Inverse differential kinematics
Statics and force transformations

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Inversion of differential kinematics

- find the joint velocity vector that realizes a \textit{desired} task/end-effector velocity ("generalized" = linear and/or angular)

\[ \dot{v} = J(q) \dot{q} \]

\[ \dot{q} = J^{-1}(q)v \]

- problems
  - near a singularity of the Jacobian matrix (too high \( \dot{q} \))
  - for \textit{redundant} robots (no standard "inverse" of a rectangular matrix)

in these cases, more \textit{robust} inversion methods are needed
Incremental solution to inverse kinematics problems

- joint velocity inversion can be used also to solve on-line and incrementally a “sequence” of inverse kinematics problems
- each problem differs by a small amount $dr$ from previous one

$r = f_r(q)$
direct kinematics

$dr = \frac{\partial f_r(q)}{\partial q} dq = J_r(q) dq$
differential kinematics
(here with a square, analytic Jacobian)

$r \rightarrow r + dr$

$r + dr = f_r(q)$
first, increment the desired task variables

$q = f_r^{-1}(r + dr)$
then, solve the inverse kinematics problem

$dq = J_r^{-1}(q) dr$
first, solve the inverse differential kinematics problem

$q \rightarrow q + dq$
then, increment the original joint variables
Behavior near a singularity

\[ \dot{q} = J^{-1}(q)v \]

- Problems arise only when commanding joint motion by inversion of a given Cartesian motion task.
- Here, a linear Cartesian trajectory for a planar 2R robot.
- There is a sudden increase of the displacement/velocity of the first joint near \( \theta_2 = -\pi \) (end-effector close to the origin), despite the required Cartesian displacement is small.
Simulation results

planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q)v \]

regular case

a line from right to left, at \( \alpha = 170° \) angle with \( x \)-axis, executed at constant speed \( v = 0.6 \) m/s for \( T = 6 \) s
Simulation results
planar 2R robot in straight line Cartesian motion

path at \( \alpha = 170^\circ \)

regular case

distance to singularity by the minimum singular value \( \sigma_{min} (= \sigma_2) > 0 \) of Jacobian \( J \)

error due only to numerical integration \( (10^{-10}) \)
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \nu \]

close to singular case

a line from right to left, at \( \alpha = 178^\circ \) angle with \( x \)-axis,
executed at constant speed \( \nu = 0.6 \) m/s for \( T = 6 \) s
Simulation results
planar 2R robot in straight line Cartesian motion

path at \( \alpha = 178^\circ \)

close to singular case

large peak of \( \dot{q}_1 \)

still very small, but increased numerical integration error (\( 2 \cdot 10^{-9} \))
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q)v \]

close to singular case
with joint velocity saturation at \( V_i = 300^\circ/s \)

a line from right to left, at \( \alpha = 178^\circ \) angle with \( x \)-axis,
executed at constant speed \( v = 0.6 \text{ m/s} \) for \( T = 6 \text{ s} \)
Simulation results
planar 2R robot in straight line Cartesian motion

path at \( \alpha = 178^\circ \)

close to singular case

saturated value of \( \dot{q}_1 \)

actual position error!! (6 cm)
Damped Least Squares method

\[ \min_{\dot{q}} H = \frac{\lambda}{2} \| \dot{q} \|^2 + \frac{1}{2} \| J \dot{q} - v \|^2, \quad \lambda \geq 0 \]

\[ \dot{q} = (\lambda I_n + J^T J)^{-1} J^T v = J^T (\lambda I_m + J J^T)^{-1} v = J_{DLS} v \]

two equivalent expressions, but the second is more convenient in redundant robots!

