expression of the $3 \times 3$ Jacobian ${}^c\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity $\dot{\boldsymbol{q}}$ to the linear velocity ${}^c\boldsymbol{v}$ of the origin of the camera frame *expressed in the camera frame* as
$$ {}^c\boldsymbol{v} = {}^c\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}. $$

### Exercise 2

Let two absolute orientations ${}^0\boldsymbol{R}_i$ (initial) and ${}^0\boldsymbol{R}_f$ (final) be assigned through their minimal representation with the $(Z, X, Y)$ Euler angles:

$$ \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4} & -\frac{\pi}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha_f & \beta_f & \gamma_f \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2} & 0 & \frac{\pi}{2} \end{pmatrix}. $$

- Design a rest-to-rest orientation trajectory that joins ${}^0\boldsymbol{R}_i$ to ${}^0\boldsymbol{R}_f$ in time $T = 1.5$ s using the *axis-angle method* and a cubic polynomial as timing law.
- Provide the expression of the orientation ${}^0\boldsymbol{R}(t)$ at a generic instant $t \in (0, T)$ of the planned motion and the associated angular velocity ${}^0\boldsymbol{\omega}(t)$, both expressed in the absolute reference frame.
- What is the maximum value $\omega_{max}$ of the norm of the angular velocity ${}^0\boldsymbol{\omega}(t)$ for $t \in [0, T]$?

**[180 minutes; open books & software]**

# Solution

April 26, 2012

**Exercise 1**

The correct frame assignment is shown in Fig. 2, where the second and third joint as well as the second link are illustrated in transparency for better clarity.

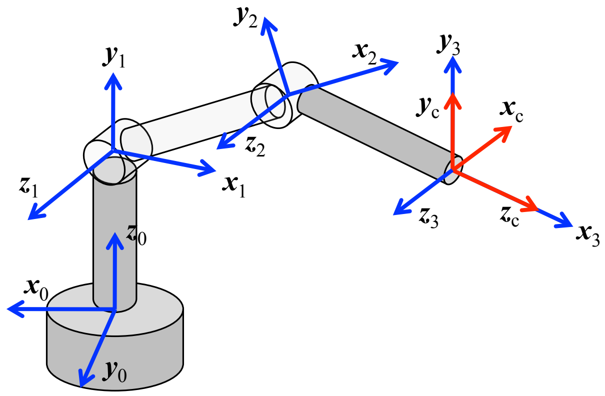


Figure 2: The DH frames for the 3R robot

For later use, we can see that the constant rotation from the end-effector to the camera frame is given by

$$ {}^3\boldsymbol{R}_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. $$

Moreover, from the DH table we can build the homogenous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$, ${}^1\boldsymbol{A}_2(q_2)$, and ${}^2\boldsymbol{A}_3(q_3)$ containing the rotation matrices

$$ {}^0\boldsymbol{R}_1 = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix} \; {}^1\boldsymbol{R}_2 = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; {}^2\boldsymbol{R}_3 = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

that will be needed in the following.

The position $\boldsymbol{p}$ of the origin $O_3$ of frame 3 can be computed (in homogeneous coordinates) as

$$ \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} $$

yielding

$$ \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ \sin q_1 (L_2 \cos q_2 + L_3 \cos(q_2 + q_3)) \\ L_1 + L_2 \sin q_2 + L_3 \sin(q_2 + q_3) \end{pmatrix}. $$

The Jacobian related to the linear velocity ${}^0\boldsymbol{v}$ $(= {}^0\boldsymbol{v}_3)$ of the origin of frame 3 and expressed in the base frame is obtained as

$$
{}^0\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} =
$$
$$
\begin{pmatrix} -\sin q_1 (L_2 \cos q_2 + L_3 \cos(q_2+q_3)) & -\cos q_1 (L_2 \sin q_2 + L_3 \sin(q_2+q_3)) & -L_3 \cos q_1 \sin(q_2+q_3)) \\ \cos q_1 (L_2 \cos q_2 + L_3 \cos(q_2+q_3)) & -\sin q_1 (L_2 \sin q_2 + L_3 \sin(q_2+q_3)) & -L_3 \sin q_1 \sin(q_2+q_3)) \\ 0 & L_2 \cos q_2 + L_3 \cos(q_2+q_3) & L_3 \cos(q_2+q_3) \end{pmatrix}.
$$

The requested Jacobian ${}^c\boldsymbol{J}(\boldsymbol{q})$ that relates $\dot{\boldsymbol{q}}$ to ${}^c\boldsymbol{v}$ $(= {}^c\boldsymbol{v}_3)$ is obtained by applying suitable rotation matrices:

$$
\begin{aligned}
{}^c\boldsymbol{J}(\boldsymbol{q}) &= {}^0\boldsymbol{R}_c^T(\boldsymbol{q}) {}^0\boldsymbol{J}(\boldsymbol{q}) = {}^3\boldsymbol{R}_c^T \left( {}^2\boldsymbol{R}_3^T(q_3) \left( {}^1\boldsymbol{R}_2^T(q_2) \left( {}^0\boldsymbol{R}_1^T(q_1)\, {}^0\boldsymbol{J}(\boldsymbol{q}) \right) \right) \right) \\
&= \begin{pmatrix} L_2 \cos q_2 + L_3 \cos(q_2+q_3) & 0 & 0 \\ 0 & L_3 + L_2 \cos q_3 & L_3 \\ 0 & L_2 \sin q_3 & 0 \end{pmatrix}.
\end{aligned}
$$

