Control of Electromechanical Systems

November 13, 2017

Exercise 1

Consider the feedback control scheme of the motor speed ω in Fig. 1, where the torque actuation includes a time constant $\tau_A = 0.1$ s and a disturbance torque T_L is present. The plant data are $J = 1$ kg m^2 and $B = 0.1$ Nm/(rad/s).

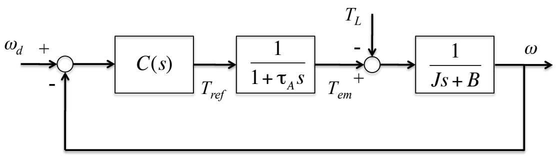

Figure 1: Speed control scheme.

Design a control law $C(s)$ in order to satisfy the following specifications.

- The closed-loop system is asymptotically stable.
- Constant torque disturbances are rejected at steady state.
- A reference acceleration profile $\dot{\omega}_d = 2$ rad/s^2 is reproduced with a maximum error $e = \omega_d - \omega$ smaller than 0.1 rad/s at steady state.
- In response to a constant reference ω_d, the transient behavior of the error is sufficiently damped.

Once a controller has been designed according to the previous specifications, determine the steady-state response of the closed-loop system to a ramp disturbance $T_L(t) = 0.5\,t$ Nm.

Exercise 2

Consider a system made of two inertias $J > 0$ and $M > 0$ connected through a transmission with torsional stiffness $K > 0$ and in the presence of viscous friction acting on the two sides of the transmission, respectively with coefficients $B > 0$ and $D > 0$. A control torque τ is applied to the first inertia, whose angular position and speed are θ and ω. The angular position and speed of the second inertia are θ_m and ω_m. The system is thus described by the following differential equations:

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\theta}_m &= \omega_m \\
J\dot{\omega} &= \tau - B\omega + K(\theta_m - \theta) \\
M\dot{\omega}_m &= -D\omega_m + K(\theta - \theta_m).
\end{aligned}
$$

- Determine the simplest feedback control law that is able to regulate the speed ω_m of the inertia M to a desired constant reference value ω_d.
- If the gain of the chosen controller is increased, will the closed-loop system become unstable? If so, which is the upper feasible limit for the control gain?
- If the chosen output ω_m needs to track accurately a desired time-varying, sufficiently smooth profile $\omega_d = \omega_d(t)$, what is the expression of the feedforward torque $\tau = \tau_{\text{ffw}}(t)$ that will guarantee zero tracking error in nominal conditions? What should be the initial state of the system (at time $t = 0$) in order to obtain perfect tracking for all $t \geq 0$?
- Draw a complete block diagram of the combined feedback/feedforward control scheme.

Exercise 3 *(in alternative to Exercise 4)*

For the same plant of Exercise 1 (with actuation dynamics), consider a PI controller with proportional action relocated in the feedback path. Draw the block diagram of the control scheme and study the asymptotic stability of the closed-loop system when varying separately the K_P and K_I gains. Whenever appropriate, use also a suitable root locus analysis.

Exercise 4 *(in alternative to Exercise 3, with an extra bonus if you do both)*

Consider the block diagram of the discrete-time control system in Fig. 4.13 of the textbook (drawn again in Fig. 2), in which

$$W_P(z) = \frac{T_c}{J}\frac{1}{z-1} \quad \text{and} \quad W_{SE}(z) = \frac{z+1}{2z} \text{ (averaging measurements of speed).}$$

Assume further that no disturbing torque T_L acts on the system. With inertia $J = 0.1$ kg m^2, sampling time $T_c = 0.005$ s, and integral gain $K_I = 1$, study the stability of the closed-loop system by using a root locus analysis for varying $K_P > 0$.

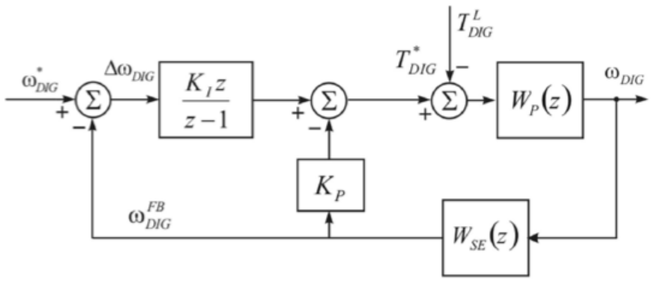

Figure 2: Fig. 4.13, taken from the textbook.