- inversion of differential kinematics as unconstrained optimization problem
- function \( H = \text{weighted} \) sum of two objectives (norm of joint velocity and error norm on achieved end-effector velocity) to be minimized
- \( J_{DLS} \) can be used for both cases: \( m = n \) (square) and \( m < n \) (redundant)
- \( \lambda = 0 \) when “far enough” from singularities: \( J_{DLS} = J^T (J J^T)^{-1} = J^{-1} \) or \( J^\# \)
- with \( \lambda > 0 \), there is a (vector) error \( \epsilon = v - J \dot{q} \) in executing the desired end-effector velocity \( v \) (check that \( \epsilon = \lambda (\lambda I_m + J J^T)^{-1} v \)), but the joint velocities are always reduced ("damped")
Simulation results
planar 2R robot in straight line Cartesian motion

a comparison of inverse and damped inverse Jacobian methods
even closer to singular case

\[
\dot{q} = J^{-1}(q)v
\]

\[
\dot{q} = J_{DLS}(q)v
\]

a line from right to left, at \( \alpha = 179.5^\circ \) angle with \( x \)-axis,
executed at constant speed \( v = 0.6 \) m/s for \( T = 6 \) s
Simulation results

planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q)v \]

path at \[ \alpha = 179.5^\circ \]

\[ \dot{q} = J_{DLS}(q)v \]

here, a very fast reconfiguration of first joint ...

a completely different inverse solution, around/after crossing the region close to the folded singularity

stroboscopic views
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \nu \]

\[ \dot{q} = J_{DLS}(q) \nu \]

extremely large peak velocity of first joint!!

smoother joint motion with limited joint velocities!
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \nu \]

increased numerical integration error (3 \times 10^{-8})

\[ \dot{q} = J_{DLS}(q) \nu \]

error (25 mm) when crossing the singularity, later recovered by a feedback action 
\( \nu \Rightarrow \nu + K_p e_p \)
with \( e_p = p_d - p(q) \))

minimum singular value of \( JJ^T \) and \( \lambda I + JJ^T \)

they differ only when damping factor is non-zero

damping factor \( \lambda \) is chosen non-zero only close to singularity!
Pseudoinverse method

A constrained optimization (minimum norm) problem

\[
\min_{\dot{q}} H = \frac{1}{2} \| \dot{q} \|^2 \quad \text{such that} \quad J \dot{q} = v
\]

\[
\iff
\min_{\dot{q} \in S} H = \frac{1}{2} \| \dot{q} \|^2
\]

\[
S = \left\{ \dot{q} \in \mathbb{R}^n : \|J\dot{q} - v\| \text{ is minimum} \right\}
\]

Solution

\[
\dot{q} = J^\# v
\]

Pseudoinverse of \( J \)

- If \( v \in \mathcal{R}(J) \), the differential constraint is satisfied (\( v \) is feasible)
- Else, \( J\dot{q} = JJ^\# v = v^\perp \), where \( v^\perp \) minimizes the error \( \|J\dot{q} - v\| \)

Orthogonal projection of \( v \) on \( \mathcal{R}(J) \)
Definition of the pseudoinverse

given $J$, is the **unique** matrix $J^\dagger$ satisfying the **four** relationships

$$JJ^\dagger J = J \quad J^\dagger JJ^\dagger = J^\dagger$$

$$(JJ^\dagger)^T = JJ^\dagger \quad (J^\dagger J)^T = J^\dagger J$$

- explicit expressions for **full rank** cases
  - if $\rho(J) = m = n$: $J^\dagger = J^{-1}$
  - if $\rho(J) = m < n$: $J^\dagger = J^T (JJ^T)^{-1}$
  - if $\rho(J) = n < m$: $J^\dagger = (J^T J)^{-1} J^T$

- $J^\dagger$ **always** exists and is computed in general numerically using the SVD = Singular Value Decomposition of $J$
  - e.g., with the MATLAB function **pinv** (which uses in turn **svd**)

Robotics 1
Numerical example

Jacobian of 2R robot with $l_1 = l_2 = 1$ at $q_2 = 0$ (rank $\rho(J) = 1$)

$$J = \begin{pmatrix} -2s_1 & -s_1 \\ 2c_1 & c_1 \end{pmatrix}$$
$$J^\# = \frac{1}{5} \begin{pmatrix} -2s_1 & 2c_1 \\ -s_1 & c_1 \end{pmatrix}$$

$$JJ^\# = \begin{pmatrix} s_1^2 & -s_1c_1 \\ -s_1c_1 & c_1^2 \end{pmatrix}$$

both symmetric ...

$$\dot{q} = J^\# v$$ is the **minimum** norm joint velocity vector that realizes exactly $v^\perp$

