Robotics 1

Trajectory planning in Cartesian space

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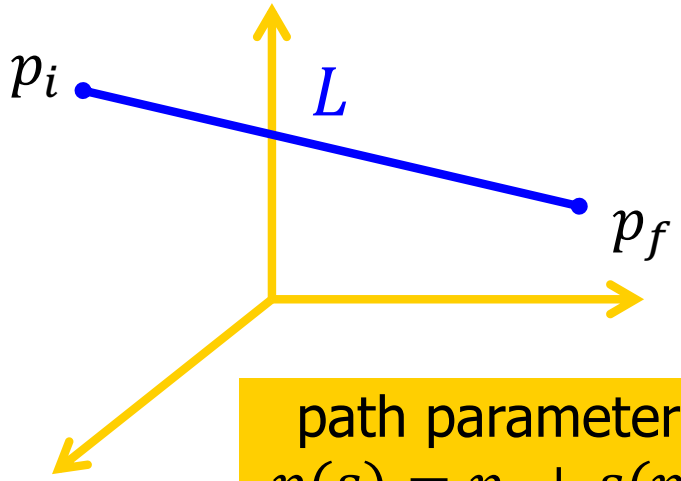
Trajectories in Cartesian space

- in general, PTP/MP trajectory planning methods developed in joint space can be applied also in Cartesian/task space
  - consider independently each component of the task vector (i.e., a position or an angle of a minimal representation of orientation)
- however, when planning a trajectory for three orientation angles, the resulting global motion cannot be intuitively visualized in advance
- moreover, if possible, we still prefer to plan Cartesian trajectories separately for position and orientation
- the number of knots to be interpolated in the Cartesian space may be small (e.g., 2 knots (PTP motion), 3 if a "via point" is added) ⇒ use simple interpolating paths, such as straight lines, arc of circles, …

Planning a linear Cartesian motion
(in position only)



GIVEN $p_i, p_f \in \mathbb{R}^3$ for the path
and $v_i, v_f \in \mathbb{R}$ (typically = 0) with
bounds $V, A \in \mathbb{R}^+$ for the timing law

$$L = \|p_f - p_i\|$$

$$\frac{p_f - p_i}{\|p_f - p_i\|} = \text{unit vector of directional cosines of the line}$$

path parameterization
$$p(s) = p_i + s(p_f - p_i)$$

$$s \in [0,1]$$

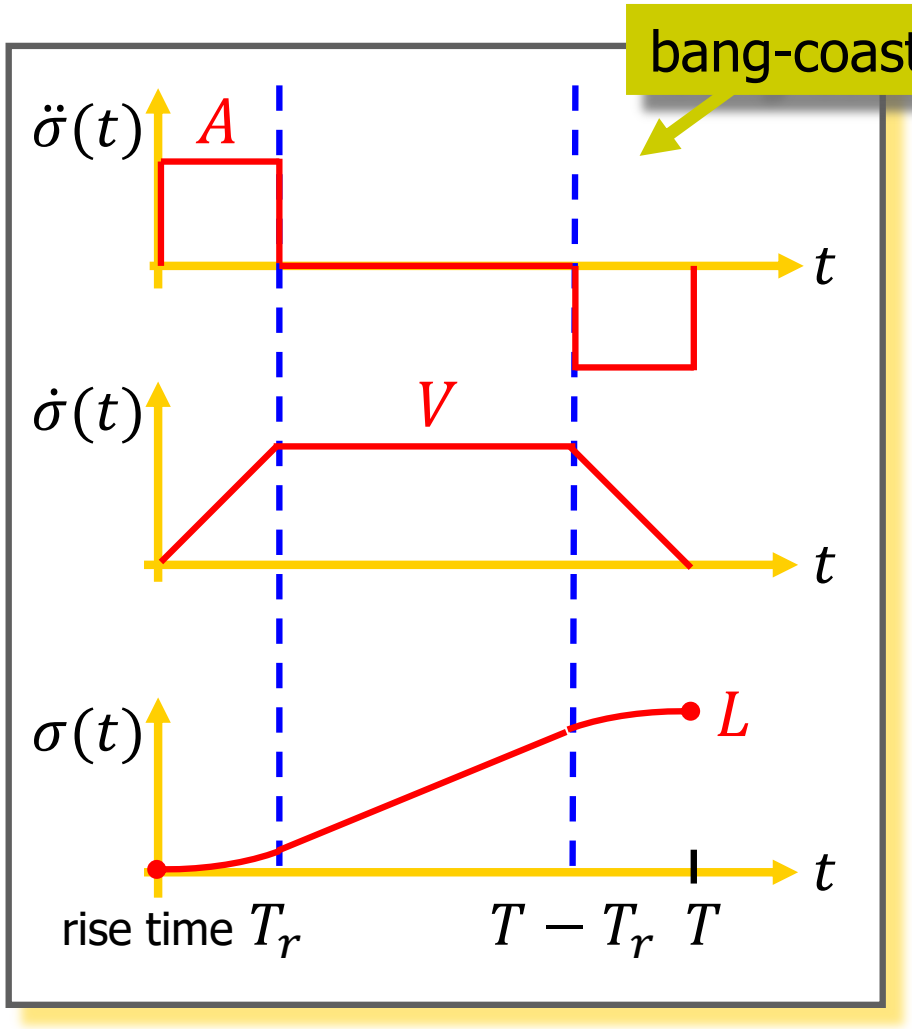
may also use $s = \sigma/L$, where $\sigma \in [0, L]$ is the arc length (= length of the path traced so far)

$$\dot{p}(s) = \frac{dp}{ds}\dot{s} = (p_f - p_i)\,\dot{s} = \frac{p_f - p_i}{L}\,\dot{\sigma}$$

$$\ddot{p}(s) = \frac{d^2p}{ds^2}\dot{s}^2 + \frac{dp}{ds}\ddot{s} = (p_f - p_i)\,\ddot{s} = \frac{p_f - p_i}{L}\,\ddot{\sigma}$$

...why?

Timing law with trapezoidal speed - 1



given*: L, V, A
find: T_r, T

$V\,(T - T_r) = L$ ← = area of the speed profile

$$T_r = \frac{V}{A}$$

$$T = \frac{A\,L + V^2}{AV}$$

a "coast" phase exists iff $L > V^2/A$

otherwise ...

triangular speed (bang-bang) profile with

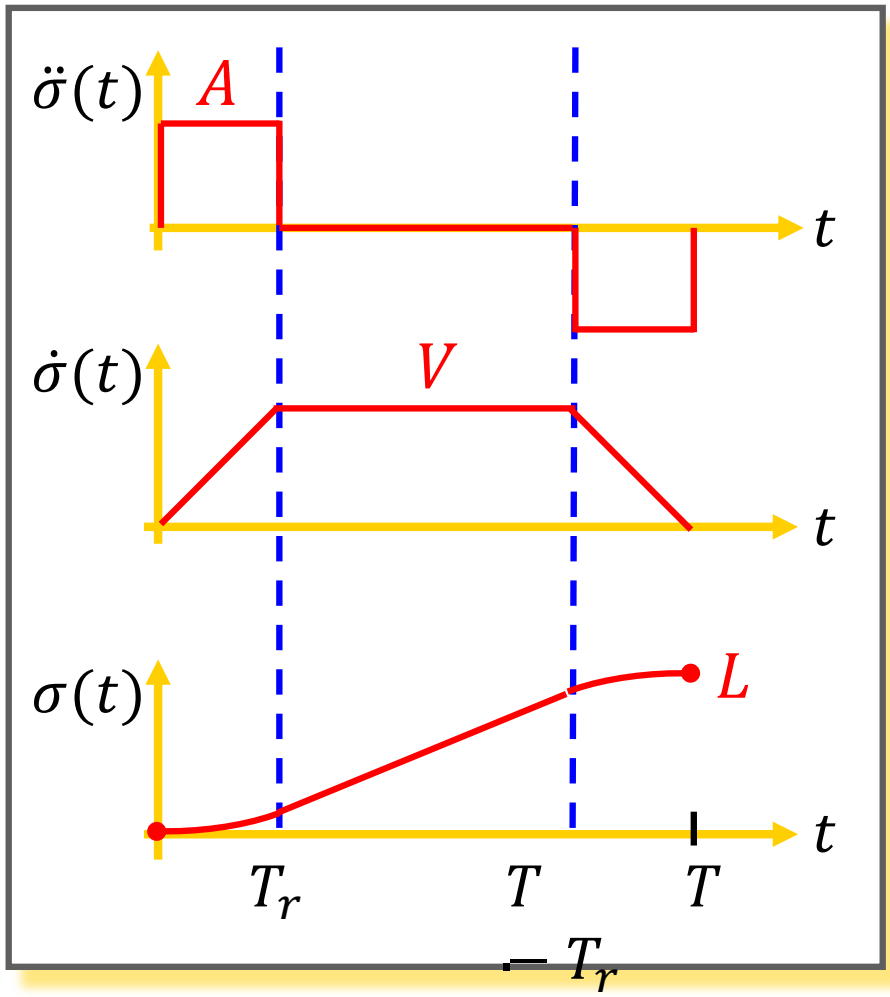
$\max\ \dot{\sigma} = \bar{V} = A\,T_r = A \cdot T/2 \leq V$

$\bar{V} \cdot T/2 = L$ (= area)

$$T = \sqrt{4L/A}$$

* = other input data combinations are possible (see textbook Sect. 4.4.2, p. 217)

Timing law with trapezoidal speed - 2



$$\sigma(t) = \begin{cases} \dfrac{At^2}{2} & t \in [0, T_r] \\ Vt - \dfrac{V^2}{2A} & t \in [T_r, T - T_r] \\ -\dfrac{A(t-T)^2}{2} + VT - \dfrac{V^2}{A} & t \in [T - T_r, T] \end{cases} = L$$

used also as timing law in joint space!

at each joint j separately, gives fastest motion time (T_j, with V_j and A_j) $\Longrightarrow$ non-coordinated motion

taking then $T = \max_{j=1,\dots,n} T_j$: the slowest joint imposes the minimum time $\Longrightarrow$ coordinated motion

it is a discontinuous acceleration profile: if required, use for instance a rest-to-rest quintic polynomial timing law

Linear trajectory bounds
from limits in Cartesian space

- for the linear trajectory

$$p(t) = p_i + \sigma(t)K$$
$$\dot{p}(t) = \dot{\sigma}(t)\,K$$
$$\ddot{p}(t) = \ddot{\sigma}(t)\,K$$

$$K = \frac{p_f - p_i}{\|p_f - p_i\|} = \frac{p_f - p_i}{L} = \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix}$$

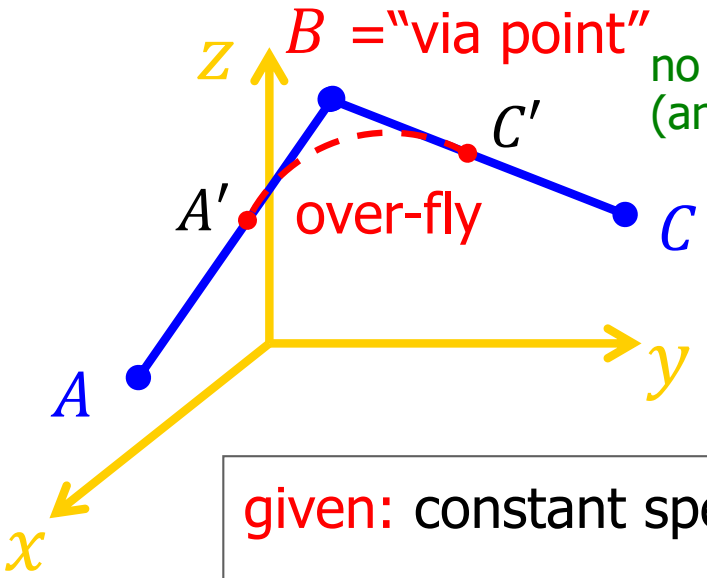
- consider task limits on velocity norm and acceleration norm

$$\|\dot{p}(t)\| \le v_{\max} \quad \Rightarrow \quad |\dot{\sigma}(t)| \le v_{\max} = V$$
$$\|\ddot{p}(t)\| \le a_{\max} \quad \Rightarrow \quad |\ddot{\sigma}(t)| \le a_{\max} = A$$

- for componentwise velocity and acceleration task limits

$$\dot{p}_{\{x,y,z\}} \le v_{\max,\{x,y,z\}} \quad \Rightarrow \quad |\dot{\sigma}(t)| \le \min\left\{\frac{v_{\max,x}}{K_x}, \frac{v_{\max,y}}{K_y}, \frac{v_{\max,z}}{K_z}\right\} = V$$
$$\ddot{p}_{\{x,y,z\}} \le a_{\max,\{x,y,z\}} \quad \Rightarrow \quad |\ddot{\sigma}(t)| \le \min\left\{\frac{a_{\max,x}}{K_x}, \frac{a_{\max,y}}{K_y}, \frac{a_{\max,z}}{K_z}\right\} = A$$

