Exercise 1

For a planar RP robot, consider a class of one-dimensional tasks defined only in terms of the $y$-component of the end-effector Cartesian position

$$y = p_y(q_1, q_2).$$

a) Study the singularity conditions for the robot performing this class of tasks.

b) Given a desired task trajectory $y_d(t)$, admitting second time derivative, provide the expression of a kinematic control law that is able to zero the task error $e = y_d - y$ in an exponential way starting from any initial robot condition $(q(0), \dot{q}(0))$, when the available control commands are the joint accelerations $\ddot{q}$.

Exercise 2

For a minimal representation of the orientation of a rigid body given by Euler angles $\phi = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $Y'X'Z''$, determine the relation

$$\omega = T(\phi)\dot{\phi}$$

between the time derivatives of the Euler angles and the angular velocity $\omega$ of the rigid body. Find the singularities of $T(\phi)$, and provide an example of an angular velocity vector $\omega$ that cannot be represented in a singularity.

[90 minutes; open books]
Exercise 1

The direct kinematics associated to the end-effector position of the RP robot is

\[ \mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \]

where a ‘natural’ set of coordinates has been chosen, with \( q_1 \) being the angle between the \( x_0 \) axis and the second link of the robot\(^1\).

Being the task defined only in terms of the \( p_y \) component, it is

\[ \dot{p}_y = ( q_2 \cos q_1 \sin q_1 ) \dot{q} = \mathbf{J}(q)\dot{q} \]

and

\[ \ddot{p}_y = \mathbf{J}(q)\ddot{q} + \mathbf{J}(q)\dot{q} = \mathbf{J}(q)\ddot{q} + ( \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \cos q_1 \dot{q}_1 ) \dot{q}. \]

The task Jacobian \( \mathbf{J} \) is then singular when

\[ \sin q_1 = 0 \text{ AND } q_2 = 0. \]

In this case, the rank of the \( \mathbf{J} \) matrix is zero and the one-dimensional task cannot be correctly performed. Out of singularities, all the joint accelerations \( \ddot{q} \) that realize a desired \( \ddot{y}_d \) can be written in the form

\[ \ddot{q} = \mathbf{J}^\#(q) \left( \ddot{y}_d - \mathbf{J}(q)\ddot{q} \right) + \left( \mathbf{I} - \mathbf{J}^\#(q)\mathbf{J}(q) \right) \ddot{q}_0, \]

being the task redundant (\( M = 1 \)) for the RP robot (\( N = 2 \)). Setting \( \ddot{q}_0 = 0 \) one obtains the solution with minimum joint acceleration norm. Assuming full rank (equal to 1) for the task Jacobian \( \mathbf{J} \), its pseudoinverse has the explicit expression

\[ \mathbf{J}^\#(q) = \frac{1}{q_2^2 \cos^2 q_1 + \sin^2 q_1} \begin{pmatrix} q_2 \cos q_1 \\ \sin q_1 \end{pmatrix}. \]

A kinematic control law with the requested performance is defined by

\[ \ddot{q} = \mathbf{J}^\#(q) \left( \ddot{y}_d + k_d(\ddot{y}_d - \ddot{p}_y) + k_p(y_d - p_y) - \mathbf{J}(q)\ddot{q} \right), \]

where \( k_d > 0 \) and \( k_p > 0 \) and we set for simplicity \( \ddot{q}_0 = 0 \). A more convenient choice would be to include an acceleration \( \ddot{q}_0 = -\mathbf{K}_D \ddot{q} \), with a diagonal, positive definite matrix \( \mathbf{K}_D \), in the null space of the task Jacobian. As a matter of fact, such additional term allows to damp possible increases of internal joint velocity without perturbing the task.

\(^1\)When using the Denavit-Hartenberg formalism, one would define \( q_2^{\text{DH}} = q_2 \pm \frac{\pi}{2} \). The rest of the developments follows accordingly in a similar way.
Exercise 2

The orientation of a rigid body is represented, using the Euler angles $\phi = (\alpha, \beta, \gamma)$ around the sequence of mobile axes $YX'Z''$, by the product of elementary rotation matrices

$$R = R_Y(\alpha)R_{X'}(\beta)R_{Z''}(\gamma).$$

The angular velocity $\omega$ due to $\dot{\phi}$ can be obtained as the sum of the three angular velocities contributed by, respectively, $\dot{\alpha}$ (along the unit vector $Y$), $\dot{\beta}$ (along $X'$), and $\dot{\gamma}$ (along $Z''$)

$$\omega = \omega_\alpha + \omega_\beta + \omega_\gamma = Y\dot{\alpha} + X'\dot{\beta} + Z''\dot{\gamma}$$

where the unit vectors $Y$, $X'$, and $Z''$ are expressed with respect to the initial reference frame. It is

$$Y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad X' = R_Y(\alpha) \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad Z'' = R_Y(\alpha)R_{X'}(\beta) \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$  

Thus, it is sufficient to compute

$$R_Y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}, \quad R_{X'}(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix},$$

$$R_Y(\alpha)R_{X'}(\beta) = \begin{bmatrix} * & * & \sin \alpha \cos \beta \\ * & * & -\sin \beta \\ * & * & \cos \alpha \cos \beta \end{bmatrix}$$

in order to obtain

$$\omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot{\alpha} + \begin{bmatrix} \cos \alpha \\ 0 \\ -\sin \alpha \end{bmatrix}\dot{\beta} + \begin{bmatrix} \sin \alpha \cos \beta \\ -\sin \beta \\ \cos \alpha \cos \beta \end{bmatrix}\dot{\gamma} = \begin{bmatrix} 0 & \cos \alpha & \sin \alpha \cos \beta \\ 1 & 0 & -\sin \beta \\ 0 & \sin \alpha & \cos \alpha \cos \beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = T(\phi)\dot{\phi}.$$

Note also, as a general property, that matrix $T$ depends only on the first two Euler angles. Matrix $T$ is singular when

$$\det T = -\cos \beta = 0 \iff \beta = \pm \frac{\pi}{2}.$$  

In this condition, an angular velocity vector (with norm $k$) of the form

$$\omega = k \begin{bmatrix} \sin \alpha \\ 0 \\ \cos \alpha \end{bmatrix} \notin \mathcal{R} \left\{ T(\alpha, \pm \frac{\pi}{2}) \right\}$$

cannot be represented by any choice of $\dot{\phi}$.  

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