- at $q_1 = \pi/6$: for $v = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}$ [m/s], $\dot{q} = J^\# v = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}$ [rad/s] \(\Rightarrow v^\perp = JJ^\# v = \begin{pmatrix} -1/8 \\ \sqrt{3}/8 \end{pmatrix}\) [m/s]

- at $q_1 = \pi/2$: $J = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$ \(\Rightarrow J^\# = \begin{pmatrix} -0.4 & 0 \\ -0.2 & 0 \end{pmatrix}\); now the same $v \in R(J)$, $\dot{q} = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}$ \(\Rightarrow v^\perp = v\) (no error!)
General solution for $m < n$

ALL solutions of the inverse differential kinematics problem can be written as

$$\dot{q} = J^\# v + (I - J^\# J) \xi$$

this is the solution of a slightly modified constrained optimization problem

(“biased” toward the joint velocity $\xi$, chosen to avoid obstacles, joint limits, etc.)

$$\min_{\dot{q}} H = \frac{1}{2} \|\dot{q} - \xi\|^2 \text{ such that } J\dot{q} = v \quad \iff \quad \min_{\dot{q} \in S} H = \frac{1}{2} \|\dot{q} - \xi\|^2$$

$$S = \left\{ \dot{q} \in \mathbb{R}^n : \|J\dot{q} - v\| \text{ is minimum} \right\}$$

verification of the actual task velocity that is being obtained

$$v_{\text{actual}} = J\dot{q} = J(J^\# v + (I - J^\# J)\xi) = J J^\# v + J(I - J^\# J)\xi = J J^\# (Jw) = Jw = v$$

if $v \in \mathcal{R}(J) \Rightarrow v = Jw$ for some $w \in \mathbb{R}^n$
Geometric interpretation for $m < n$

a simple case with $n = 2$, $m = 1$
at a given configuration

$$J\dot{q} = [j_1 \ j_2] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = v \in \mathbb{R}$$

$J\dot{q}$ space of joint velocities (at a configuration $q$)

associated homogeneous equation

$J\dot{q} = 0$

task equality constraint

$J\dot{q} = v$

minimum norm solution $J^\# v$

solution with minimum $\|\dot{q} - \xi\|^2$

“biasing” joint velocity (in general, not a solution)

space of joint velocities (at a configuration $q$)

orthogonal projection of $\xi$ on $\mathcal{N}(J)$

linear subspace $\mathcal{N}(J) = \{\dot{q} \in \mathbb{R}^2 : J\dot{q} = 0\}$
Higher-order differential inversion

- inversion of motion from task to joint space can be performed also at a higher differential level
- acceleration-level: given $q, \dot{q}$
  \[ \ddot{q} = J_r^{-1}(q)(\ddot{r} - \dot{J}_r(q)\dot{q}) \]
- jerk-level: given $q, \dot{q}, \ddot{q}$
  \[ \dddot{q} = J_r^{-1}(q)(\dddot{r} - \dot{J}_r(q)\ddot{q} - 2\ddot{J}_r(q)\dot{q}) \]
- (pseudo-)inverse of the Jacobian is always the leading term
- smoother joint motions are expected (at least, due to the existence of higher-order time derivatives $\dddot{r}, \dddot{r}, ...$)
Generalized forces and torques

- \( \tau \) = forces/torques exerted by the motors at the robot joints
- \( F \) = equivalent forces/torques exerted by the robot end-effector
- \( F_e \) = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: \( F_e = -F \)

reaction from environment is equal and opposite to the robot action on it
Transformation of forces – Statics

- what is the transformation between $F$ at robot end-effector and $\tau$ at joints?
- in static equilibrium conditions (i.e., no motion):
  - what $F$ will be exerted on environment by a $\tau$ applied at the robot joints?
  - what $\tau$ at the joints will balance a $F_e (= -F)$ exerted by the environment?

in a given configuration

all equivalent formulations
Virtual displacements and works

infinitesimal \( dq \) (or “virtual” \( \delta q \), i.e., satisfying all possible constraints imposed on the system) displacements at an equilibrium

- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the virtual work is the work done by all forces/torques acting on the system for a given virtual displacement

\[
\left( \frac{dp}{\omega \ dt} \right) = J \ dq
\]
Principle of virtual work

The sum of the virtual works done by all forces/torques acting on the system = 0

$$\tau^T dq - F^T \begin{pmatrix} dp \\ \omega dt \end{pmatrix} = \tau^T dq - F^T J dq = 0 \quad \forall dq$$

$$\tau = J^T(q)F$$
Duality between velocity and force

velocity $\dot{q}$
(or displacement $dq$)
in the joint space

generalized velocity $v$
(or e-e displacement $(dp/\omega dt)$)
in the Cartesian space

forces/torques $\tau$
at the joints

generalized forces $F$
at the Cartesian e-e

the singular configurations for the velocity map are the same as those for the force map

$\rho(J) = \rho(J^T)$
Dual subspaces of velocity and force
summary of definitions

\[ \mathcal{R}(J) = \{ v \in \mathbb{R}^m : \exists \dot{q} \in \mathbb{R}^n, J \dot{q} = v \} \]

\[ \mathcal{N}(J^T) = \{ F \in \mathbb{R}^m : J^T F = 0 \} \]

\[ \mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^m \]

\[ \mathcal{R}(J^T) = \{ \tau \in \mathbb{R}^n : \exists F \in \mathbb{R}^m, J^T F = \tau \} \]

\[ \mathcal{N}(J) = \{ \dot{q} \in \mathbb{R}^n : J \dot{q} = 0 \} \]

\[ \mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^n \]
Velocity and force singularities  
list of possible cases

\[ \rho = \text{rank}(J) = \text{rank}(J^T) \leq \min(m, n) \]

1. \( \rho = m \)
   - \( \exists \dot{q} \neq 0 : J\dot{q} = 0 \)
   - \( \mathcal{N}(J^T) = \{0\} \)

2. \( \rho < m \)
   - \( \exists \dot{q} \neq 0 : J\dot{q} = 0 \)
   - \( \exists \dot{F} \neq 0 : J^T F = 0 \)

1. \( \det J \neq 0 \)
   - \( \mathcal{N}(J) = \{0\} \)
   - \( \mathcal{N}(J^T) = \{0\} \)

2. \( \det J = 0 \)
   - \( \exists \dot{q} \neq 0 : J\dot{q} = 0 \)
   - \( \exists \dot{F} \neq 0 : J^T F = 0 \)

1. \( \rho = n \)
   - \( \mathcal{N}(J) = \{0\} \)
   - \( \exists F \neq 0 : J^T F = 0 \)

2. \( \rho < n \)
   - \( \exists \dot{q} \neq 0 : J\dot{q} = 0 \)
   - \( \exists \dot{F} \neq 0 : J^T F = 0 \)
Singularity analysis

planar 2R arm with link lengths $l_1$ and $l_2$

\[ J(q) = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \quad \text{det } J(q) = l_1 l_2 s_2 \]

singularity at $q_2 = 0$ (arm straight) \[ J = \begin{pmatrix} -(l_1 + l_2) s_1 & -l_2 s_1 \\ (l_1 + l_2) c_1 & l_2 c_1 \end{pmatrix} \]

\[ \mathcal{R}(J) = \alpha \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix} \quad \mathcal{N}(J^T) = \alpha \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} \]

\[ \mathcal{R}(J^T) = \beta \begin{pmatrix} l_1 + l_2 \\ l_2 \end{pmatrix} \quad \mathcal{N}(J) = \beta \begin{pmatrix} l_2 \\ -(l_1 + l_2) \end{pmatrix} \]

singularity at $q_2 = \pi$ (arm folded) \[ J = \begin{pmatrix} (l_2 - l_1) s_1 & l_2 s_1 \\ -(l_2 - l_1) c_1 & -l_2 c_1 \end{pmatrix} \]

\[ \mathcal{R}(J) \text{ and } \mathcal{N}(J^T) \text{ as above} \]

\[ \mathcal{R}(J^T) = \beta \begin{pmatrix} l_2 - l_1 \\ l_2 \end{pmatrix} \text{ (for } l_1 = l_2: \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \quad \mathcal{N}(J) = \beta \begin{pmatrix} l_2 \\ -(l_2 - l_1) \end{pmatrix} \text{ (for } l_1 = l_2: \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \]