Concatenation of linear segments



no need to pass (and stop!) there

$$\frac{B-A}{\|B-A\|} = K_{AB}$$

$$\frac{C-B}{\|C-B\|} = K_{BC}$$

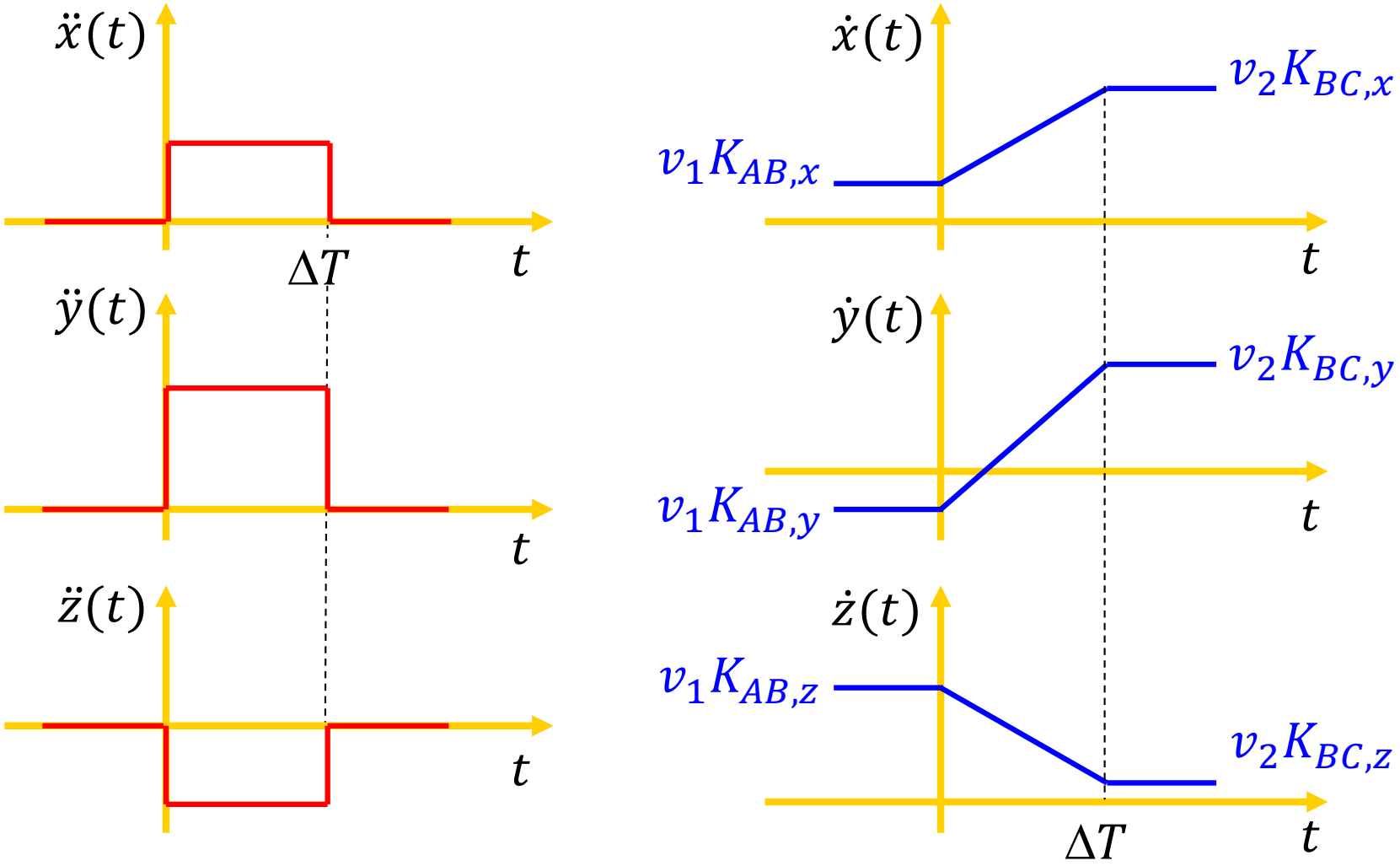
unit vectors of directional cosines

given: constant speeds v_1 on linear segment AB
v_2 on linear segment BC
desired transition: blending with constant acceleration for a time ΔT

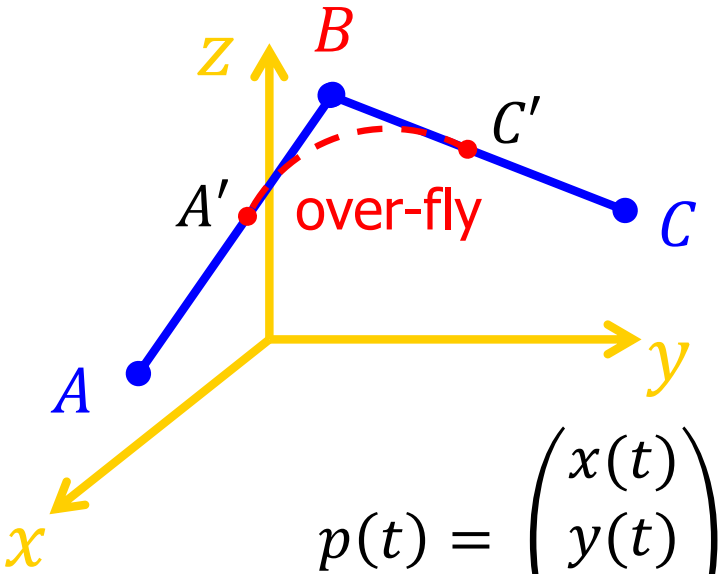
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad t \in [0, \Delta T] \quad \text{(transition starts at } t = 0)$$

note: during over-fly, the path remains always in the plane specified by the two lines intersecting at B (in essence, it is a planar problem)

Time profiles on components



Timing law during transition



$$\frac{B-A}{\|B-A\|} = K_{AB} \qquad \frac{C-B}{\|C-B\|} = K_{BC}$$

unit vectors of directional cosines

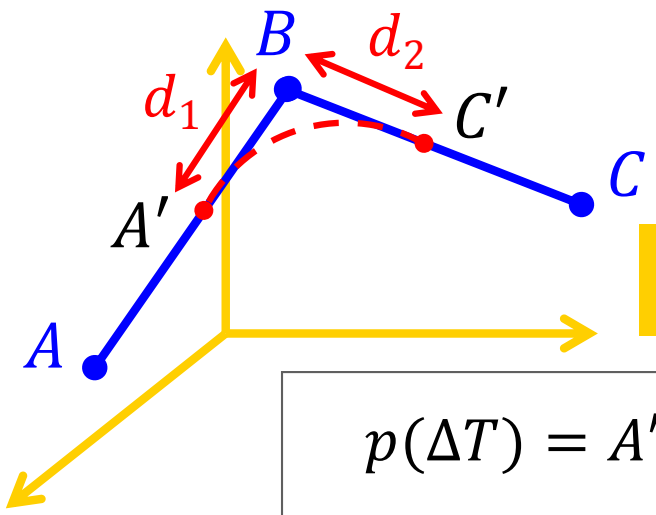
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad t \in [0, \Delta T] \quad \text{(transition starts at } t = 0\text{)}$$

$$\ddot{p}(t) = (v_2 K_{BC} - v_1 K_{AB})/\Delta T \xrightarrow{\int} \dot{p}(t) = v_1 K_{AB} + (v_2 K_{BC} - v_1 K_{AB})t/\Delta T$$

$$\dot{p}(t) \xrightarrow{\int} p(t) = A' + v_1 K_{AB} t + (v_2 K_{BC} - v_1 K_{AB})t^2/(2\Delta T)$$

thus, we obtain a parabolic blending (same approach as in joint space!)

Solution
(various options)



$$B - A' = d_1 K_{AB}$$
$$C' - B = d_2 K_{BC}$$

(1)

$$p(t) = A' + v_1 K_{AB} t + (v_2 K_{BC} - v_1 K_{AB}) t^2/(2\Delta T)$$

$$p(\Delta T) = A' + (\Delta T/2)(v_1 KAB + v_2 K_{BC}) = C'$$

$$\Rightarrow \; -B + A' + (\Delta T/2)\,(v_1 K_{AB} + v_2 K_{BC}) = C' - B$$

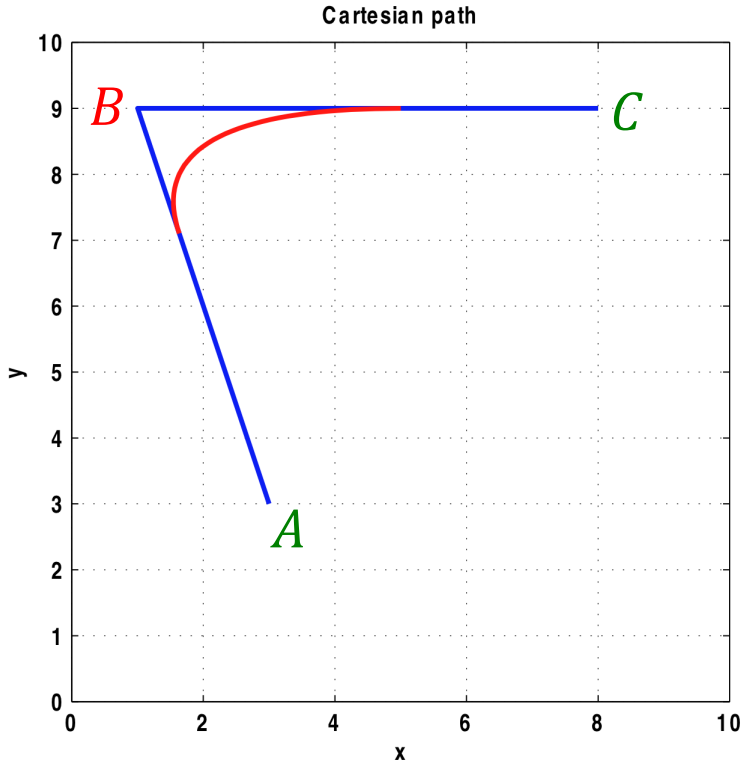
$$(1) \Rightarrow \; d_1 K_{AB} + d_2 K_{BC} = (\Delta T/2)(v_1 K_{AB} + v_2 K_{BC})$$

$$\Rightarrow \; d_1 = v_1 \Delta T/2 \qquad d_2 = v_2 \Delta T/2$$

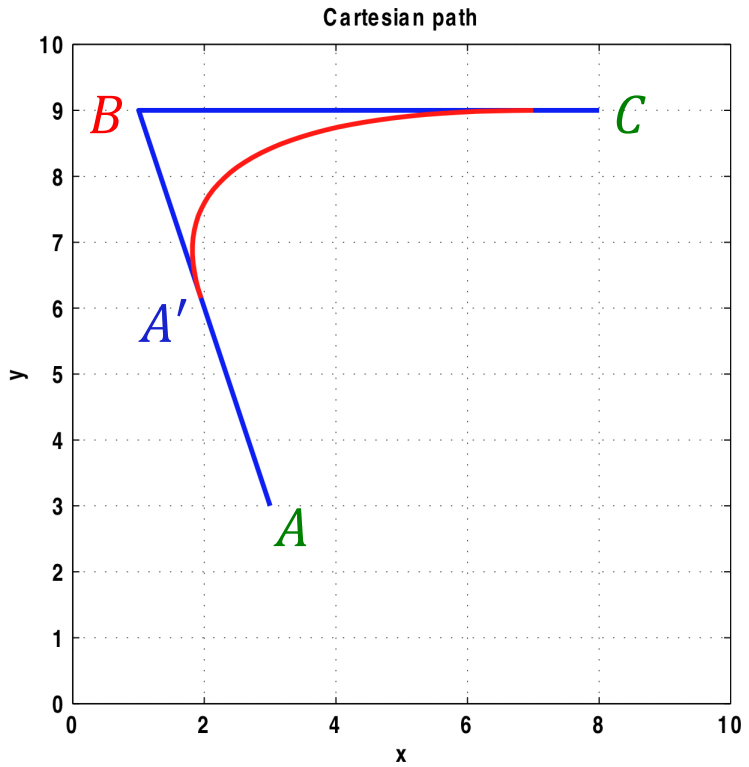
by choosing, e.g., d_1 (namely A') $\rightarrow$ $\Delta T = 2d_1/v_1 \Rightarrow d_2 = d_1 v_2/v_1$

