Robotics I

July 4, 2011

Several views of the Barrett 4-dof WAM arm are shown in Fig. 1, together with the frame assignment used by the manufacturer. Six reference frames are considered, including one attached to the Base (fixed platform) and one attached at the Tool (end-effector). All data are in mm. The only information missing in the drawings is that the displacement at the elbow joint is equal to 45 mm. Also, the origin of the Tool frame is placed at the end of link 4 (the drawing may be misleading).

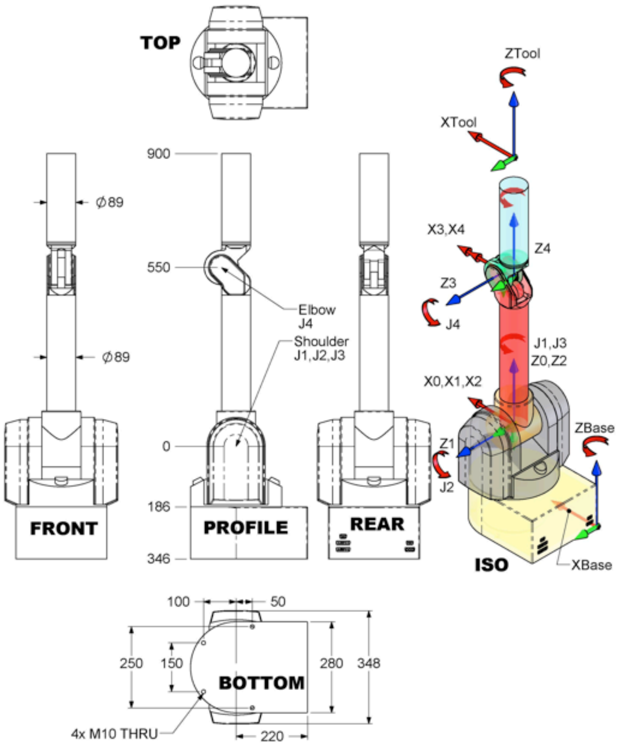

Figure 1: The Barrett 4-dof WAM arm, with the frame assignment used by the manufacturer

1. Check whether the frame assignment used by the manufacturer is fully consistent with the Denavit-Hartenberg (D-H) convention, and modify it if needed. Derive the correct table of D-H parameters.

2. Determine the direct kinematic map for the position ${}^B\boldsymbol{p}_T$ of the origin of the Tool frame with respect to the Base frame (use symbols for geometric quantities, specifying separately their numerical values):

$$\begin{pmatrix} {}^B\boldsymbol{p}_T \\ \\ 1 \end{pmatrix} = {}^BT_0\,{}^0T_1(\theta_1)\,{}^1T_2(\theta_2)\,{}^2T_3(\theta_3)\,{}^3T_4(\theta_4)\,{}^4T_T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (1)$$

3. Four motors are used to drive the four joints through suitable transmission elements, with two motors (located at the shoulder) cooperating for the motion of joints 2 and 3. The mapping between the motor velocity vector $\dot{\boldsymbol{\theta}}_m$ and the joint velocity vector $\dot{\boldsymbol{\theta}}$ is

$$\dot{\boldsymbol{\theta}} = \begin{pmatrix} -\frac{1}{N_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2N_2} & -\frac{1}{2N_2} & 0 \\ 0 & -\frac{n_3}{2N_2} & -\frac{n_3}{2N_2} & 0 \\ 0 & 0 & 0 & -\frac{1}{N_4} \end{pmatrix} \dot{\boldsymbol{\theta}}_m, \tag{2}$$

with $N_1 = 42$, $N_2 = 28.25$, $n_3 = 1.68$, $N_4 = 18$. Determine the mapping *from* the torque vector $\boldsymbol{\tau}_m$ at the output shaft of each motor *to* the torque vector $\boldsymbol{\tau}$ producing work on the joint coordinates $\boldsymbol{\theta}$.

