Position and orientation of rigid bodies

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Position and orientation

- **position**: \(^A p_{AB}\) (vector \(\in \mathbb{R}^3\)), expressed in \(RF_A\) (use of coordinates other than Cartesian is possible, e.g., cylindrical or spherical)

- **orientation**: orthonormal 3x3 matrix
  
  \[ \begin{bmatrix} A x_B & A y_B & A z_B \end{bmatrix} \]

  - \(x_A y_A z_A\) (\(x_B y_B z_B\)) are unit vectors (with unitary norm) of frame \(RF_A\) (\(RF_B\))
  - components in \(^A R_B\) are the *direction cosines* of the axes of \(RF_B\) with respect to (w.r.t.) \(RF_A\)

\[ A R_B = \begin{bmatrix} A x_B & A y_B & A z_B \end{bmatrix} \]
Rotation matrix

\[ A_R^B = \begin{bmatrix}
  x_A^T x_B & x_A^T y_B & x_A^T z_B \\
  y_A^T x_B & y_A^T y_B & y_A^T z_B \\
  z_A^T x_B & z_A^T y_B & z_A^T z_B
\end{bmatrix} \]

Orthogonal, with det = +1

Chain rule property:

\[ kR_i \cdot iR_j = kR_j \]

NOTE: in general, the product of rotation matrices does not commute!
Change of coordinates

\[ \begin{bmatrix} \mathbf{0p}_x \\ \mathbf{0p}_y \\ \mathbf{0p}_z \end{bmatrix} = \begin{bmatrix} \mathbf{1p}_x \end{bmatrix} \mathbf{0x}_1 + \begin{bmatrix} \mathbf{1p}_y \end{bmatrix} \mathbf{0y}_1 + \begin{bmatrix} \mathbf{1p}_z \end{bmatrix} \mathbf{0z}_1 \]

\[ = \begin{bmatrix} \mathbf{0x}_1 & \mathbf{0y}_1 & \mathbf{0z}_1 \end{bmatrix} \begin{bmatrix} \mathbf{1p}_x \\ \mathbf{1p}_y \\ \mathbf{1p}_z \end{bmatrix} = \mathbf{0R}_1 \mathbf{1p} \]

the rotation matrix \( \mathbf{0R}_1 \) (i.e., the orientation of \( \mathbf{RF}_1 \) w.r.t. \( \mathbf{RF}_0 \)) represents also the change of coordinates of a vector from \( \mathbf{RF}_1 \) to \( \mathbf{RF}_0 \).
Orientation of frames in a plane
(elementary rotation around z-axis)

x = OB – xB = u \cos \theta - v \sin \theta
y = OC + Cy = u \sin \theta + v \cos \theta
z = w

or...

\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix} = R_z(\theta)

R_z(-\theta) = R_z^T(\theta)

\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}

R_x(\theta) =
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}

R_y(\theta) =
Ex: Rotation of a vector around z

\[
x' = |v| \cos (\alpha + \theta) = |v| (\cos \alpha \cos \theta - \sin \alpha \sin \theta)
= x \cos \theta - y \sin \theta
\]

\[
y' = |v| \sin (\alpha + \theta) = |v| (\sin \alpha \cos \theta + \cos \alpha \sin \theta)
= x \sin \theta + y \cos \theta
\]

\[z' = z\]

or...

\[
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]...as before!
Equivalent interpretations of a rotation matrix

The same rotation matrix, e.g., $R_z(\theta)$, may represent:

1. The orientation of a rigid body with respect to a reference frame $RF_0$
   \[ ^0x_c \, ^0y_c \, ^0z_c \] = $R_z(\theta)$

2. The change of coordinates from $RF_C$ to $RF_0$
   \[ ^0P = R_z(\theta) \, ^CP \]

3. The vector rotation operator
   \[ v' = R_z(\theta) \, v \]

The rotation matrix $^0R_C$ is an operator superposing frame $RF_0$ to frame $RF_C$.
Composition of rotations

brings $RF_0$ on $RF_1$

$0R_1$

$1R_2$ brings $RF_1$ on $RF_2$

$2R_3$ brings $RF_2$ on $RF_3$

$p_{01} = 0$

$p_{12} = 0$

$p_{23} = 0$

a comment on computational complexity

$0p = (0R_1 1R_2 2R_3)^3p = 0R_3^3p$

63 products
42 summations

$0p = 0R_1 (1R_2 (2R_3^3p))$

27 products
18 summations
Axis/angle representation

DATA

- unit vector $r$ ($\|r\| = 1$)
- $\theta$ (positive if counterclockwise, as seen from an “observer” oriented like $r$ with the head placed on the arrow)