A numerical example

- transition: $A = (3,3)$ to $C = (8,9)$ via $B = (1,9)$, with speed from $v_1 = 1$ to $v_2 = 2$
- exploiting two options for solution (resulting in different paths!)
  - assign transition time: $\Delta T = 4$
  - assign distance from B for departing: $d_1 = 3$ (assigning d_2 for landing handled similarly)



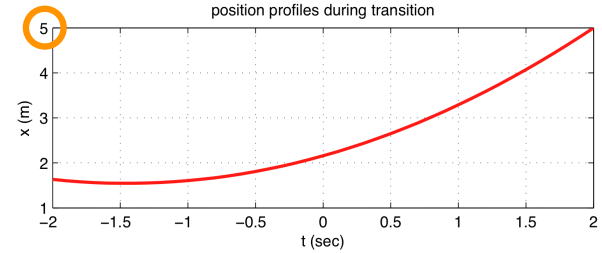
$\Delta T = 4$

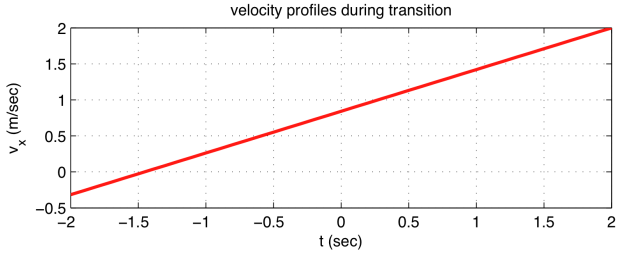


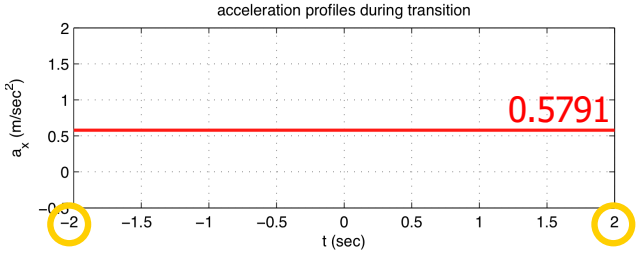
$d_1 = 3$

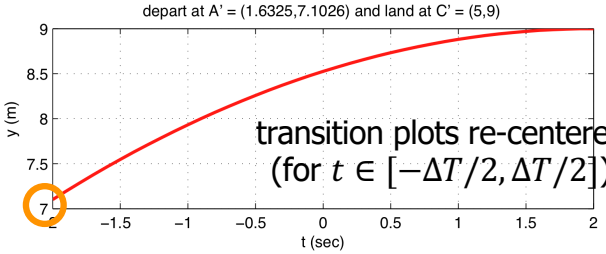
A numerical example (cont'd)

first option: $\Delta T = 4$ (resulting in $d_1 = 2, d_2 = 4$) $\Rightarrow \|\ddot{p}\| = 0.39$ m/s²

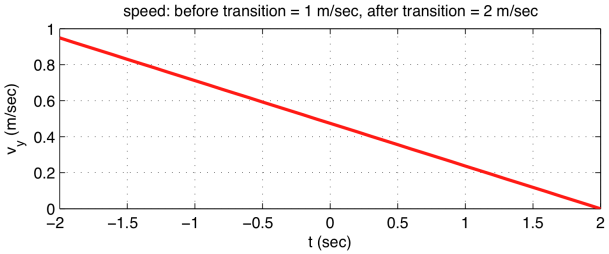


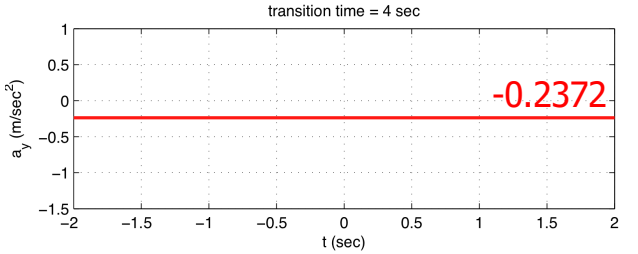




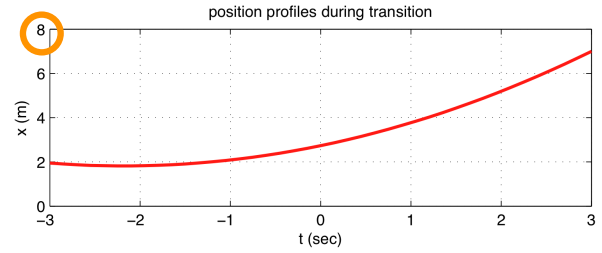


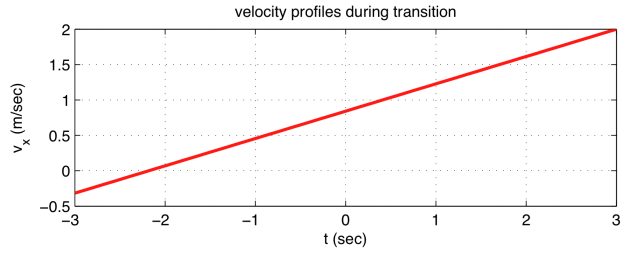
transition plots re-centered
(for $t \in [-\Delta T/2, \Delta T/2]$)

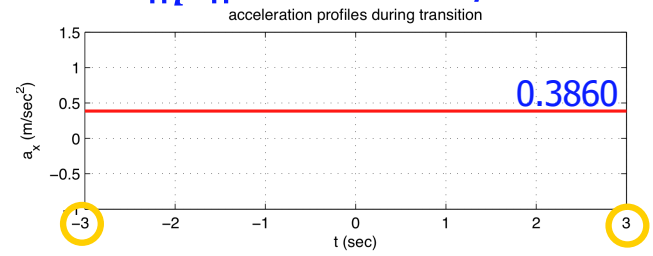


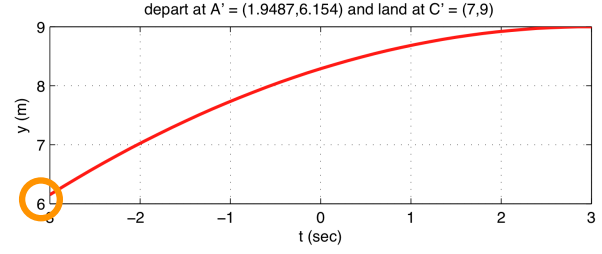


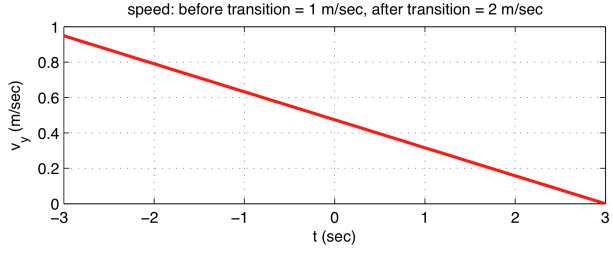
second option: $d_1 = 3$ (resulting in $\Delta T = 6, d_2 = 6$) $\Rightarrow \|\ddot{p}\| = 0.17$ m/s²

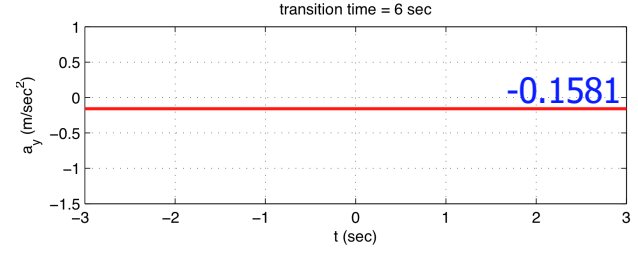






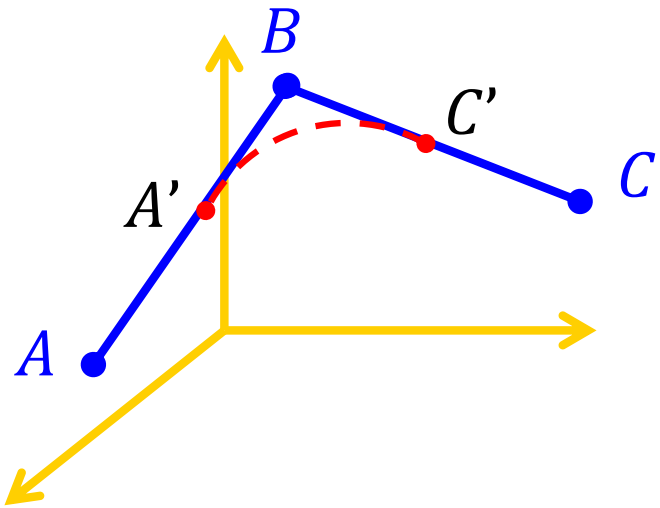






actually: similar velocity/acceleration profiles, but with a different time scale!!

Alternative solution
(imposing acceleration)



$$\ddot{p}(t) = (v_2 K_{BC} - v_1 K_{AB})/\Delta T$$

$v_1 = v_2 = v_{\max}$ (for simplicity)

$\|\ddot{p}(t)\| = a_{\max}$

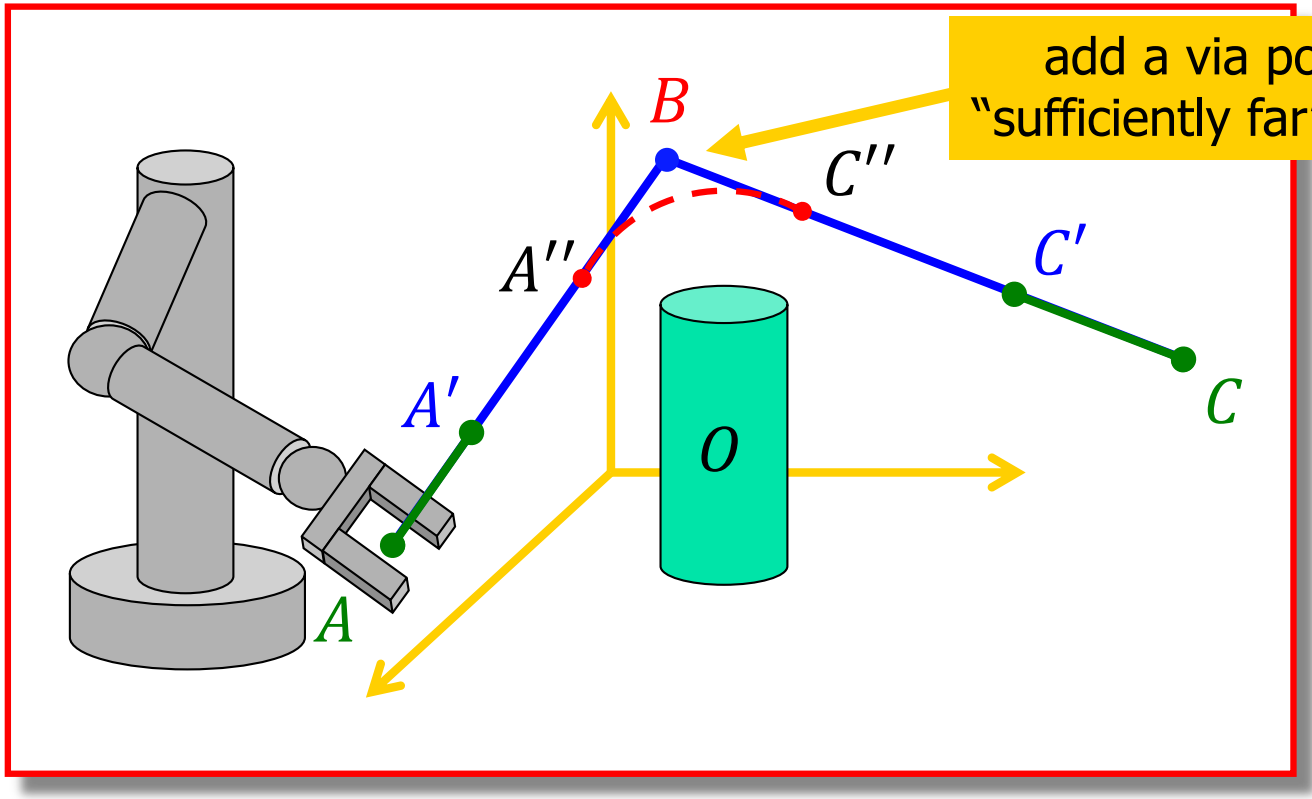
$$\Delta T = (v_{\max}/a_{\max}) \, \|K_{BC} - K_{AB}\|$$

$$= (v_{\max}/a_{\max}) \sqrt{2(1 - K_{BC,x}K_{AB,x} - K_{BC,y}K_{AB,y} - K_{BC,z}K_{AB,z})}$$

then, $d_1 = d_2 = v_{\max} \Delta T/2$

Application example

plan a Cartesian trajectory from A to C (rest-to-rest)
that avoids the obstacle O, with $a \leq a_{\max}$ and $v \leq v_{\max}$



on $\overline{AA'} \rightarrow a_{\max}$; on $\overline{A'B}$ and $\overline{BC'} \rightarrow v_{\max}$; on $\overline{C'C} \rightarrow -a_{\max}$;
+ over-fly between A'' e C'' (e.g., with $a_{\max}$ in norm)