4. Find the 3×4 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ relating the joint velocity vector $\dot{\boldsymbol{\theta}}$ to the velocity $\boldsymbol{v}_T$ of the origin of the Tool frame

$$\boldsymbol{v}_T = \dot{\boldsymbol{p}}_T = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}.$$

Choose your preferred reference frame for expressing $\boldsymbol{v}_T$. *Note: In order to obtain the full expression of the associated Jacobian, the use of a symbolic manipulation program is appropriate.*

5. Determine one singularity of $\boldsymbol{J}(\boldsymbol{\theta})$. Verify whether this singularity is within the following joint limits of the robot or not:

$$-150° \leq \theta_1 \leq +150°, \quad -113° \leq \theta_2 \leq +113°, \quad -157° \leq \theta_3 \leq +157°, \quad -50° \leq \theta_4 \leq +180°.$$

[150 minutes; open books]

Solution

July 4, 2011

For item 1., the frame assignment used by the manufacturer is fully consistent with the Denavit-Hartenberg convention and there is no need of modifications. The D-H parameters are given in Tab. 1, with $a_3 = 45$ and $d_3 = 550$ [mm]. In Fig. 1, the arm is shown in its "zero configuration" ($\boldsymbol{\theta} = \mathbf{0}$).

i	α_i	a_i	d_i	θ_i
1	$-\frac{\pi}{2}$	0	0	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	$-\frac{\pi}{2}$	a_3	d_3	θ_3
4	$\frac{\pi}{2}$	$-a_3$	0	θ_4

Table 1: D-H parameters for the Barrett 4-dof WAM arm

For item 2., Table 1 is used to compute the four homogeneous transformations ${}^{i-1}T_i(\theta_i)$, for $i = 1, \ldots, 4$. The remaining two constant transformations are

$$ {}^{B}T_0 = \begin{pmatrix} & & {}^{B}p_{0x} \\ \boldsymbol{I} & & {}^{B}p_{0y} \\ & & {}^{B}p_{0z} \\ \mathbf{0}^T & & 1 \end{pmatrix}, $$

with ${}^{B}p_{0x} = 220$, ${}^{B}p_{0y} = 140$, ${}^{B}p_{0z} = 346$ [mm], and

$$ {}^{4}T_T = \begin{pmatrix} & & 0 \\ \boldsymbol{I} & & 0 \\ & & L \\ \mathbf{0}^T & & 1 \end{pmatrix}, $$

with $L = 350$ [mm]. Note that these constant transformations may not be expressed in general as D-H homogeneous matrices (i.e., in terms of the usual four, now all constant, D-H parameters). In particular, ${}^{4}T_T$ has the structure of a D-H matrix (with $\alpha_T = \theta_T = 0$, $a_T = 0$, and $d_T = L$), whereas ${}^{B}T_0$ cannot be associated to D-H parameters.

In order to find the expression of ${}^{B}\boldsymbol{p}_T$ in (1), there is no need to compute the full product of all the transformation matrices, i.e., ${}^{B}T_T$ (or even just ${}^{0}T_4$). In fact, expressing all vectors in homogenous coordinates, we can compute recursively:

$$\begin{aligned}
{}^4\boldsymbol{p}_T &= {}^4T_T{}^T\boldsymbol{p}_T = {}^4T_T \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} = \begin{pmatrix} 0\\0\\L\\1 \end{pmatrix}\\
{}^3\boldsymbol{p}_T &= {}^3T_4(\theta_4){}^4\boldsymbol{p}_T\\
&\vdots\\
{}^0\boldsymbol{p}_T &= {}^0T_1(\theta_1){}^1\boldsymbol{p}_T\\
{}^B\boldsymbol{p}_T &= {}^BT_0{}^4\boldsymbol{p}_T.
\end{aligned}$$

It is then tedious but straightforward to obtain

$${}^B\boldsymbol{p}_T = \begin{pmatrix} {}^Bp_{0x} + (c_1c_2c_3 - s_1s_3)(a_3 + Ls_4 - a_3c_4) + c_1s_2(d_3 - a_3s_4 - Lc_4)\\ {}^Bp_{0y} + (s_1c_2c_3 + c_1s_3)(a_3 + Ls_4 - a_3c_4) + s_1s_2(d_3 - a_3s_4 - Lc_4)\\ {}^Bp_{0z} + s_2c_3(a_3 + Ls_4 - a_3c_4) + c_2(d_3 + a_3s_4 + Lc_4) \end{pmatrix}, \tag{3}$$

where the short notations $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ have been used. At this point, we can substitute the numerical values previously defined.

