Minimal representations of orientation
(Euler and roll-pitch-yaw angles)
Homogeneous transformations

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“Minimal” representations

- rotation matrices: 9 elements
  - 3 orthogonality relationships
  - 3 unitary relationships
  = 3 independent variables

- sequence of 3 rotations around independent axes
  - fixed \((a_i)\) or moving/current \((a'_i)\) axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12 + 12 possible different sequences (e.g., XYX)
  - actually, only 12 since

\[
\{(a_1 \alpha_1), (a_2 \alpha_2), (a_3 \alpha_3)\} \equiv \{(a'_3 \alpha_3), (a'_2 \alpha_2), (a'_1 \alpha_1)\}
\]

Robotics 1
**ZX’Z” Euler angles**

1. \( R_z(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

2. \( R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \)

3. \( R_z(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \)
ZX’Z” Euler angles

- **direct problem**: given \( \phi , \theta , \psi \); find \( R \)

\[
R_{ZX'Z''}(\phi, \theta, \psi) = R_Z(\phi) R_X'(\theta) R_Z''(\psi)
\]

given a vector \( v'''' = (x'''', y'''', z'''') \) expressed in \( RF'''' \), its expression in the coordinates of \( RF \) is

\[
v = R_{ZX'Z''}(\phi, \theta, \psi) v''''
\]

- the orientation of \( RF'''' \) is the same that would be obtained with the sequence of rotations:

\[
\psi \text{ around } z, \theta \text{ around } x \text{ (fixed), } \phi \text{ around } z \text{ (fixed)}
\]
ZX’Z” Euler angles

- **inverse problem:** given $R = \{r_{ij}\}$; find $\phi$, $\theta$, $\psi$

\[
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix}
= \begin{bmatrix}
C\phi \ C\psi - S\phi \ C\theta \ S\psi & - C\phi \ S\psi - S\phi \ C\theta \ C\psi & S\phi \ S\theta \\
S\phi \ C\psi + C\phi \ C\theta \ S\psi & - S\phi \ S\psi + C\phi \ C\theta \ C\psi & - C\phi \ S\theta \\
S\theta \ S\psi & S\theta \ C\psi & C\theta
\end{bmatrix}
\]

- $r_{13}^2 + r_{23}^2 = s^2\theta$, $r_{33} = c\theta \Rightarrow \theta = \text{ATAN2}\{\pm \sqrt{r_{13}^2 + r_{23}^2}, r_{33}\}$
  - two values differing just for the sign

- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)
  - $r_{31}/s\theta = s\psi$, $r_{32}/s\theta = c\psi \Rightarrow \psi = \text{ATAN2}\{r_{31}/s\theta, r_{32}/s\theta\}$
  - $\phi = \text{ATAN2}\{r_{13}/s\theta, -r_{23}/s\theta\}$

- similarly...

- there is always a pair of solutions

- there are always singularities (here $\theta = 0, \pm \pi$)

*Robotics 1*
Roll-Pitch-Yaw angles

**ROLL**

\[ R_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{bmatrix} \]

**PITCH**

\[ C_1R_Y(\theta)C_1^T \]

with \( R_Y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \)

**YAW**

\[ C_2R_Z(\phi)C_2^T \]

with \( R_Z(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \)
Roll-Pitch-Yaw angles (fixed XYZ)

- **direct problem:** given $\psi$, $\theta$, $\phi$; find $R$
  \[
  R_{R PY}(\psi, \theta, \phi) = R_Z(\phi) \ R_Y(\theta) \ R_X(\psi)
  \]
  \[
  \text{order of definition} = \begin{bmatrix}
  \cos \phi \ \cos \theta \ & \cos \phi \ \sin \theta \ \sin \psi - \sin \phi \ \cos \psi \ & \cos \phi \ \sin \theta \ \cos \psi + \sin \phi \ \sin \psi \\
  \sin \phi \ \cos \theta \ & \sin \phi \ \sin \theta \ \sin \psi + \cos \phi \ \cos \psi \ & \sin \phi \ \sin \theta \ \cos \psi - \cos \phi \ \sin \psi \\
  -\sin \theta \ & \cos \theta \ \sin \psi \ & \cos \theta \ \cos \psi
  \end{bmatrix}
  \]

- **inverse problem:** given $R = \{r_{ij}\}$; find $\psi$, $\theta$, $\phi$

  - $r_{32}^2 + r_{33}^2 = c^2 \theta$, $r_{31} = -s \theta \Rightarrow \theta = \text{ATAN2}\{-r_{31}, \pm \sqrt{r_{32}^2 + r_{33}^2}\}$
    - for $r_{31} < 0$, two symmetric values w.r.t. $\pi/2$
  
  - if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c \theta \neq 0$)
    \[
    r_{32}/c \theta = s \psi, \quad r_{33}/c \theta = c \psi \Rightarrow
    \psi = \text{ATAN2}\{r_{32}/c \theta, r_{33}/c \theta\}
    \]

  - similarly ...