Other Cartesian paths

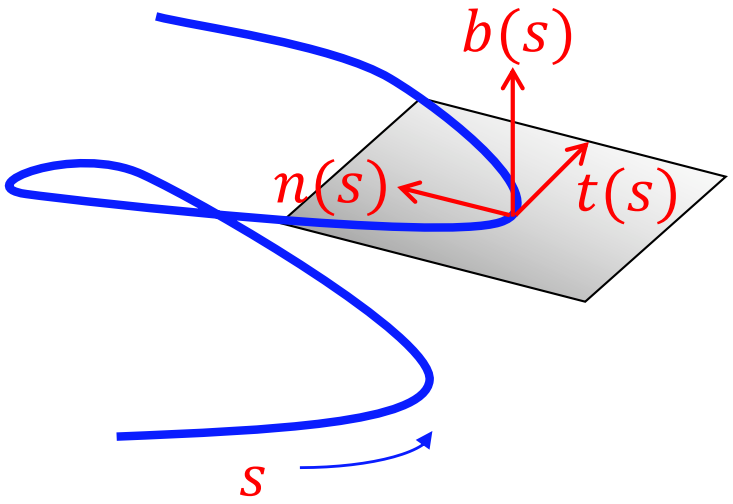
- circular path through 3 points in 3D (built-in feature) see textbook Ex. 4.3
- linear path for the end-effector with constant orientation (available in KRL)
- in robots with spherical wrist: planning may be decomposed into a path for wrist center and one for E-E orientation, with a common timing law
- though more complex, it may be convenient to parameterize the Cartesian geometric path $p(s)$ in terms of its arc length (e.g., $\sigma = R\theta$ for circular paths), similarly to what seen in joint space for $q(s)$; the following holds:
  - velocity $\dot{p} = dp/dt = (dp/d\sigma)(d\sigma/dt) = p'\dot{\sigma}$
    - p' = unit vector ($\|\cdot\| = 1$) tangent to the path ⇒ tangent direction $t(\sigma)$
    - $\dot{\sigma} \geq 0$ is the absolute value of the tangential velocity (= speed)
  - acceleration $\ddot{p} = (dp/d\sigma)(d^2\sigma/dt^2) + (d^2p/d\sigma^2)(d\sigma/dt)^2 = p'\ddot{\sigma} + p''\dot{\sigma}^2$
    - $\|p''\|$ = curvature $\kappa(\sigma)$ (= 1/radius of curvature)
    - $p''\dot{\sigma}^2$ = centripetal acceleration ⇒ normal direction $n(\sigma)$ $\perp$ to the path, on the osculating plane; the binormal direction is $b(\sigma) = t(\sigma) \times n(\sigma)$
    - $\ddot{\sigma}$ = scalar value (with any sign) of the tangential acceleration

Definition of Frenet frame

- for a smooth and non-degenerate curve $p(s) \in \mathbb{R}^3$, parameterized by s (not necessarily its arc length), one can define a reference frame as shown

$$p' = dp/ds \quad p'' = d^2p/ds^2$$

derivatives w.r.t. the parameter s



unit tangent vector

$$t(s) = p'(s)/\|p'(s)\|$$

unit normal vector ($\in$ osculating plane)

$$n(s) = t'(s)/\|t'(s)\|$$
$$= p'(s) \times (p''(s) \times p'(s))/(\|p'(s)\| \cdot \|p''(s) \times p'(s)\|)$$

unit binormal vector

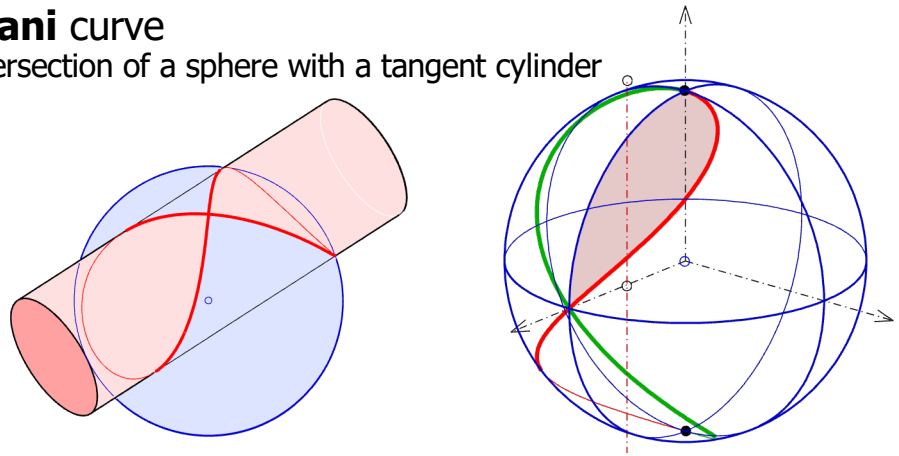
$$b(s) = t(s) \times n(s)$$
$$= p'(s) \times p''(s)/\|p'(s) \times p''(s)\|$$

- general expressions of path curvature and torsion (at a path point $p(s)$)

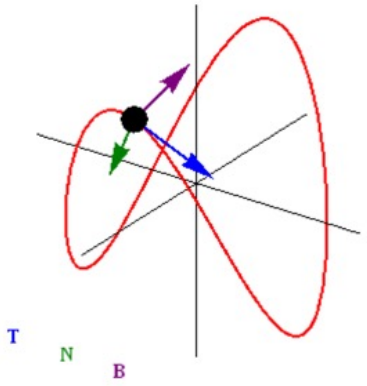
$$\kappa(s) = \|p'(s) \times p''(s)\|/\|p'(s)\|^3$$
$$\tau(s) = [p'(s) \cdot (p''(s) \times p'''(s))]/\|p'(s) \times p''(s))\|^2$$

Examples of paths with Frenet frame

Viviani curve
= intersection of a sphere with a tangent cylinder



$$x = r\cos^2 s$$
$$y = r\cos s \sin s$$
$$z = r\sin s$$
$$s \in [-\pi/2, \pi/2]$$
$$x = r\cos^2 s$$
$$y = -r\cos s \sin s$$
$$z = -r\sin s$$



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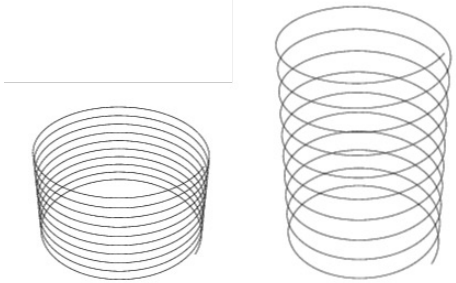
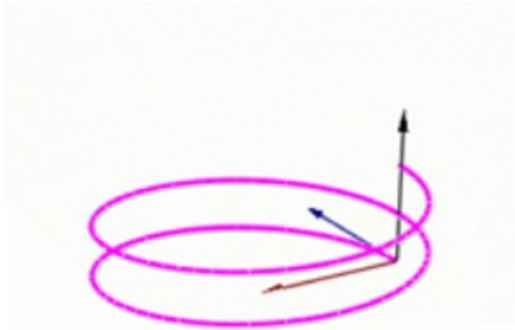
Helix curve (right handed)



$$x = r\cos s$$
$$y = r\sin s$$
$$z = h\,s$$
$$s \in [0, 2\pi]$$

$$\kappa = \frac{r}{r^2 + h^2}$$
$$\tau = \frac{h}{r^2 + h^2}$$



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Exercise

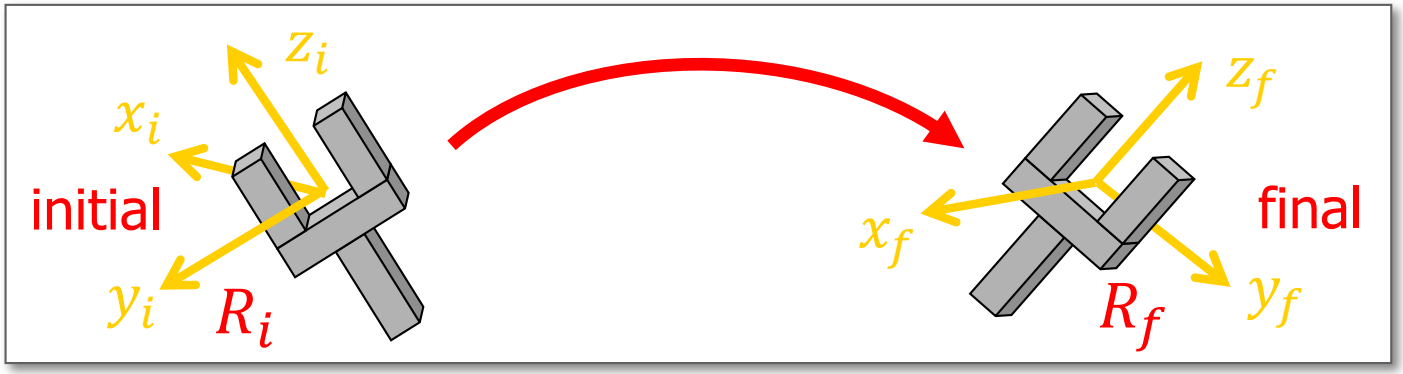
given the path $p(s) = \begin{pmatrix} 6s + 2 \\ 5s^2 \\ -8s \end{pmatrix}, \quad s \in [0,1]$

a) define the Frenet frame $\{t(s), n(s), b(s)\}$
b) compute the curvature $\kappa(s)$ and the torsion $\tau(s)$

Optimal trajectories

- for Cartesian robots (e.g., PPP joints)
  1. the straight line joining two position points in the Cartesian space is one path that can be executed in minimum time under velocity/acceleration constraints (but other such paths exist, if (joint) motion is not coordinated)
  2. the optimal timing law is of the bang-coast-bang type in acceleration (in this special case, also in terms of motor torques)
- for articulated robots (with at least one R joint)
  - 1. e 2. are no longer true in general in the Cartesian space, but time-optimality still holds in the joint space when assuming bounds on joint velocity/acceleration
    - straight line paths in the joint space do not correspond to straight line paths in the Cartesian space, and vice-versa
  - bounds on joint acceleration are conservative (though kinematically tractable) w.r.t. actual bounds on motor torques, which involve the full robot dynamics
    - when changing robot configuration/state, different torque values are needed to impose the same joint accelerations …

Planning orientation trajectories



- using minimal representations of orientation (α, β, γ), e.g., ZXZ Euler angles, we can plan a trajectory for each component independently
  - e.g., a linear path in space $\alpha(s), \beta(s), \gamma(s)$, with a cubic timing law $s(t)$
    $\Rightarrow$ but poor prediction/understanding of the resulting intermediate orientations
- alternative method based on the axis/angle representation
  - determine (unit) axis ${}^i r$ and angle θ_{if}: $Rot({}^i r, \theta_{if}) = {}^i R_f = R_i^T R_f$ (relative rotation between initial and final orientation) $\Rightarrow$ inverse axis-angle problem
  - plan a timing law $\theta(t)$ for the (scalar) angle interpolating $\theta = 0$ with $\theta = \theta_{if}$ in time T (with possible constraints/boundary conditions on its time derivatives)
  - $\forall t \in [0, T], R_i Rot({}^i r, \theta(t))$ specifies the actual end-effector orientation at time t

Orientation trajectory bounds
from limits in Cartesian space

- for an orientation path $\phi(s)$ using a minimal representation (e.g., Euler angles), consider a limit on the angular velocity norm

$$\|\omega(t)\| \leq \omega_{\max}$$

since $\omega(t) = T\big(\phi(t)\big)\dot{\phi}(t) = T\big(\phi(s)\big)\phi'(s)\dot{s}(t)$

$$\Rightarrow \quad |\dot{s}(t)| \leq \frac{\omega_{\max}}{\max\limits_{s\in[0,1]} \big\|T\big(\phi(s)\big)\phi'(s)\big\|}$$