Robotics 1
Velocity manipulability

- in a given configuration, evaluate how effective is the transformation between joint and end-effector velocities
  - “how easily” can the end-effector be moved in various directions of the task space
  - equivalently, “how far” is the robot from a singular condition
- we consider all end-effector velocities that can be obtained by choosing joint velocity vectors of unit norm

\[
\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad \nu^T J^\#^T J^\# \nu = 1
\]

if \( \rho(J) = m \)

\[
(J J^T)^{-1}
\]

note: the “core” matrix of the ellipsoid equation \( \nu^T A^{-1} \nu = 1 \) is the matrix \( A \)!
Manipulability ellipsoid
in velocity

planar 2R arm with unitary links

length of principal (semi-)axes
singular values $\sigma_i$ of $J$ (in its SVD)

$$\sigma_i(J) = \sqrt{\lambda_i(J^T J)}$$

in a singularity, the ellipsoid loses a dimension
(for $m = 2$, it becomes a segment)

direction of principal axes
eigenvectors associated to $\lambda_i$

$$w = \sqrt{\det(J^T J)} = \prod_{i=1}^{m} \sigma_i \geq 0$$

proportional to the volume of the ellipsoid (for $m = 2$, to its area)
Manipulability measure

planar 2R arm (with $l_1 = l_2 = 1$):

$$\sqrt{\det(J J^T)} = \sqrt{\det(J) \cdot \det(J^T)} = |\det J| = \prod_{i=1}^{2} \sigma_i$$

- max at $\theta_2 = \pi/2$
- max at $r = \sqrt{2}$

best posture for manipulation (similar to a human arm!)

no full isotropy here, since it is always $\sigma_1 \neq \sigma_2$
in a given configuration, evaluate how effective is the transformation between joint torques and end-effector forces

“how easily” can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space

in singular configurations, there are directions in the task space where external forces are balanced without the need of any joint torque

we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

\[ \tau^T \tau = 1 \]

\[ F^T J J^T F = 1 \]

same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths

task force manipulability ellipsoid
Velocity and force manipulability
dual comparison of actuation vs. control

planar 2R arm with unitary links

note: velocity and force ellipsoids have a different scale for a better view

Cartesian **actuation** task (joint-to-task high transformation ratio):
preferred velocity (or force) directions are those where the ellipsoid stretches

Cartesian **control** task (low transformation ratio = high resolution):
preferred velocity (or force) directions are those where the ellipsoid shrinks
Velocity and force transformations

- Same reasoning made for relating end-effector to joint forces/torques (virtual work principle + static equilibrium) used also transforming forces and torques applied at different places of a rigid body and/or expressed in different reference frames.

Transformation among generalized velocities:

\[
\begin{bmatrix}
A\nu_A \\
A\omega
\end{bmatrix} = \begin{bmatrix}
AR_B & -AR_B S(Br_{BA}) \\
0 & AR_B
\end{bmatrix} \begin{bmatrix}
B\nu_B \\
B\omega
\end{bmatrix} = J_{BA} \begin{bmatrix}
B\nu_B \\
B\omega
\end{bmatrix}
\]

Transformation among generalized forces:

\[
\begin{bmatrix}
Bf_B \\
Bm
\end{bmatrix} = J_{BA}^T \begin{bmatrix}
Af_A \\
Am
\end{bmatrix} = \begin{bmatrix}
BR_A & 0 \\
-S^T(Br_{BA})BR_A & BR_A
\end{bmatrix} \begin{bmatrix}
Af_A \\
Am
\end{bmatrix}
\]

For skew-symmetric matrices, it is: 
\(-S^T(r) = S(r)\)
Example: 6D force/torque sensor

frame of **measure** for the forces/torques (attached to the wrist sensor)

frame of **interest** for evaluating forces/torques in a task with environment contact
Example: Gear reduction at joints

Transmission element with motion reduction ratio $N_r: 1$

One of the simplest applications of the principle of virtual work:

\[ P_m = u_m \dot{\theta}_m = u \dot{\theta} = P \]

\[ u = N_r u_m \]

here, $J = J^T = N_r$ (a scalar!)