  - **singularities** for $\theta = \pm \pi/2$
...why this order in the product?

$$R_{\text{RPY}}(\psi, \theta, \phi) = R_Z(\phi) R_Y(\theta) R_X(\psi)$$

order of definition  “reverse” order in the product (pre-multiplication...)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use of the angle/axis technique for each rotation in the sequence: $C R(\alpha) C^T$, with $C$ being the rotation matrix reverting the previously made rotations (= go back to the original axes)

concatenating three rotations: $[ ] [ ] [ ]$ (post-multiplication...)

$$R_{\text{RPY}}(\psi, \theta, \phi) = [R_X(\psi)] [R_X^T(\psi) R_Y(\theta) R_X(\psi)]$$

$$[R_X^T(\psi) R_Y^T(\theta) R_Z(\phi) R_Y(\theta) R_X(\psi)]$$

$$= R_Z(\phi) R_Y(\theta) R_X(\psi)$$
Homogeneous transformations

\[ \mathbf{A}p = \mathbf{A}p_{AB} + \mathbf{A}R_B \mathbf{B}p \]

"applied" position vector (with specific origin) \( \mathbf{A}p_{0A} \)

"applied" position vector (with specific origin) \( \mathbf{B}p_{0B} \)

RF_A

RF_B

\[ \mathbf{A}p_{\text{hom}} = \begin{bmatrix} \mathbf{A}p \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}R_B & \mathbf{A}p_{AB} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{B}p \\ 1 \end{bmatrix} = \mathbf{A}T_B \mathbf{B}p_{\text{hom}} \]

vector in homogeneous coordinates

4x4 matrix of homogeneous transformation

linear relationship

'affine' relationship
Properties of T matrix

- describes the relation between reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from a given frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible \((^A T_B)^{-1} = ^B T_A\)
- can be composed, i.e., \(^A T_C = ^A T_B \cdot ^B T_C\) ← note: it does not commute!
Inverse of a homogeneous transformation

\[ \begin{align*}
Ap &= Ap_{AB} + AR_B Bp \\
Bp &= Bp_{BA} + BR_A Ap = -AR_B^T Ap_{AB} + AR_B^T Ap
\end{align*} \]
Defining a robot task

- Absolute definition of task
- Task definition relative to the robot end-effector
- Direct kinematics of the robot arm (function of q)

\[
^B T_E(q) = \frac{W_T}{W_B} ^B T_E \frac{E_T}{E_T}^{-1} = \text{constant}
\]
Example of task definition

- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis downward and being aligned with the table sides

\[
E_R_T = \begin{pmatrix}
E_x_T & E_y_T & E_z_T \\
0 & 1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}
\]

- point P is known in the table frame RF_T

\[
T_p = \begin{pmatrix}
p_x \\
p_y \\
0 \\
\end{pmatrix}
\]

- the depth camera proceeds centering point P in its image until it senses a distance h from the table (in RF_E)

\[
e_p = \begin{pmatrix}
0 \\
0 \\
h \\
\end{pmatrix}
\]

with

\[
E_T^{-1} = T_E = \begin{pmatrix}
T_R_E & T_p_{TE} \\
0 & 1 \\
\end{pmatrix}
\]

\[
T_R_E = \left( E_R_T \right)^T = E_R_T
\]

\[
T_p_{TE} = T_p - T_R_E e_p
\]
Final comments on T matrices

- they are the main tool for computing the **direct kinematics** of robot manipulators
- they are used in many application areas (in robotics and beyond)
  - in positioning/orienting a vision camera (matrix $b^T_c$ with extrinsic parameters of the camera pose)
  - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

\[ A_T B = \begin{bmatrix} A_R_B & A_P_{AB} \\ \alpha_x & \alpha_y & \alpha_z & \sigma \end{bmatrix} \]

- all zero in robotics

- coefficients of perspective deformation

- scaling coefficient

- always unitary in robotics