Consider a 3R anthropomorphic robot mounted on the floor and characterized by the Denavit-Hartenberg parameters in Table 1, where $D$, $L_1$, $L_2$, and $L_3$ are all strictly positive values.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$a_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>$D$</td>
<td>$L_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$L_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$L_3$</td>
<td>$q_3$</td>
</tr>
</tbody>
</table>

Table 1: Table of DH parameters

1. Obtain the $3 \times 3$ Jacobian matrix $^0J_L(q)$ relating the joint velocity $\dot{q}$ to the linear velocity $^0v$ of the origin $O_3$ of frame 3 expressed in frame 0.

2. Characterize the singular configurations $\mathbf{q}$ of the Jacobian $^3J_L(q)$ relating $\dot{q}$ to the linear velocity $^3v$ of the origin $O_3$ of frame 3 expressed in frame 3.

3. Obtain the $3 \times 3$ Jacobian matrix $^0J_A(q)$ relating the joint velocity $\dot{q}$ to the angular velocity $^0\omega$ of frame 3 expressed in frame 0. Show that this matrix is always singular and provide an explanation of this result.

4. Assume that the robot is in the configuration

$$\mathbf{q}^* = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T \text{ [rad]}$$

with a joint velocity

$$\dot{\mathbf{q}}^* = \begin{pmatrix} \dot{q}_1^* & 0 & 0 \end{pmatrix}^T \text{ [rad/s]}, \quad \text{with } \dot{q}_1^* \neq 0.$$  

Determine the joint acceleration $\ddot{q}$ that should be imposed so that the resulting linear Cartesian acceleration of the origin $O_3$ is directed along $y_3$ and has an intensity $A \neq 0$. Provide some comment on the structure of the obtained solution. In particular, is there a value $A$ such that only one joint needs to accelerate?
For item 1, we are interested in the velocity of point \( O_3 \), whose position \( p = 0^0 p \) is given by the direct kinematics map

\[
p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos q_1 (L_1 + L_2 \cos q_2 + L_3 \cos (q_2 + q_3)) \\ \sin q_1 (L_1 + L_2 \cos q_2 + L_3 \cos (q_2 + q_3)) \\ D + L_2 \sin q_2 + L_3 \sin (q_2 + q_3) \end{pmatrix} = f(q). \quad (1)
\]

The Jacobian \( ^0J_L(q) \) can be obtained either by analytical differentiation of \( f(q) \) in (1) w.r.t. \( q \) or by using the expression of the first three rows of the geometric Jacobian. Using the usual short notation for trigonometric functions, the result is in both cases

\[
^0J_L(q) = \begin{pmatrix} -s_1(L_1 + L_2c_2 + L_3c_{23}) & -c_1(L_2s_2 + L_3s_{23}) & -L_3c_1s_{23} \\ c_1(L_1 + L_2c_2 + L_3c_{23}) & -s_1(L_2s_2 + L_3s_{23}) & -L_3s_1s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \end{pmatrix}. \quad (2)
\]

For item 2, we have that

\[
\det ^3J_L(q) = \det \left( ^2R_3^T(q_3)\ ^1R_2^T(q_2)\ ^0R_1^T(q_1)\ ^0J_L(q) \right) = \det ^0J_L(q).
\]

Nonetheless, it is useful to rewrite the Jacobian in the successive frames 1, 2, and 3, because the resulting expressions will be simplified. From Table 1, we have

\[
^0R_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix}, \quad ^1R_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ^2R_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

From these we obtain

\[
^1J_L(q) = ^0R_1^T(q_1)\ ^0J_L(q) = \begin{pmatrix} 0 & -(L_2s_2 + L_3s_{23}) & -L_3s_{23} \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},
\]

\[
^2J_L(q) = ^1R_2^T(q_2)\ ^1J_L(q) = \begin{pmatrix} 0 & -L_3s_3 & -L_3s_3 \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix},
\]

and

\[
^3J_L(q) = ^2R_3^T(q_3)\ ^2J_L(q) = \begin{pmatrix} 0 & L_2s_3 & 0 \\ 0 & L_3 + L_2c_3 & L_3 \\ -(L_1 + L_2c_2 + L_3c_{23}) & 0 & 0 \end{pmatrix}.
\]

In particular from the last expression, it is immediate to see that for any \( i \in \{1, 2, 3\} \)

\[
\det ^iJ_L(q) = -(L_2L_3(L_1 + L_2c_2 + L_3c_{23})s_3). \quad (3)
\]
Therefore, the singular configurations of $J_L(q)$ are:

$$s_3 = 0 \iff q_3 = \{0, \pm \pi\} \quad \text{(third link is stretched or folded)}$$

$$L_1 + L_2c_2 + L_3c_{23} = 0 \iff p_x = p_y = 0 \quad (O_3 \text{ is on the axis } z_0 \text{ of joint 1})$$