For item 3., writing eq. (2) as $\dot{\boldsymbol{\theta}} = \boldsymbol{A}\dot{\boldsymbol{\theta}}_m$, we have from the principle of virtual work

$$\boldsymbol{\tau}^T\dot{\boldsymbol{\theta}} = \boldsymbol{\tau}^T\boldsymbol{A}\dot{\boldsymbol{\theta}}_m = \boldsymbol{\tau}_m^T\dot{\boldsymbol{\theta}}_m, \qquad \forall\dot{\boldsymbol{\theta}}_m,$$

and thus

$$\boldsymbol{\tau}_m = \boldsymbol{A}^T\boldsymbol{\tau} \qquad \Longleftrightarrow \qquad \boldsymbol{\tau} = \boldsymbol{A}^{-T}\boldsymbol{\tau}_m.$$

Therefore, the mapping is

$$\boldsymbol{\tau} = \begin{pmatrix} -N_1 & 0 & 0 & 0\\ 0 & N_2 & -N_2 & 0\\ 0 & -\dfrac{N_2}{n_3} & -\dfrac{N_2}{n_3} & 0\\ 0 & 0 & 0 & -N_4 \end{pmatrix}\boldsymbol{\tau}_m.$$

We can now replace in this expression the numerical data provided in the text.

For item 4., the Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$ of interest is computed either geometrically or by analytic differentiation of eq. (3) w.r.t. $\boldsymbol{\theta}$. The result can be obtained by hand or by using symbolic/algebraic manipulation tools (e.g., the *Matlab Symbolic Toolbox*). However, very complex expressions are typically obtained which are difficult to be simplified automatically. A possible relatively simpler form for $\boldsymbol{J}(\boldsymbol{\theta})$ is obtained when looking at the expression of the task velocity $\boldsymbol{v}_T$ (naturally defined in the zero-th reference frame) in a suitable reference frame. In fact, we have

$${}^i\boldsymbol{v}_T = {}^0\boldsymbol{R}_i^T(\theta_1,\ldots,\theta_i)\,{}^0\boldsymbol{v}_T = {}^0\boldsymbol{R}_i^T(\theta_1,\ldots,\theta_i)\,{}^0\boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \Rightarrow \qquad {}^i\boldsymbol{J}(\boldsymbol{\theta}) = {}^0\boldsymbol{R}_i^T(\theta_1,\ldots,\theta_i)\,{}^0\boldsymbol{J}(\boldsymbol{\theta})$$

where the Jacobian ${}^i\boldsymbol{J}(\boldsymbol{\theta})$ is expressed in the i-th frame of the robot, and

$${}^0\boldsymbol{R}_i(\theta_1,\ldots,\theta_i) = {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\ldots{}^{i-1}\boldsymbol{R}_i(\theta_i)$$

is the composition of the rotation matrices defined by the D-H parameters of the robot. For the Barrett 4-dof WAM arm, a convenient choice is to refer the Jacobian to frame 2, i.e.,

$$^2\boldsymbol{J}(\boldsymbol{\theta}) = {}^1\boldsymbol{R}_2^T(\theta_2)\, {}^0\boldsymbol{R}_1^T(\theta_1)\, {}^0\boldsymbol{J}(\boldsymbol{\theta}).$$

For illustration, we report in the Appendix the $3 \times 4 = 12$ elements j_{ik} of $^2\boldsymbol{J}(\boldsymbol{\theta})$, i.e.,

$$^2\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} j_{11} & j_{12} & j_{13} & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ j_{31} & j_{32} & j_{33} & j_{34} \end{pmatrix}, \tag{4}$$

provided as output of a *Matlab Symbolic Toolbox* program (available upon request).

For item 5., it is rather immediate to see from Fig. 1 that the robot configuration having $\theta_2 = \theta_3 = \theta_4 = 0$ (and arbitrary θ_1) is certainly singular, since the arm is fully stretched. Plugging in these values of joint angles, the symbolic Jacobian matrix becomes

$$^2\boldsymbol{J}(\boldsymbol{\theta})|_{\theta_2=\theta_3=\theta_4=0} = \begin{pmatrix} 0 & (L+d_3)\,c_1 & 0 & L\,c_1 \\ 0 & (L+d_3)\,s_1 & 0 & L\,s_1 \\ 0 & 0 & 0 & a_3 \end{pmatrix},$$

having, as expected, rank 2 for any θ_1. In particular, this is true at the zero configuration (i.e., with $\theta_1 = 0$, in addition to the previously selected values for the other joints) illustrated in Fig. 1. This singularity is clearly within the assigned joint limits.