The following is a symbolic Matlab script performing intermediate and final computations.

```
clear all
clc
syms L1 L2 L3 q1 q2 q3 alfa d a theta pi real
% DH parameters
alfa1=pi/2; alfa2=0; alfa3=0; d1=L1; d2=0; d3=0; a1=0; a2=L2; a3=L3;
% DH homogeneous matrix
A=[cos(theta) -sin(theta)*cos(alfa) sin(theta)*sin(alfa) a*cos(theta);
sin(theta) cos(theta)*cos(alfa) -cos(theta)*sin(alfa) a*sin(theta);
0 sin(alfa) cos(alfa) d;
0 0 0 1];
% evaluations
A1=subs(A, {alfa,d,a,theta}, {alfa1,d1,a1,q1})
A2=subs(A, {alfa,d,a,theta}, {alfa2,d2,a2,q2})
A3=subs(A, {alfa,d,a,theta}, {alfa3,d3,a3,q3})
R1=A1(1:3,1:3); R2=A2(1:3,1:3); R3=A3(1:3,1:3);
% camera frame
Rc=[0 0 1; 0 1 0; -1 0 0]
% position of O3
phom=A1*(A2*(A3*[0 0 0 1]'));
p=simplify(phom(1:3))
% Jacobian in frame 0
q=[q1 q2 q3]';
J=jacobian(p,q)
% Jacobian in frame 1,2,3
J1=simplify(R1'*J)
J2=simplify(R2'*J1)
J3=simplify(R3'*J2)
% final Jacobian in camera frame
Jc=simplify(Rc'*J3)
% end
```

**Exercise 2**

The rotation matrix associated to the $(\alpha, \beta, \gamma)$ angles in the $(Z, X, Y)$ Euler representation, i.e., for a sequence of rotations around the axes $Z$, $X'$ (moved), and $Y''$ (moved), is obtained from the elementary rotation matrices

$$\boldsymbol{R}_Z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{R}_X(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \quad \boldsymbol{R}_Y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix},$$

as

$$\boldsymbol{R}_{ZXY}(\alpha, \beta, \gamma) = \boldsymbol{R}_Z(\alpha)\boldsymbol{R}_X(\beta)\boldsymbol{R}_Y(\gamma),$$

or

$$\boldsymbol{R}_{ZXY}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}.$$

Thus, we can compute the rotation matrices associated to the given $(\alpha_i, \beta_i, \gamma_i)$

$${}^0\boldsymbol{R}_i = \boldsymbol{R}_{ZXY}(\alpha_i, \beta_i, \gamma_i) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \end{pmatrix},$$

and to $(\alpha_f, \beta_f, \gamma_f)$

$${}^0\boldsymbol{R}_f = \boldsymbol{R}_{ZXY}(\alpha_f, \beta_f, \gamma_f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

The relative rotation between the initial and final orientation is thus

$$\boldsymbol{R}_{if} = {}^0\boldsymbol{R}_i^T\,{}^0\boldsymbol{R}_f = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Note that this rotation matrix is defined with respect to the initial orientation ${}^0\boldsymbol{R}_i$ (or $\boldsymbol{R}_{if} = {}^i\boldsymbol{R}_{if}$).

We extract then the angle $\theta_{if}$ and the invariant axis $\boldsymbol{r}$ (a unit vector) from the elements $R_{ij}$ of the rotation matrix $\boldsymbol{R}_{if}$:

$$\theta_{if} = \text{ATAN2}\left\{\sqrt{(R_{21} - R_{12})^2 + (R_{31} - R_{13})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1\right\} = 2.5936 \text{ [rad]}$$

(or, in degrees, $\theta_{if} = 148.6°$). Being $\sin\theta_{if} \neq 0$, we have

$$\boldsymbol{r} = \frac{1}{2\sin\theta_{if}} \begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} = \frac{1}{1.042} \begin{pmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \\ 1 - (\sqrt{2}/2) \end{pmatrix} = \begin{pmatrix} -0.6786 \\ -0.6786 \\ 0.2811 \end{pmatrix}.$$

Again, this vector is expressed in the frame defined by the initial orientation ${}^0\boldsymbol{R}_i$ (or $\boldsymbol{r} = {}^i\boldsymbol{r}$).