For item 3, we compute the expression of the lower three rows of the geometric Jacobian. It is

$$0J_A(q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_1 \\ 0 & z_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & -c_1 \end{pmatrix} R_1(q_1) R_2(q_2) R_3(q_3).$$

(4)

Matrix $0J_A(q)$ is always singular, having constant rank equal to 2. This can be easily explained as follows. The three degrees of freedom of the considered manipulator allow placing the end-effector in any point of the robot primary workspace, and imposing a linear velocity in any direction when the arm is out of singularities. However, the orientation of the end-effector frame can never be changed around the unitary axis $n(q_1) = (c_1 \quad s_1 \quad 0)^T$. In fact, $\omega = \alpha n(q_1) \notin R\{0J_A(q)\}$, for every $q$ and for any scalar $\alpha$.

Finally, for item 4 we use the second-order differential map

$$\ddot{p} = 0J_L(q)^{-1} \ddot{q} + 0J_L(q) \dot{q},$$

(5)

evaluated at $q = q^*$, $\dot{q} = \dot{q}^*$. The Cartesian acceleration is specified as

$$\ddot{p} = 0R_3(q)^{\ddot{p}} = 0R_1(q_1)^{1} R_2(q_2)^{2} R_3(q_3) \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} = \begin{pmatrix} -Ac_1s_{23} \\ -As_1c_{23} \\ A_{c_{23}} \end{pmatrix},$$

which, when evaluated at $q = q^*$, yields the desired value

$$\ddot{p}_d = \ddot{p}|_{q=q^*} = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix},$$

(6)

i.e., the acceleration of the end-effector should be directed along $z_0$, the vertical direction. Since the determinant (3) of the associated Jacobian is nonzero at the given configuration, the solution for the joint acceleration is obtained from (5) as

$$\ddot{q} = 0J_L^{-1}(q^*) \begin{pmatrix} \ddot{p}_d - 0J_L(q^*) \dot{q}^* \end{pmatrix},$$

where

$$0J_L^{-1}(q^*) = \begin{pmatrix} 0 & -L_2 \frac{\sqrt{2}}{2} & 0 \\ L_1 + L_2 \frac{\sqrt{2}}{2} + L_3 & 0 & 0 \\ 0 & L_2 \frac{\sqrt{2}}{2} + L_3 & L_3 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{L_2} & 0 \\ -L_1 + L_2 \frac{\sqrt{2}}{2} + L_3 & 0 & 0 \\ -L_1 + L_2 \frac{\sqrt{2}}{L_2} & 0 & \frac{1}{L_3} \end{pmatrix}.$$
Let \( ^0J_1 \) be the first column of the Jacobian \( ^0J_L \). Thanks to the simple structure of \( \dot{q}^* \), for the term involving the time derivative of the Jacobian we need only to compute

\[
\left( ^0J_L(q) \dot{q} \right)_{q=q^*, \dot{q}=\dot{q}^*} = \left( ^0J_1(q) \right)_{q=q^*, \dot{q}=\dot{q}^*} \left( \frac{\partial ^0J_1(q)}{\partial q_1} \right)_{q=q^*} \dot{q}^*_1
\]

\[
= \begin{pmatrix} -c_1(L_1 + L_2 c_2 + L_3 c_{23}) \\ -s_1(L_1 + L_2 c_2 + L_3 c_{23}) \\ 0 \end{pmatrix}_{q=q^*} \dot{q}^*_1^2 = \begin{pmatrix} -(L_1 + L_2 \frac{\sqrt{2}}{2} + L_3) \\ 0 \\ 0 \end{pmatrix} \dot{q}^*_1^2.
\]

As a result, from (6-8) the final solution is

\[
\ddot{q} = A \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_3} \end{pmatrix} + (L_1 + L_2 \frac{\sqrt{2}}{2} + L_3) (\dot{q}^*_1)^2 \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{L_3} \\ \frac{1}{L_3 + \frac{\sqrt{2}}{L_3}} \end{pmatrix}.
\]

We note that no acceleration should be applied to the first joint (\( \ddot{q}_1 = 0 \)), as could be argued already from (6). In fact, any angular acceleration imposed to joint 1 (along the vertical joint axis \( z_0 \)) would produce a centrifugal acceleration on the end-effector, which is in contrast with the requested zero acceleration along the \( x_0 \) and \( y_0 \) axes in (6). Moreover, if

\[
A = - \left( 1 + \frac{L_3}{L_2} \frac{\sqrt{2}}{2} \right) (L_1 + L_2 \frac{\sqrt{2}}{2} + L_3) (\dot{q}^*_1)^2
\]

then \( \ddot{q}_1 = \ddot{q}_3 = 0 \) in the solution.

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