- conservative handling of limit $\|\dot{\omega}(t)\| \leq \dot{\omega}_{\max}$ as a bound on $|\ddot{s}(t)|$
- with the axis/angle representation, it is easy to transfer limits on angular velocity/acceleration norms to bounds on timing law $\theta(t)$

$$\begin{matrix} \|\omega(t)\| \leq \omega_{\max} \\ \|\dot{\omega}(t)\| \leq \dot{\omega}_{\max} \end{matrix} \Rightarrow \begin{matrix} \omega(t) = \dot{\theta}(t)\, r = \theta_{if}\dot{s}(t)\, r \\ \dot{\omega}(t) = \ddot{\theta}(t)\, r = \theta_{if}\ddot{s}(t)\, r \end{matrix} \Rightarrow \begin{matrix} |\dot{s}(t)| \leq \omega_{\max}/\theta_{if} \\ |\ddot{s}(t)| \leq \dot{\omega}_{\max}/\theta_{if} \end{matrix}$$

$$(\|r\| = \|R_i\, {}^i r\| = \|{}^i r\|)$$

A complete position/orientation Cartesian trajectory (simulation)

position $p_i = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow p_f = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ $L = \|p_f - p_i\| = 5.385$ m

orientation $R_i = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \longrightarrow R_f = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \end{pmatrix}$

$\updownarrow$ ${}^iR_f = R_i^T R_f = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 1/2 & -1/2 & -\sqrt{2}/2 \\ -1/2 & 1/2 & -\sqrt{2}/2 \end{pmatrix}$

using ZYZ Euler angles [rad]

absolute $\phi_i = \begin{pmatrix} -\pi/2 \\ \pi/4 \\ -\pi/2 \end{pmatrix} \longrightarrow \phi_f = \begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi/4 \end{pmatrix}$ relative $\phi_{if} = \begin{pmatrix} -\pi/2 \\ 3\pi/4 \\ \pi/4 \end{pmatrix}$

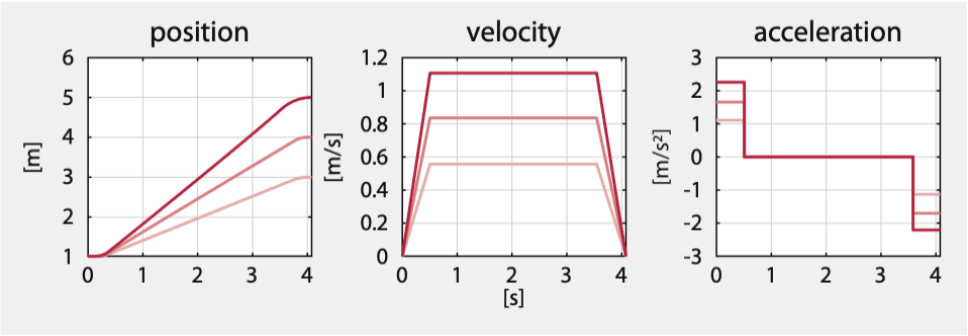
using axis/angle representation (from ${}^iR_f$) ${}^ir = \begin{pmatrix} 0.912 \\ 0.378 \\ -0.156 \end{pmatrix}$ $\theta_{if} = 2.419$ rad

linear parametrized paths

$$p(s) = p_i + s(p_f - p_i) \qquad \begin{cases} \phi(s) = \phi_i + s(\phi_f - \phi_i) \\ {}^i\phi(s) = s\phi_{if} \\ \theta(s) = s\theta_{if} \end{cases} \qquad s \in [0,1]$$

three alternatives for orientation

A complete position/orientation Cartesian trajectory (simulation)



position trajectory

task limits

$\|\dot{p}\| \le v_{\max} = 1.5$ m/s $\quad \|\ddot{p}\| \le a_{\max} = 3$ m/s²

$\Longrightarrow \quad T_{r,\text{pos}} = 0.5$ s $\qquad T_{\text{pos}} = 4.09$ s

$\|\omega\| \le \omega_{\max} = \pi/4$ rad/s

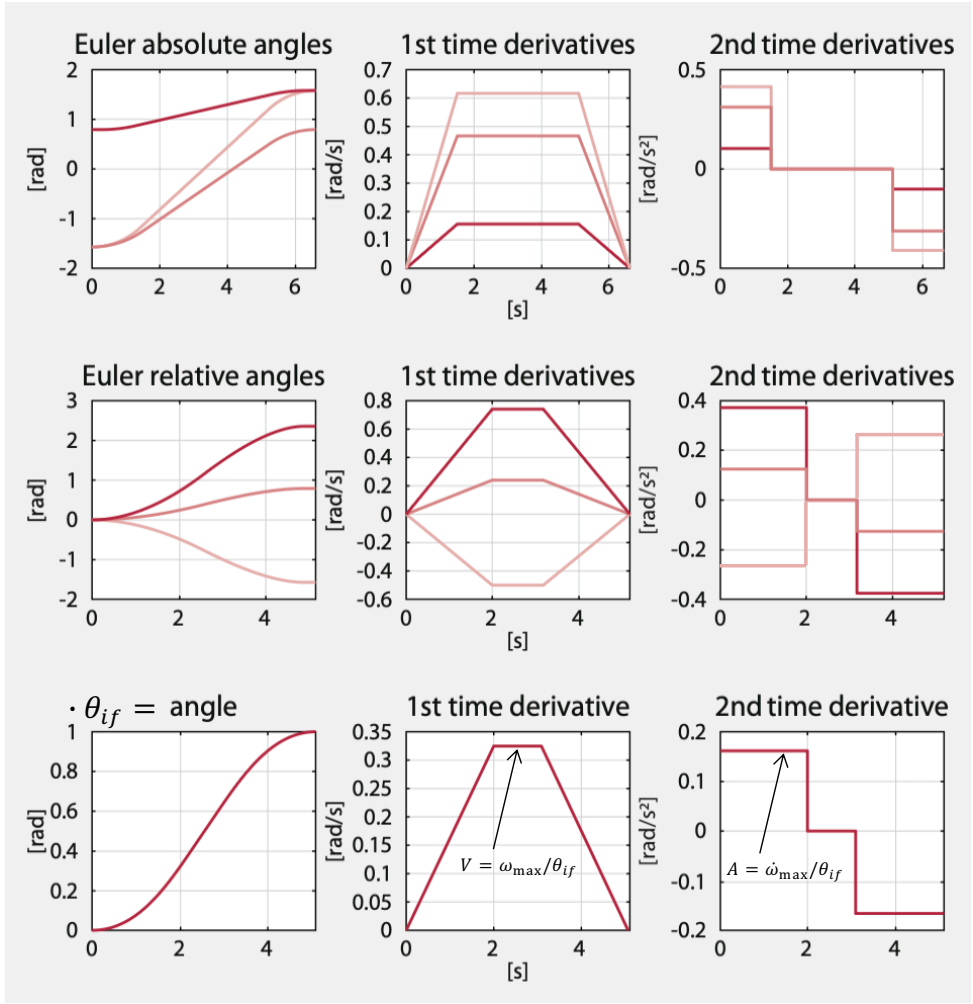
$\|\dot{\omega}\| \le \dot{\omega}_{\max} = \pi/8$ rad/s²

via explicit formulas, exact or approximated (see also Ex. 4.4 in textbook)

$\Longrightarrow \quad T_{r,\text{Eabs}} = 1.50$ s $\qquad T_{\text{Eabs}} = 6.60$ s

$\Longrightarrow \quad T_{r,\text{Erel}} = 2$ s $\qquad T_{\text{Erel}} = 5.16$ s

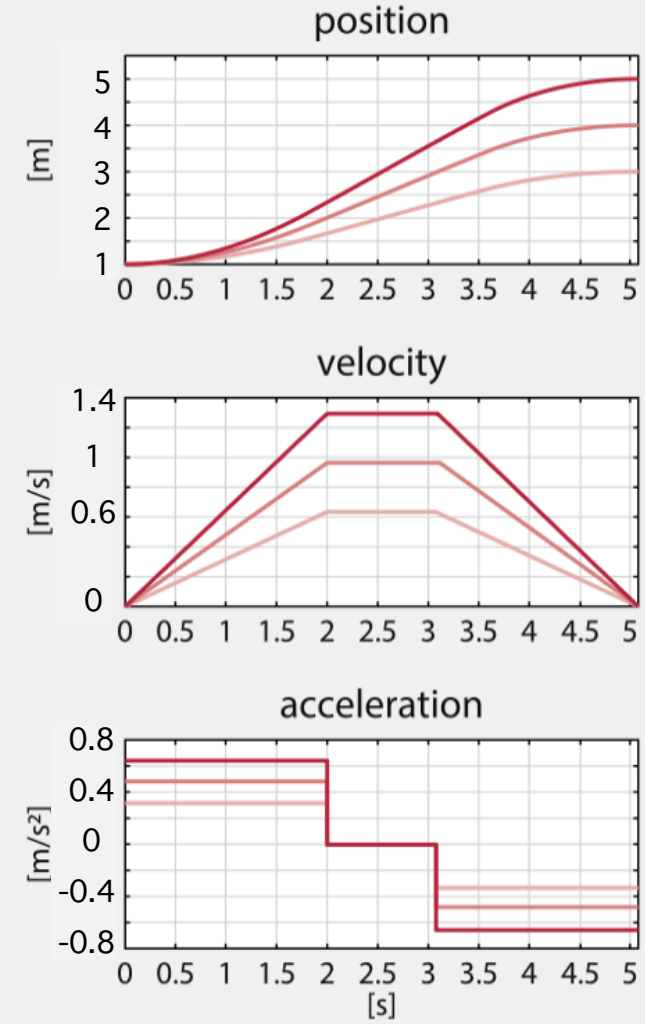
$\Longrightarrow \quad T_{r,\text{a\&a}} = 2$ s $\qquad T_{\text{a\&a}} = 5.08$ s

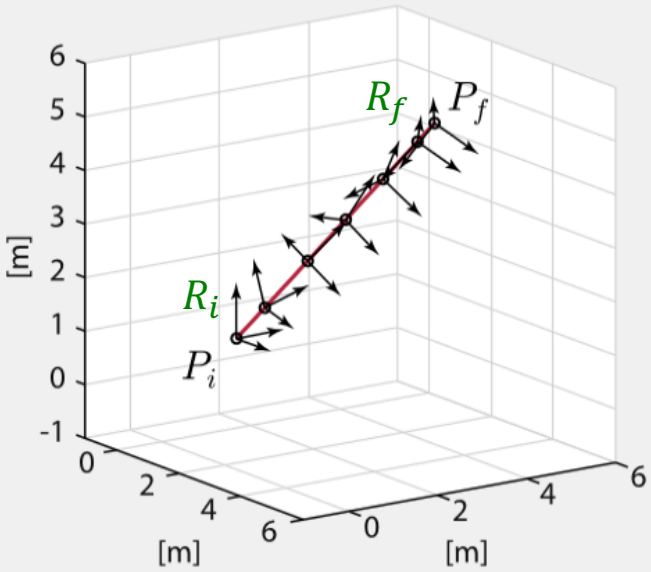


three orientation trajectories

bounds on bang-coast-bang timing law should be derived
– first treating separately position and each of the three possible orientation paths

A complete position/orientation Cartesian trajectory (simulation)





… and where is the robot?

scaled position trajectory in time

best time of orientation methods

$$T_{\text{ang}} = \min\{T_{\text{Erel}}, T_{\text{Eabs}}, T_{\text{a\&a}}\} = T_{\text{a\&a}} = 5.08 \text{ s}$$

minimum feasible time

$$T = \max\{T_{\text{pos}}, T_{\text{ang}}\} = 5.08 \text{ s}$$

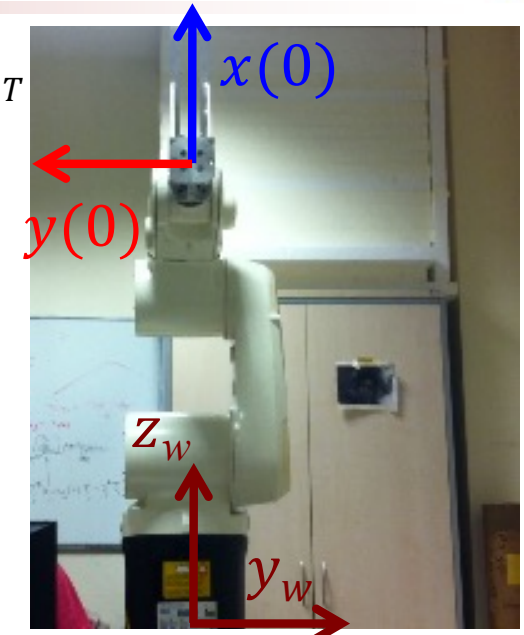
coordinated Cartesian motion by a special form of time scaling

the position trajectory is slowed down
a) keeping the same switching times of the axis/angle acceleration trajectory
b) tuning max velocity/acceleration levels to obtain the same displacements

invariant axis of rotation (in absolute coordinates)

$$r = R_i {}^i r = -(0.912, 0.157, 0.378)$$

A complete position/orientation Cartesian trajectory (experimental)