DIRECT PROBLEM

find

$$R(\theta, r) = [0x_1 \ 0y_1 \ 0z_1]$$

such that

$$^0P = R(\theta, r)^1P \quad ^0v' = R(\theta, r)^0v$$
Axis/angle: Direct problem

sequence of 3 rotations that bring frame RF₀ to superpose with frame RF₁

$$R(\theta, r) = C R_z(\theta) C^T$$

sequence of three rotations

$$C = \begin{bmatrix} n & s & r \end{bmatrix}$$

after the first rotation the z-axis coincides with r

n and s are orthogonal unit vectors such that
$$n \times s = r,$$
$$n_y s_z - s_y n_z = r_x,$$
$$n_z s_x - s_z n_x = r_y,$$
$$n_x s_y - s_x n_y = r_z.$$
Axis/angle: Direct problem

solution

\[ R(\theta, r) = C R_z(\theta) C^T \]

\[
R(\theta, r) = \begin{bmatrix} n & s & r \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n^T \\ s^T \\ r^T \end{bmatrix}
\]

\[ = r r^T + (n n^T + s s^T) c\theta + (s n^T - n s^T) s\theta \]

taking into account that

\[ C C^T = n n^T + s s^T + r r^T = I \]

and that

\[
s n^T - n s^T = \begin{bmatrix} 0 & -r_z & r_y \\ 0 & 0 & -r_x \\ \text{skew-sym} & 0 \end{bmatrix} = S(r)
\]

skew-symmetric\( (r) \):

\[ r \times v = S(r)v = -S(v)r \]

depends only on \( r \) and \( \theta \) !!

\[ R(\theta, r) = r r^T + (I - r r^T) c\theta + S(r) s\theta \]

\[ = R^T(-\theta, r) = R(-\theta, -r) \]
Final expression of $R(\theta, r)$

developing computations...

$$R(\theta, r) =$$

$$\begin{bmatrix}
  r_x^2(1-\cos\theta)+\cos\theta & r_xr_y(1-\cos\theta)-r_z\sin\theta & r_xr_z(1-\cos\theta)+r_y\sin\theta \\
  r_xr_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_yr_z(1-\cos\theta)-r_x\sin\theta \\
  r_xr_z(1-\cos\theta)-r_y\sin\theta & r_yr_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta
\end{bmatrix}$$
Axis/angle: a simple example

\[ R(\theta, r) = rr^T + (I - rr^T) c\theta + S(r) s\theta \]

\[ r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = z_0 \]

\[
R(\theta, r) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} c\theta + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s\theta
\]

\[
= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_z(\theta)
\]
Axis/angle: proof of Rodriguez formula

\[ v' = R(\theta,r) \ v \]

\[ v' = v \cos \theta + (r \times v) \sin \theta + (1 - \cos \theta)(r^T v) \ r \]

proof:

\[ R(\theta,r) \ v = (rr^T + (I - rr^T) \cos \theta + S(r) \sin \theta)v \]

\[ = rr^Tv (1 - \cos \theta) + v \cos \theta + (r \times v) \sin \theta \]

q.e.d.
Properties of $R(\theta, r)$

1. $R(\theta, r)r = r$ (r is the invariant axis in this rotation)
2. when $r$ is one of the coordinate axes, $R$ boils down to one of the known elementary rotation matrices
3. $(\theta, r) \rightarrow R$ is not an injective map: $R(\theta, r) = R(-\theta, -r)$
4. $\det R = +1 = \Pi \lambda_i$ (eigenvalues)
5. $\text{tr}(R) = \text{tr}(rr^T) + \text{tr}(I - rr^T)c\theta = 1 + 2c\theta = \Sigma \lambda_i$

1. $\Rightarrow \lambda_1 = 1$
4. & 5. $\Rightarrow \lambda_2 + \lambda_3 = 2c\theta \Rightarrow \lambda^2 - 2c\theta \lambda + 1 = 0$
   $\Rightarrow \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta - 1} = c\theta \pm is\theta = e^{\pm i\theta}$
all eigenvalues $\lambda$ have unitary module ($\Leftrightarrow R$ orthonormal)
Axis/angle: Inverse problem