[180 minutes, open books]

Solution

October 27, 2017

Exercise 1

The considered plant

$$G(s) = \frac{1}{(Js+B)(1+\tau_A s)} = \frac{1/B}{(1+(J/B)s)(1+\tau_A s)} = \frac{1/(J\tau_A)}{(s+(B/J))(s+(1/\tau_A))} = \frac{10}{(s+0.1)(s+10)} \tag{1}$$

has gain $K_G = 1/B = 10$, two asymptotically stable poles, a slow one in $s = -B/J = -0.1$ and a fast one in $s = -1/\tau_A = -10$, and no zeros —different equivalent representations are used in (1). The open-loop response to a unitary step reaches the steady-state value very slowly (see Fig. 3).

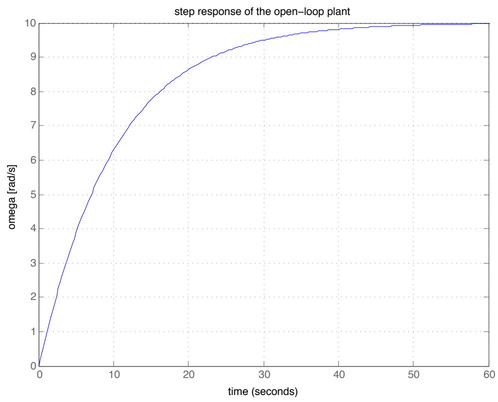

Figure 3: Step response of the open-loop plant (1).

To satisfy the steady-state requirement on the input-output behavior (control system of type 1, with limited steady-state error in response to a ramp reference input) and on the disturbance-output behavior (astatism, i.e., rejection of constant disturbances), the controller should introduce a pole in $s = 0$ (integral action). In order to have additional parameters left for control synthesis, a PI controller will be considered in the first place,

$$PI(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} = K_I \frac{1+\tau_z s}{s} = K_P \frac{s + (1/\tau_z)}{s}, \tag{2}$$

with control gain $K_I > 0$, integral action, and a negative zero with time constant $\tau_z = K_P/K_I > 0$. In (2), different equivalent forms are presented for the PI controller. With the loop transfer function $F(s) = PI(s)G(s)$, the closed-loop system becomes

$$W(s) = \frac{\omega(s)}{\omega_d(s)} = \frac{F(s)}{1+F(s)} = \frac{K_I(1+\tau_z s)}{s(Js+B)(1+\tau_A s) + K_I(1+\tau_z s)}. \tag{3}$$

On the other hand, the error transfer function in the closed loop is

$$W_e(s) = \frac{e(s)}{\omega_d(s)} = \frac{\omega_d(s) - \omega(s)}{\omega_d(s)} = 1 - W(s) = \frac{s(Js+B)(1+\tau_A s)}{s(Js+B)(1+\tau_A s) + K_I(1+\tau_z s)}. \tag{4}$$

The asymptotic stability of $W(s)$ in (3) can be checked by using the Routh criterion on the third-degree polynomial at the denominator. The Routh table is:

$$\begin{array}{c|ll} 3 & J\tau_A & B+K_I\tau_z \\ 2 & J+B\tau_A & K_I \\ 1 & (J+B\tau_A)(B+K_I\tau_z)-K_IJ\tau_A & \\ 0 & K_I & \end{array}$$

Being always $J > 0$, $B > 0$, and $\tau_A > 0$, the necessary and sufficient conditions for asymptotic stability are

$$K_I > 0, \qquad K_I\left(J\tau_A - \tau_z(J+B\tau_A)\right) < B(J+B\tau_A). \tag{5}$$

Assuming that both inequalities (5) are satisfied, the steady-state error of our type 1 control system to a ramp velocity reference (with rate 2 rad/s^2) will be constant but different from zero. It can be computed using (4) and the final value theorem as

$$e_1 = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,W_e(s)\,\frac{2}{s^2} = 2\cdot\lim_{s\to 0}\frac{(Js+B)(1+\tau_A s)}{s(Js+B)(1+\tau_A s)+K_I(1+\tau_z s)} = \frac{2B}{K_I}. \tag{6}$$

Therefore,

$$e_1 \le 0.1 \qquad \Rightarrow \qquad K_I \ge \frac{2B}{0.1} = \frac{0.2}{0.1} = 2. \tag{7}$$