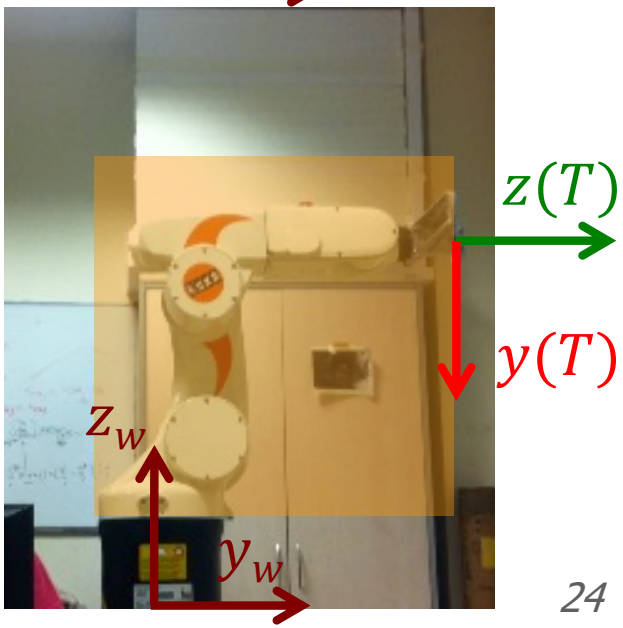


- initial given configuration $q(0) = (0 \quad \pi/2 \quad 0 \quad 0 \quad 0 \quad 0)^T$
- initial end-effector position $p(0) = (0.540 \quad 0 \quad 1.515)^T$
- initial orientation

$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

linear path for position

axis-angle method for orientation



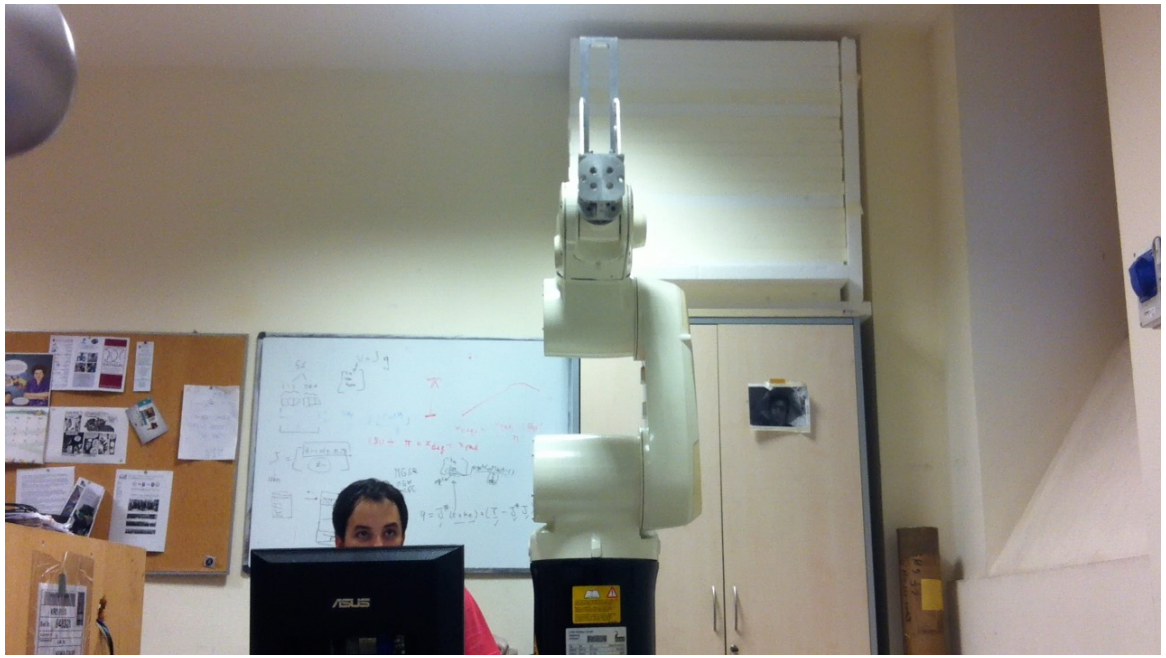
- final end-effector position $p(T) = (0 \quad 0.540 \quad 1.515)^T$
- final orientation

$$R(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- the final configuration is NOT specified a priori

Axis-angle orientation trajectory

video



$$L = \|p_f - p_i\| = 0.763 \text{ m}$$

$$\omega = {}^i r\dot{\theta} \rightarrow \|\omega\| = |\dot{\theta}|$$

$$\dot{\omega} = {}^i r\ddot{\theta} \rightarrow \|\dot{\omega}\| = |\ddot{\theta}|$$

coordinated
Cartesian motion
with bounds

$v_{\max} = 0.4$ m/s
$a_{\max} = 0.1$ m/s^2
$\omega_{\max} = 2\pi$ rad/s
$\dot{\omega}_{\max} = 4\pi$ rad/s^2

triangular
speed profile $\dot{s}(t)$
with minimum
time $T = 5.52$ s
(imposed by the bounds
on linear motion)

$$p(s) = p_i + s(p_f - p_i) = (0.540 \quad 0 \quad 1.515)^T + s(-0.540 \quad 0.540 \quad 0)^T, \qquad s \in [0,1]$$

$$R_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = R_i^T \qquad R_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R_i^T R_f = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = Rot({}^i r, \theta_{if})$$

$${}^i r = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \theta_{if} = \frac{2\pi}{3} \text{ [rad]} (= 120°)$$

$$R(s) = R_i Rot({}^i r, \theta(s))$$

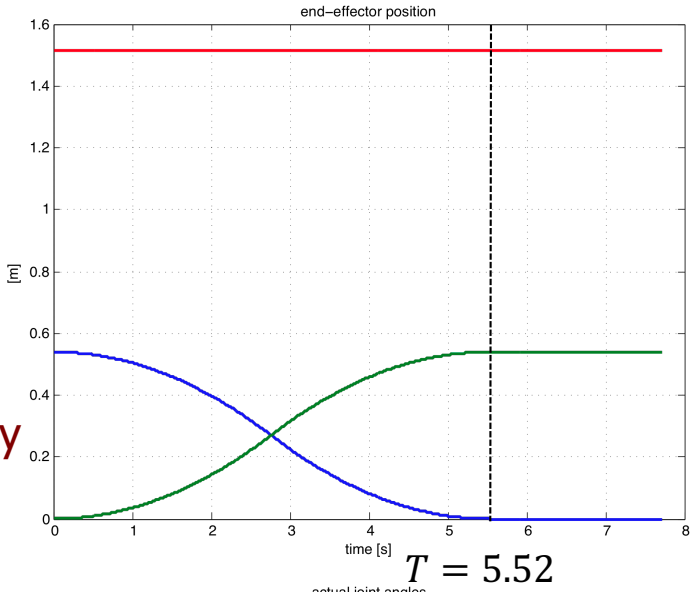
$$\theta(s) = s\theta_{if}, \qquad s \in [0,1]$$

$$\boldsymbol{s = s(t),\ t \in [0, T]}$$

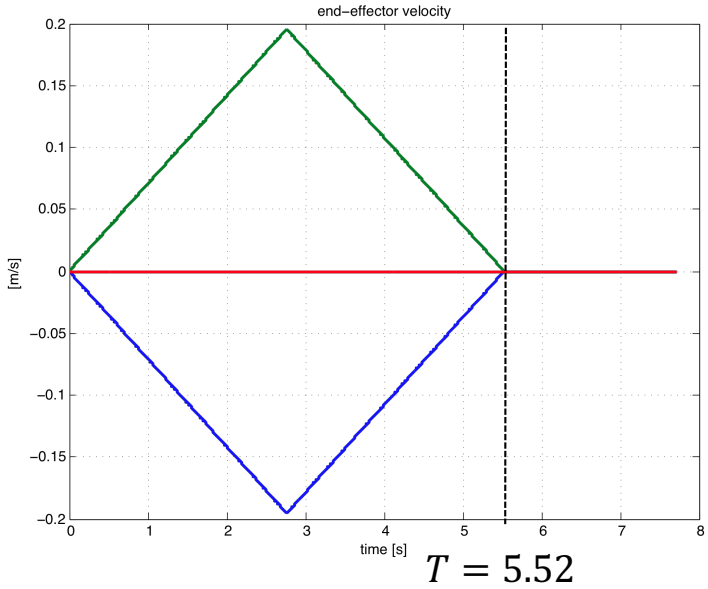
Axis-angle orientation trajectory

x =
y =
z =

planned motion of Cartesian position and velocity



$T = 5.52$

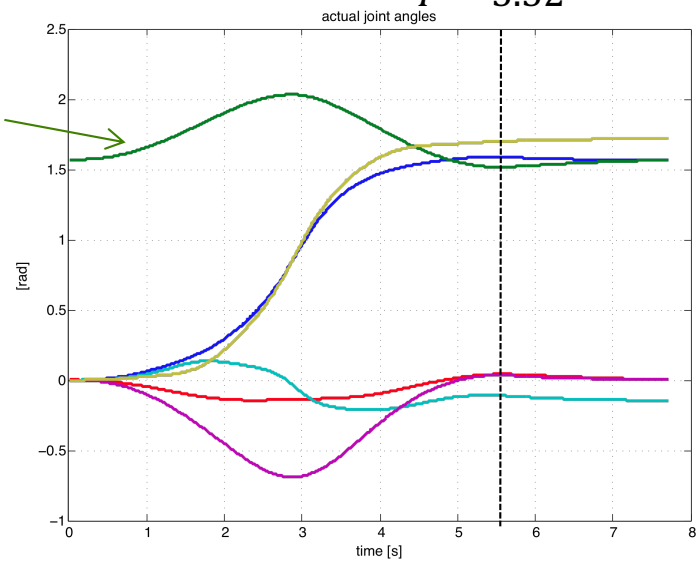


$T = 5.52$

triangular profile for linear speed $T = 5.52$ s

joint 2

actual joint motion



$T = 5.52$

- the robot joint velocity was commanded by inversion of the **geometric** Jacobian
- a **user** program, via KUKA RSI interface at $T_c = 12$ ms sampling time (two-way communication)
- robot motion execution is ≈ what was planned, but only thanks to an external **kinematic control** loop (at **task** level)

Comparison of orientation trajectories
Euler angles vs. axis-angle method (experimental)

- initial configuration $q(0) = (0 \quad \pi/2 \quad \pi/2 \quad 0 \quad -\pi/2 \quad 0)^T$
- initial end-effector position $p(0) = (0.151 \quad 0 \quad 1.720)^T$
- initial orientation

$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

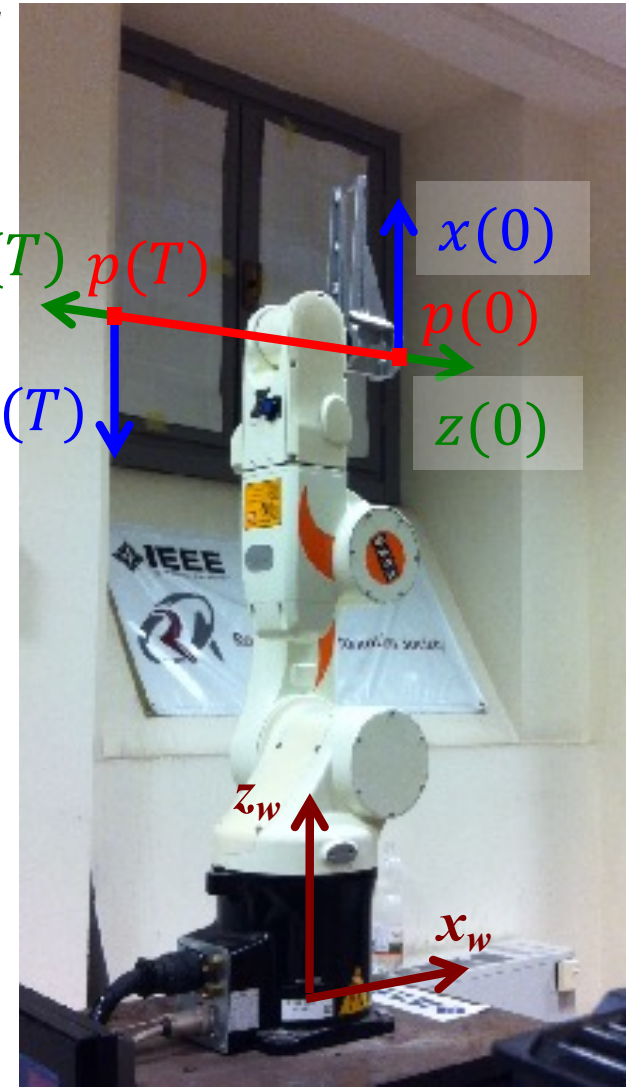
- initial Euler ZYZ (α, β, γ) angles $\phi_{ZYZ}(0) = (0 \quad \pi/2 \quad \pi)^T$

via a **linear path** (for position)

- final end-effector position $p(T) = (-0.172 \quad 0 \quad 1.720)^T$
- final orientation

$$R(T) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- final Euler ZYZ angles $\phi_{ZYZ}(T) = (-\pi \quad \pi/2 \quad 0)^T$



Comparison of orientation trajectories
Euler angles vs. axis-angle method (experimental)

$$R_i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \phi_{ZYZ,i} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi \end{pmatrix}$$

$$R_f = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \phi_{ZYZ,f} = \begin{pmatrix} -\pi \\ \pi/2 \\ 0 \end{pmatrix}$$

(singularity at $\beta = 0$ avoided!)