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Appendix

With reference to (4), using the short notations for$s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ $(i = 1, \ldots, 4)$, we have:

$$\begin{aligned}
j_{11} &= -c_2\left(c_1\left(c_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) + s_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\right.\\
&\quad \left. + s_1\left(s_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) - c_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\right)\\
j_{12} &= s_2\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right) - c_2\left(c_1\left(c_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right.\right.\\
&\quad \left.\left. - c_1c_2\left(d_3 + Lc_4 + a_3s_4\right)\right) - s_1\left(c_2s_1\left(d_3 + Lc_4 + a_3s_4\right) - s_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\\
j_{13} &= -s_3\left(2a_3\sin\left(\tfrac{\theta_4}{2}\right)^2 + Ls_4\right)s_2{}^2 - c_2\left(c_1\left(c_3s_1 + c_1c_2s_3\right)\left(a_3 + Ls_4 - a_3c_4\right)\right.\\
&\quad \left. - s_1\left(c_1c_3 - c_2s_1s_3\right)\left(a_3 + Ls_4 - a_3c_4\right)\right)\\
j_{14} &= s_2\left(c_2\left(Ls_4 - a_3c_4\right) + c_3s_2\left(Lc_4 + a_3s_4\right)\right) - c_2\left(c_1\left(c_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right)\right.\right.\\
&\quad \left.\left. + s_1s_3\left(Lc_4 + a_3s_4\right)\right) + s_1\left(s_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right) - c_1s_3\left(Lc_4 + a_3s_4\right)\right)\right)\\
j_{21} &= s_1\left(c_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) + s_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\\
&\quad - c_1\left(s_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) - c_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\\
j_{22} &= s_1\left(c_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right) - c_1c_2\left(d_3 + Lc_4 + a_3s_4\right)\right) + c_1\left(c_2s_1\left(d_3 + Lc_4 + a_3s_4\right)\right.\\
&\quad \left. - s_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\\
j_{23} &= c_1\left(c_1c_3 - c_2s_1s_3\right)\left(a_3 + Ls_4 - a_3c_4\right) + s_1\left(c_3s_1 + c_1c_2s_3\right)\left(a_3 + Ls_4 - a_3c_4\right)\\
j_{24} &= s_1\left(c_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right) + s_1s_3\left(Lc_4 + a_3s_4\right)\right)\\
&\quad - c_1\left(s_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right) - c_1s_3\left(Lc_4 + a_3s_4\right)\right)\\
j_{31} &= -s_2\left(c_1\left(c_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) + s_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\right.\\
&\quad \left. + s_1\left(s_1\left(a_3s_3 + s_3\left(Ls_4 - a_3c_4\right)\right) - c_1\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right)\right)\\
j_{32} &= -s_2\left(c_1\left(c_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right) - c_1c_2\left(d_3 + Lc_4 + a_3s_4\right)\right) - s_1\left(c_2s_1\left(d_3 + Lc_4 + a_3s_4\right)\right.\right.\\
&\quad \left.\left. - s_1s_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\right) - c_2\left(s_2\left(d_3 + Lc_4 + a_3s_4\right) + c_2\left(a_3c_3 + c_3\left(Ls_4 - a_3c_4\right)\right)\right)\\
j_{33} &= c_2s_2s_3\left(2a_3\sin\left(\tfrac{\theta_4}{2}\right)^2 + Ls_4\right) - s_2\left(c_1\left(c_3s_1 + c_1c_2s_3\right)\left(a_3 + Ls_4 - a_3c_4\right)\right.\\
&\quad \left. - s_1\left(c_1c_3 - c_2s_1s_3\right)\left(a_3 + Ls_4 - a_3c_4\right)\right)\\
j_{34} &= -c_2\left(c_2\left(Ls_4 - a_3c_4\right) + c_3s_2\left(Lc_4 + a_3s_4\right)\right) - s_2\left(c_1\left(c_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right)\right.\right.\\
&\quad \left.\left. + s_1s_3\left(Lc_4 + a_3s_4\right)\right) + s_1\left(s_1\left(s_2\left(Ls_4 - a_3c_4\right) - c_2c_3\left(Lc_4 + a_3s_4\right)\right) - c_1s_3\left(L\cos\theta_4 + a_3\sin\theta_4\right)\right)\right).
\end{aligned}$$

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