We verify next if and how a sufficiently damped transient behavior can be obtained with the PI controller (2), under the constraint (7). A convenient choice is to proceed by (stable) pole-zero cancelation, removing the slow dynamics of the plant with the zero of the PI law. Setting in (2)

$$\tau_z = \frac{J}{B} = 10 \quad\Rightarrow\quad F(s) = \frac{K_P/(J\tau_A)}{s(s+(1/\tau_A))} \quad\Rightarrow\quad W(s) = \frac{K_P/(J\tau_A)}{s(s+(1/\tau_A))+K_P/(J\tau_A)}, \tag{8}$$

the closed-loop system is reduced to a second-order, asymptotically stable dynamics with gain $W(0) = 1$, without zeros, and with two poles in s_1 and s_2:

$$s^2 + \frac{1}{\tau_A}s + \frac{K_P}{J\tau_A} = 0 \quad\Rightarrow\quad s_{1,2} = -\frac{1}{2\tau_A} \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau_A}\right)^2 - \frac{4K_P}{J\tau_A}} = -5 \pm \frac{1}{2}\sqrt{100-40K_P}. \tag{9}$$

The choice $\tau_z = 10$ in (8) implies $K_P = 10K_I$ and, from (7), also $K_P \ge 20$. Therefore, the two poles in (9) will be complex conjugate. Moreover, their natural frequency will increase and their damping uniformly decrease when taking larger values of K_P. Thus, we should choose the minimum value that satisfies also the steady-state condition, i.e., $K_P = 20$. The final PI law is

$$PI(s) = K_P + \frac{K_I}{s} = 20 + \frac{2}{s} = 20\,\frac{s+0.1}{s}, \tag{10}$$

yielding

$$s_{1,2} = -5 \pm j13.22 \qquad\Rightarrow\qquad \omega_n = 14.14, \qquad \zeta = 0.35. \tag{11}$$

Note that we could get to the same result by looking at the simple root locus of the denominator of $W(s)$ in (8) when varying $K' = K_P/(J\tau_A) > 0$, i.e., by changing the gain K_P in the PI controller.

In practice, the obtained damping coefficient $\zeta = 0.35$ is the best that can be obtained when using only a PI control design. Such damping may be considered adequate when looking at the step response in Fig. 4: the overshoot is about 30%, with only few oscillations, whereas the rise time is $t_r \simeq 0.15$ s. The associated control effort $T_{ref}(t)$ reported in Fig. 4 has a peak value of 20 at $t = 0$,

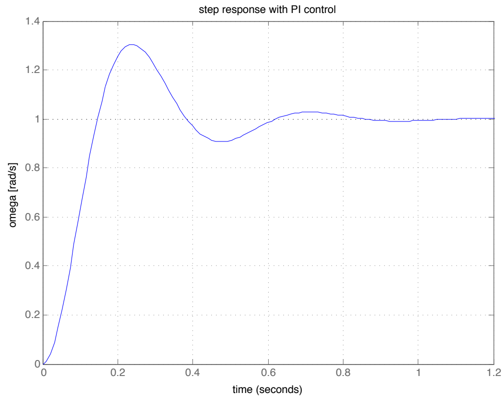

Figure 4: Step response for the closed-loop system obtained with the PI control law (10).

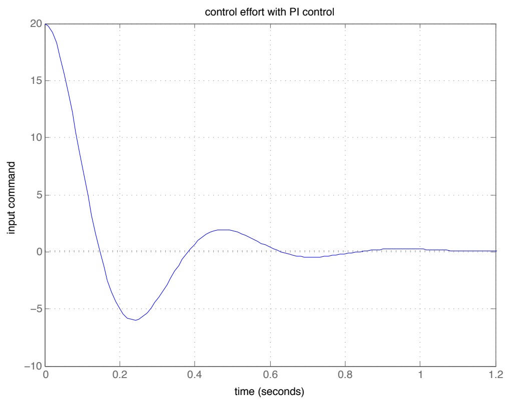

Figure 5: Control effort during the step response of Fig. 4.