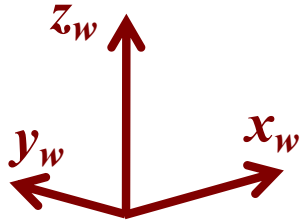
video

using ZYZ Euler angles

using axis-angle method

$$R_i^T R_f = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Longrightarrow r = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \theta = \pi$$

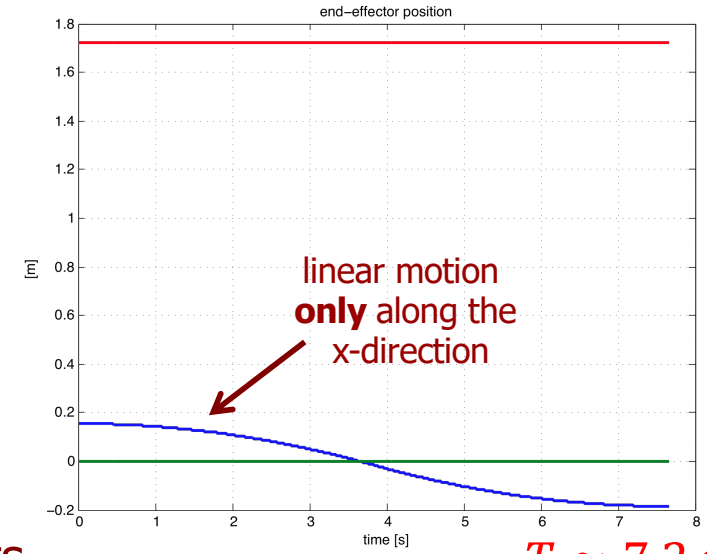


video

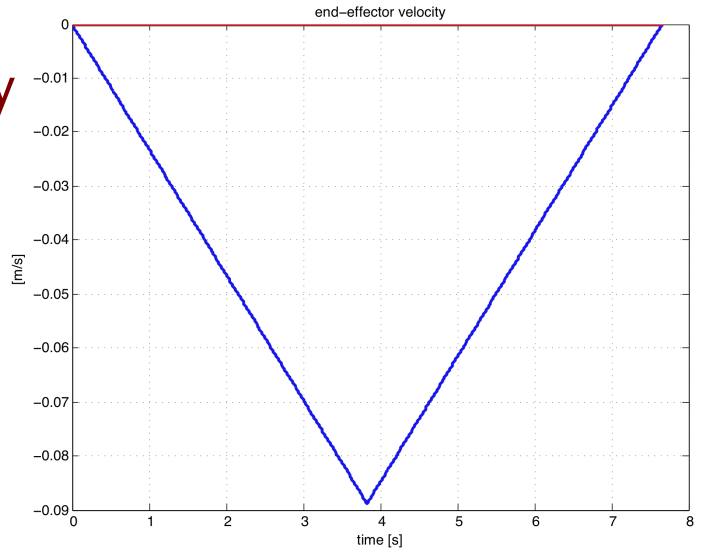
Comparison of orientation trajectories
Euler angles vs. axis-angle method

x = —
y = —
z = —

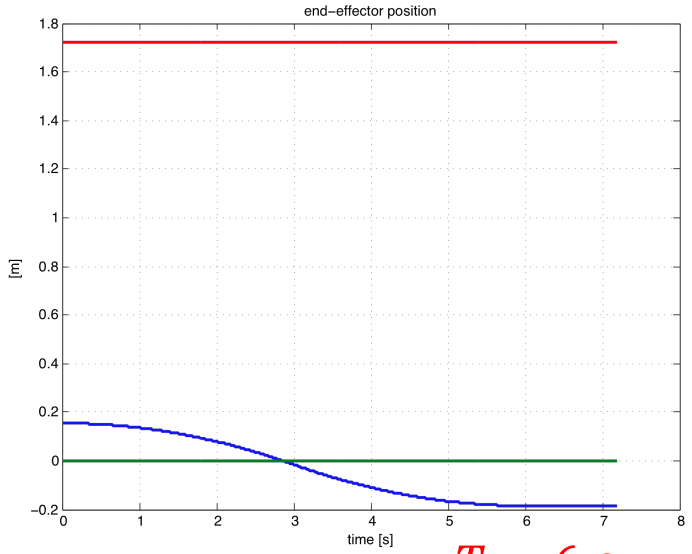
planned Cartesian components of position and velocity



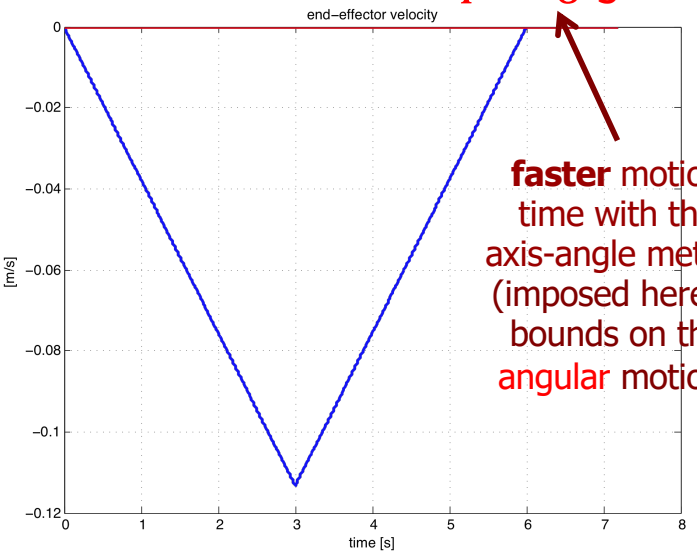
$T \approx 7.2$ s



using ZYZ Euler angles



$T \approx 6$ s



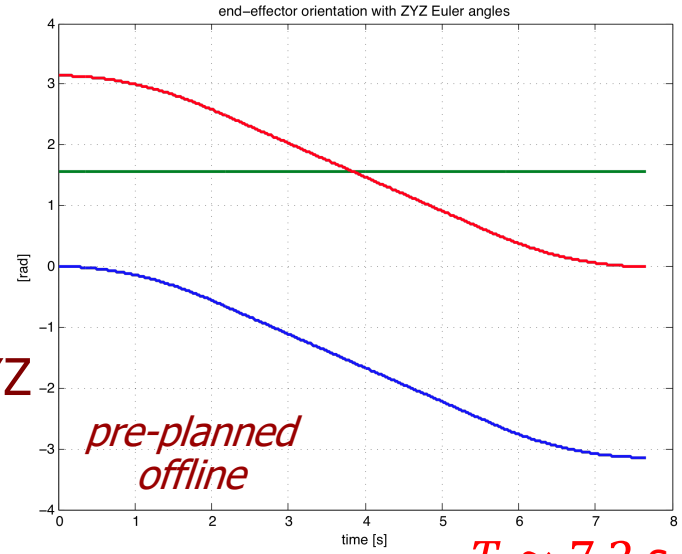
using axis-angle method

Comparison of orientation trajectories

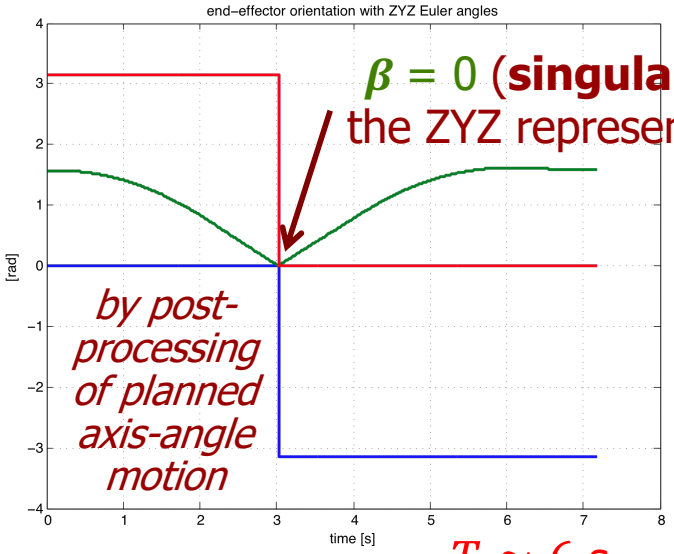
Euler angles vs. axis-angle method (experimental)