For the rest-to-rest rotation in time $T$, the cubic polynomial (in normalized time $\tau = t/T$)

$$\theta(t) = \theta_{if}\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right)$$

is such that $\theta(0) = 0$ and $\theta(T) = \theta_{if}$, and its time derivative

$$\dot\theta(t) = \frac{\theta_{if}}{T}\left(6\left(\frac{t}{T}\right) - 6\left(\frac{t}{T}\right)^2\right),$$

satisfies $\dot\theta(0) = \dot\theta(T) = 0$ as required. The maximum rotation speed is attained at $t = T/2$:

$$\dot\theta\left(\frac{T}{2}\right) = \frac{3\theta_{if}}{2T} \;(> 0) \qquad \Rightarrow \qquad (\text{for } T = 1.5) \quad \dot\theta_{max} = \dot\theta(0.75) = 2.5936 \text{ [rad/s]}.$$

Using the obtained $\boldsymbol{r}$, the orientation at a generic instant $t \in [0, T]$ is

$$\boldsymbol{R}(\boldsymbol{r}, \theta(t)) =$$

$$\begin{pmatrix} r_x^2(1-\cos\theta(t))+\cos\theta(t) & r_x r_y(1-\cos\theta(t))-r_z\sin\theta(t) & r_x r_z(1-\cos\theta(t))+r_y\sin\theta(t) \\ r_x r_y(1-\cos\theta(t))+r_z\sin\theta(t) & r_y^2(1-\cos\theta(t))+\cos\theta(t) & r_y r_z(1-\cos\theta(t))-r_x\sin\theta(t) \\ r_x r_z(1-\cos\theta(t))-r_y\sin\theta(t) & r_y r_z(1-\cos\theta(t))+r_x\sin\theta(t) & r_z^2(1-\cos\theta(t))+\cos\theta(t) \end{pmatrix} =$$

$$\begin{pmatrix} 0.5395\cos\theta(t)+0.4605 & 0.4605(1-\cos\theta(t))-0.2811\sin\theta(t) & -0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) \\ 0.4605(1-\cos\theta(t))+0.2811\sin\theta(t) & 0.5395\cos\theta(t)+0.4605 & -0.1907(1-\cos\theta(t))+0.6786\sin\theta(t) \\ -0.1907(1-\cos\theta(t))+0.6786\sin\theta(t) & -0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) & 0.9210\cos\theta(t)+0.07901 \end{pmatrix}.$$

Indeed, this orientation is relative to the initial one ${}^0\boldsymbol{R}_i$, or $\boldsymbol{R}(\boldsymbol{r}, \theta(t)) = {}^i\boldsymbol{R}({}^i\boldsymbol{r}, \theta(t))$. For check, it is easy to see that at $t = 0$ ($\theta(0) = 0$) it is $\boldsymbol{R}(\boldsymbol{r}, 0) = \boldsymbol{I}$. Similarly, at $t = T$ ($\theta(T) = \theta_{if}$) it is $\boldsymbol{R}(\boldsymbol{r}, \theta_{if}) = \boldsymbol{R}_{if}$. The absolute orientation is simply obtained as

$${}^0\boldsymbol{R}(\boldsymbol{r}, \theta(t)) = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{R}({}^i\boldsymbol{r}, \theta(t)) = \boldsymbol{R}({}^0\boldsymbol{R}_i\,{}^i\boldsymbol{r}, \theta(t)) = \boldsymbol{R}({}^0\boldsymbol{r}, \theta(t)) =$$

$$\begin{pmatrix} 0.2466\cos\theta(t)-0.4798\sin\theta(t)+0.4605 & 0.4605(1-\cos\theta(t))+0.2811\sin\theta(t) & -0.1907-0.5164\cos\theta(t))-0.4798\sin\theta(t) \\ 0.5164\cos\theta(t)+0.4798\sin\theta(t)+0.1907 & 0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) & 0.7861\cos\theta(t)-0.4798\sin\theta(t)-0.07901 \\ -0.4605(1-\cos\theta(t))-0.2811\sin\theta(t) & -0.5395\cos\theta(t)-0.4605 & 0.1907(1-\cos\theta(t))-0.6786\sin\theta(t) \end{pmatrix}.$$

Finally, the angular velocity associated to the planned motion expressed in the frame ${}^0\boldsymbol{R}_i$ is

$${}^i\boldsymbol{\omega}(t) = {}^i\boldsymbol{r}\,\dot\theta(t) = \begin{pmatrix} -0.6786 \\ -0.6786 \\ 0.2811 \end{pmatrix} \dot\theta(t),$$

and in the absolute frame

$${}^0\boldsymbol{\omega}(t) = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{\omega}(t) = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{r}\,\dot\theta(t) = {}^0\boldsymbol{r}\,\dot\theta(t) = \begin{pmatrix} -0.6786 \\ -0.2811 \\ 0.6786 \end{pmatrix} \dot\theta(t).$$

Its maximum value in norm (invariant with respect to the frame of definition) is simply

$$\max_{t\in[0,T]} \left\| {}^{0}\boldsymbol{\omega}(t) \right\| = \max_{t\in[0,T]} \left\| {}^{i}\boldsymbol{\omega}(t) \right\| = \left\| {}^{i}\boldsymbol{r} \right\| \cdot \max_{t\in[0,T]} |\dot{\theta}(t)| = 1 \cdot \dot{\theta}_{max} = 2.5936 \text{ [rad/s]}.$$

A symbolic/numeric Matlab script supporting the computations of Exercise 2 is available.

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