GIVEN a rotation matrix $R$, 
FIND a unit vector $r$ and an angle $\theta$ such that

$$R = r r^T + (I - r r^T) \cos \theta + S(r) \sin \theta = R(\theta, r)$$

Note first that $\text{tr}(R) = R_{11} + R_{22} + R_{33} = 1 + 2 \cos \theta$; so, one could solve

$$\theta = \arccos \frac{R_{11} + R_{22} + R_{33} - 1}{2}$$

but:
- provides only values in $[0, \pi]$ (thus, never negative angles $\theta \ldots$)
- loss of numerical accuracy for $\theta \rightarrow 0$
Axis/angle: Inverse problem solution

from

\[ R - R^T = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ 0 & 0 & R_{23} - R_{32} \\ \text{skew-symm} & \text{skew-symm} & 0 \end{bmatrix} = 2 \sin \theta \begin{bmatrix} 0 & -r_z & r_y \\ -r_z & 0 & -r_x \\ r_y & r_x & 0 \end{bmatrix} \]

it follows

\[ \|r\| = 1 \Rightarrow \sin \theta = \pm \frac{1}{2} \sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2} \quad (*) \]

\[ (**) \]

\[ \theta = \text{ATAN2} \left\{ \pm \sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}, R_{11} + R_{22} + R_{33} - 1 \right\} \]

\[ r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2 \sin \theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \]

can be used only if \[ \sin \theta \neq 0 \]

(test made in advance on the expression \((*)\) of \(\sin \theta\) in terms of the \(R_{ij}\)'s)
ATAN2 function

- arctangent with output values “in the four quadrants”
  - two input arguments
  - takes values in \([-\pi, +\pi]\)
  - undefined only for (0,0)
- uses the sign of both arguments to define the output quadrant
- based on \(\text{arctan}\) function with output values in \([-\pi/2, +\pi/2]\)
- available in main languages (C++, Matlab, ...)

\[
\text{atan2}(y, x) = \begin{cases} 
\arctan\left(\frac{y}{x}\right) & x > 0 \\
\pi + \arctan\left(\frac{y}{x}\right) & y \geq 0, x < 0 \\
-\pi + \arctan\left(\frac{y}{x}\right) & y < 0, x < 0 \\
\frac{\pi}{2} & y > 0, x = 0 \\
-\frac{\pi}{2} & y < 0, x = 0 \\
\text{undefined} & y = 0, x = 0 
\end{cases}
\]
Singular cases
(use when \( \sin \theta = 0 \))

- if \( \theta = 0 \) from (**), there is no given solution for \( r \)
  (rotation axis is undefined)
- if \( \theta = \pm \pi \) from (**), then set \( \sin \theta = 0, \cos \theta = -1 \)
  \[ R = 2r r^T - I \]

\[
\begin{bmatrix}
  r_x \\
  r_y \\
  r_z
\end{bmatrix} = \begin{bmatrix}
  \pm \sqrt{(R_{11} + 1)/2} \\
  \pm \sqrt{(R_{22} + 1)/2} \\
  \pm \sqrt{(R_{33} + 1)/2}
\end{bmatrix}
\]

exercise: determine the two solutions \((r, \theta)\) for \( R = \begin{pmatrix}
  -1 & 0 & 0 \\
  0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
  0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \)
Unit quaternion

- to eliminate undetermined and singular cases arising in the axis/angle representation, one can use the unit quaternion representation
  \[ Q = \{\eta, \varepsilon\} = \{\cos(\theta/2), \sin(\theta/2) \mathbf{r}\} \]
  - a scalar \( \eta^2 + \|\varepsilon\|^2 = 1 \) (thus, “unit ...”)
  - \((\theta, \mathbf{r})\) and \((-\theta, -\mathbf{r})\) gives the same quaternion \( Q \)
  - the absence of rotation is associated to \( Q = \{1, \mathbf{0}\} \)
  - unit quaternions can be composed with special rules (in a similar way as in a product of rotation matrices)
    \[ Q_1 \ast Q_2 = \{\eta_1\eta_2 - \varepsilon_1^T\varepsilon_2, \eta_1\varepsilon_2 + \eta_2\varepsilon_1 + \varepsilon_1 \times \varepsilon_2\} \]