α = ——
β = ——
γ = ——

orientation in terms of ZYZ Euler angles

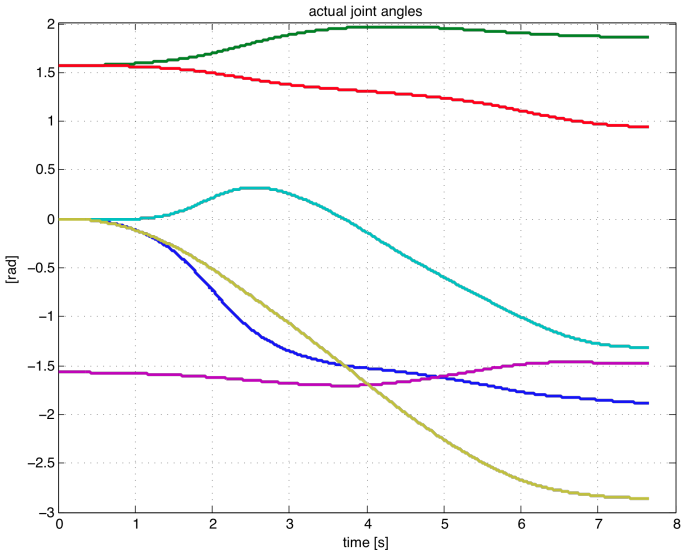


$T \approx 7.2$ s

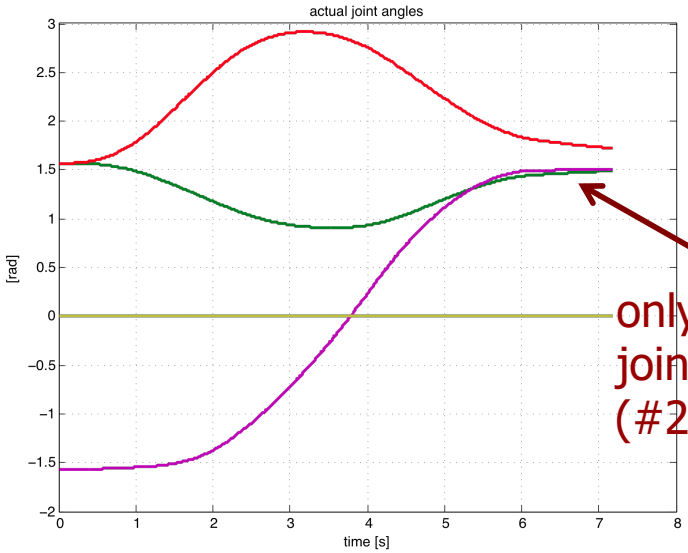


$T \approx 6$ s

actual joint motion



using ZYZ Euler angles



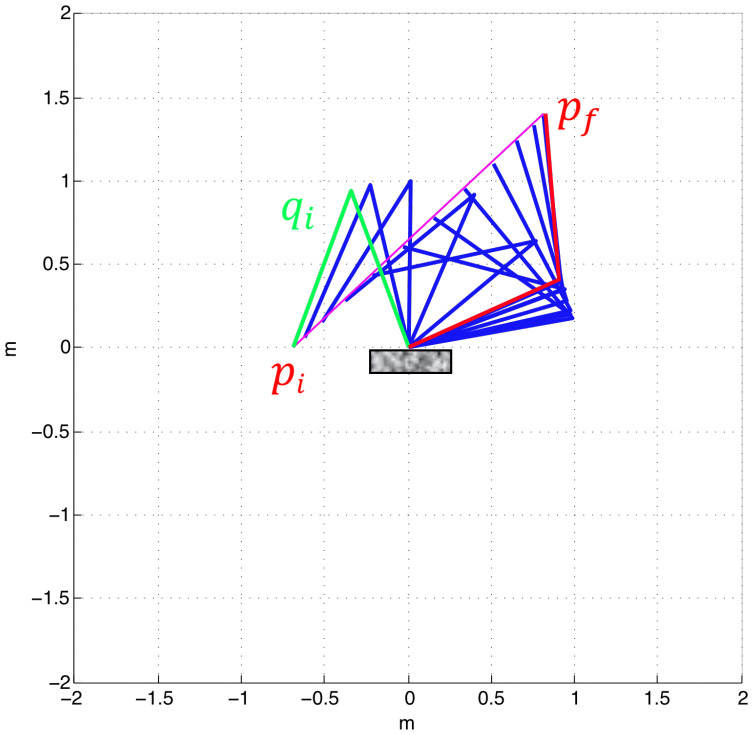
using axis-angle method

Uniform time scaling

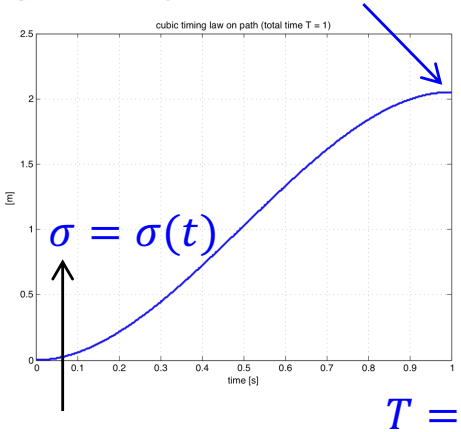
- for a given path $p(s)$ (in joint or Cartesian space) and timing law $s(\tau)$ ($\tau = t/T$, T="motion time"), we need to check if existing bounds $v_{\max}$ on (joint) velocity and/or $a_{\max}$ on (joint) acceleration are violated or not
  - … unless such constraints have already been taken into account during the trajectory planning, e.g., by using a bang-coast-bang acceleration timing law
- velocity scales **linearly** with motion time
  - $\dot{p} = dp/dt = (dp/ds)\,(ds/d\tau) \cdot 1/T$
- acceleration scales **quadratically** with motion time
  - $\ddot{p} = d^2p/dt^2 = \left((d^2p/ds^2)(ds/d\tau)^2 + (dp/ds)(d^2s/d\tau^2)\right) \cdot 1/T^2$
- if motion is unfeasible, increase time $T \rightarrow T_s = kT$ $(k > 1)$, based on the "most violated" constraint (max of the ratios $|v|/v_{\max}$ and $|a|/a_{\max}$)
- if motion is "too slow" for the robot capabilities, decrease $T \rightarrow T_s$ $(k < 1)$
  - in both cases, after scaling, there will be (at least) one instant of saturation (for at least one variable)
  - no need to re-compute motion profiles from scratch!

Numerical example - 1

- 2R planar robot with links of unitary length (1 [m])
- linear Cartesian path $p(s)$: $q_i = (110°, 140°) \Rightarrow p_i = f(q_i) = (-0.684, 0)$ $\Rightarrow p_f = (0.816, 1.4)$ [m], with rest-to-rest cubic timing law $\sigma(t)$, $T = 1$ s
- joint space bounds: max (absolute) velocity $v_{\max,1} = 2, v_{\max,2} = 2.5$ [rad/s], max (absolute) acceleration $a_{\max,1} = 5, a_{\max,2} = 7$ [rad/s^2]



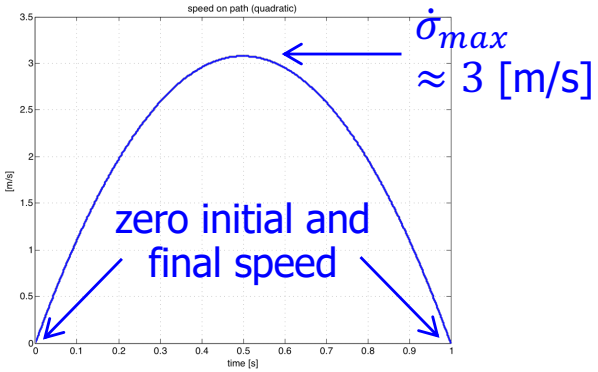
path length $L = 2.0518$ [m]

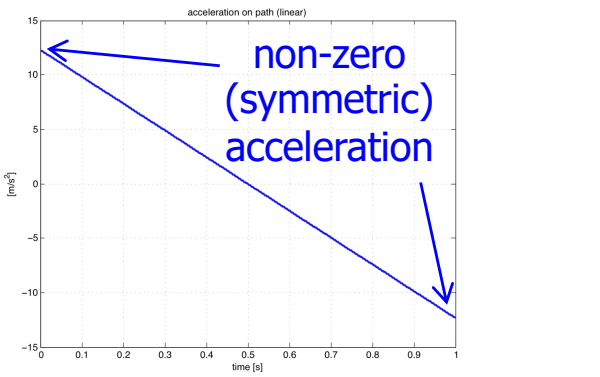


$T = 1$

arc length

$$p(\sigma) = p_i + \frac{p_f - p_i}{L}\,\sigma, \quad \sigma \in [0, L]$$

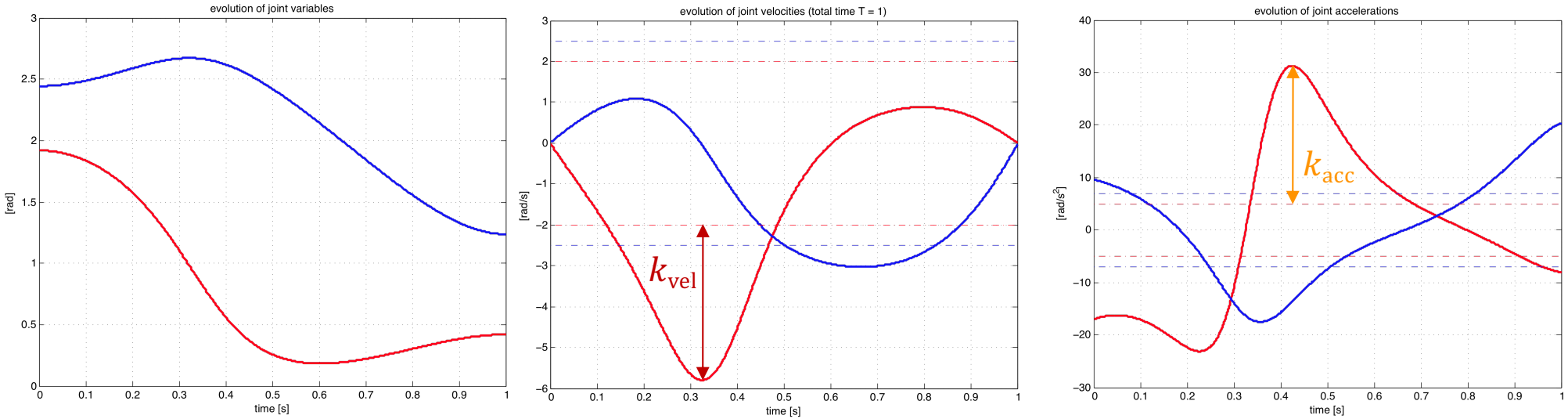




Numerical example - 2

- violation of both joint velocity and acceleration bounds with $T = 1$ s
  - max relative violation of joint velocities: $k_{\text{vel}} = 2.898 = \max\left\{1, |\dot{q}_1|/v_{\max,1}, |\dot{q}_2|/v_{\max,2}\right\}$
  - …. and of joint accelerations: $k_{\text{acc}} = 6.2567 = \max\left\{1, |\ddot{q}_1|/a_{\max,1}, |\ddot{q}_2|/a_{\max,2}\right\}$
- minimum uniform time scaling of Cartesian trajectory to recover feasibility

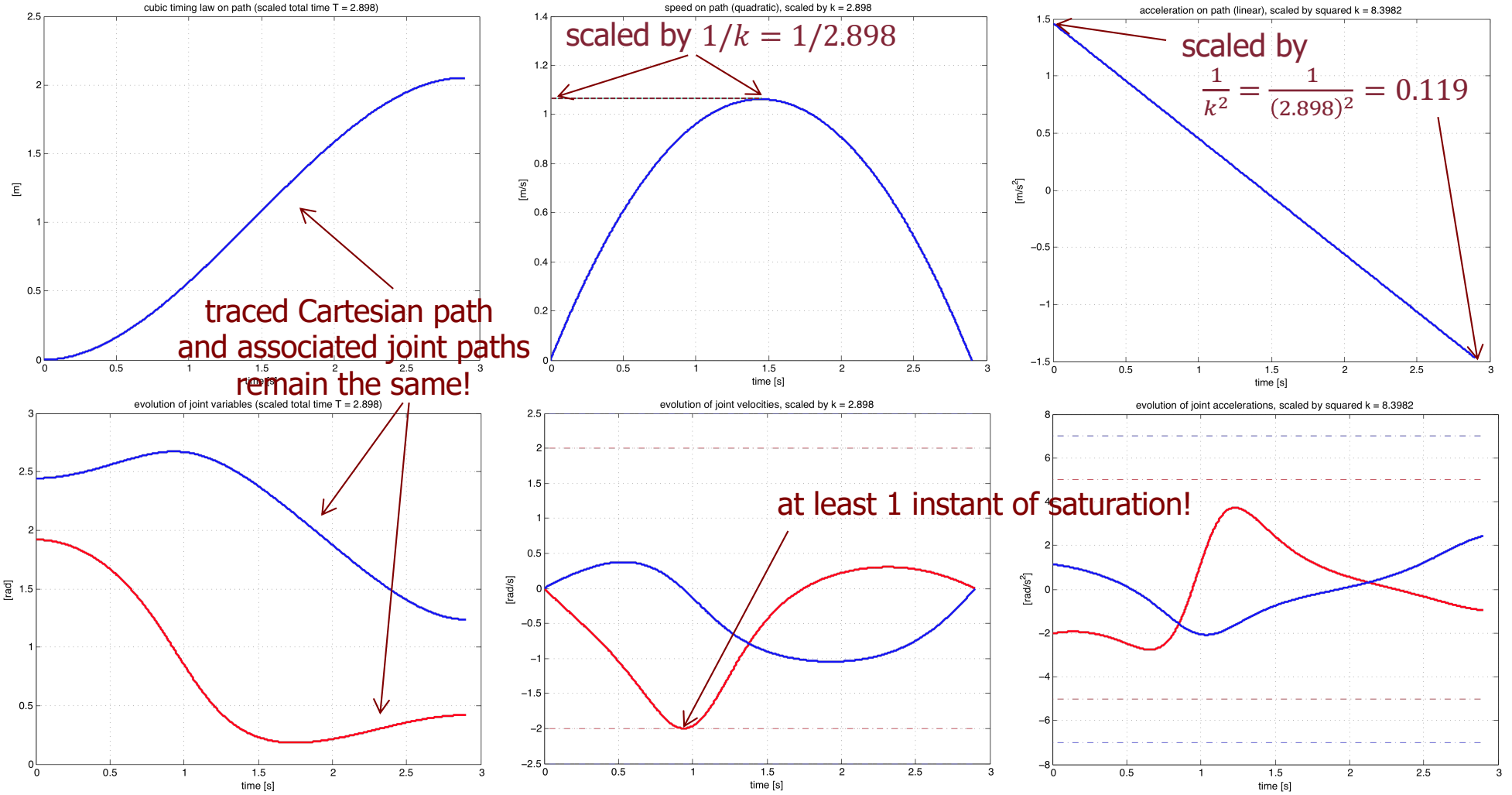
$$k = \max\left\{1, k_{\text{vel}}, \sqrt{k_{\text{acc}}}\right\} = \max\{1, 2.898, 2.5023\} = 2.898 \;\Rightarrow\; T_s = kT = 2.898 > T$$



= joint 1 = joint 2

Numerical example - 3

- scaled trajectory with $T_s = 2.898$ s
  - speed [acceleration] on path and joint velocities [accelerations] scale linearly [quadratically]



= joint 1 = joint 2