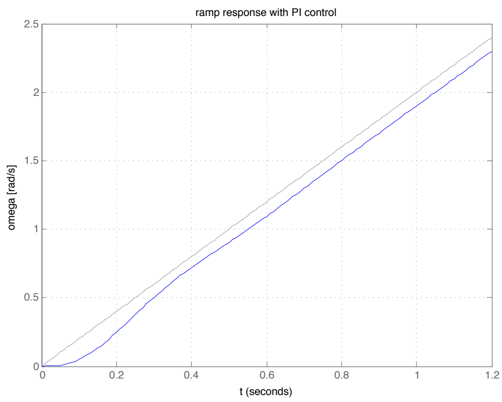

Figure 6: Response to a ramp input $\omega_d(t) = 2t$ for the same controlled system as in Fig. 4.

still moderate. The response to a ramp input $\omega_d(t) = 2t$ is shown in Fig. 6: the steady-state error is $e_1 = 0.1$, as predicted.

Finally, with the chosen PI controller (10), we obtain the following disturbance-output transfer function for the closed-loop system:

$$W_{dist}(s) = \frac{\omega(s)}{T_L(s)} = -\frac{s(1+\tau_A s)}{s(Js+B)(1+\tau_A s)+K_I(1+\tau_z s)}. \tag{12}$$

Therefore, the steady-state response to a ramp disturbance $T_L(t) = 0.5\,t$ [Nm] is computed as

$$e_L = \lim_{s\to 0} s\, W_{dist}(s)\,\frac{0.5}{s^2} = -0.5 \cdot \lim_{s\to 0} \frac{(1+\tau_A s)}{s(Js+B)(1+\tau_A s)+K_I(1+\tau_z s)} = -\frac{0.5}{K_I} = -0.25. \tag{13}$$

Wishing to increase the damping characteristics of the closed-response, we should add a further action in the form of a lead compensator (made by a zero preceding a pole in the frequency domain, with unitary gain). A simple design choice is to cancel also the other pole of the plant with the zero of this additional compensator, replacing it with a pole that has a smaller time constant, say reduced by a factor of 5:

$$C(s) = \left(K_P + \frac{K_I}{s}\right)\frac{1+\tau_A s}{1+0.2\tau_A s} = \left(20 + \frac{2}{s}\right)\frac{1+0.1s}{1+0.02s}. \tag{14}$$

This would lead to

$$F(s) = C(s)P(s) = \frac{5K_P/(J\tau_A)}{s(s+(5/\tau_A))} \quad\Rightarrow\quad W(s) = \frac{5K_P/(J\tau_A)}{s(s+(5/\tau_A))+5K_P/(J\tau_A)}, \tag{15}$$

with the closed-loop poles moved to

$$s_{1,2} = -\frac{5}{2\tau_A} \pm \frac{1}{2}\sqrt{\left(\frac{5}{\tau_A}\right)^2 - \frac{20K_P}{J\tau_A}}. = -25 \pm \frac{1}{2}\sqrt{2500-200K_P}, \tag{16}$$

For the same $K_P = 20$, two complex poles are obtained with an associated response that is now (slightly more than) critically damped:

$$s_{1,2} = -25 \pm \frac{1}{2}\sqrt{-1500} = -25 \pm j19.36 \qquad\Rightarrow\qquad \omega_n = 31.62, \qquad \zeta = 0.79. \tag{17}$$

Figure 7 shows the output of the closed-loop system in response to a unitary step. The rise time has been reduced to $t_r = 0.125$ s, whereas overshoot and oscillations are now practically absent. The largely improved transient behavior of the controlled output is indeed counterbalanced by a larger control effort during the first few instants of the response, with a peak of 100 at $t = 0$ —see Fig. 8). In Fig. 9, the response to a ramp input $\omega_d(t) = 2t$ shows the same previous steady-state error $e_1 = 0.1$, but a faster transient behavior.

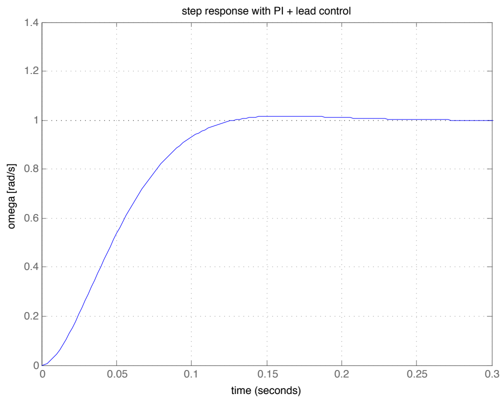

Figure 7: Step response for the closed-loop system obtained with the control law (14).

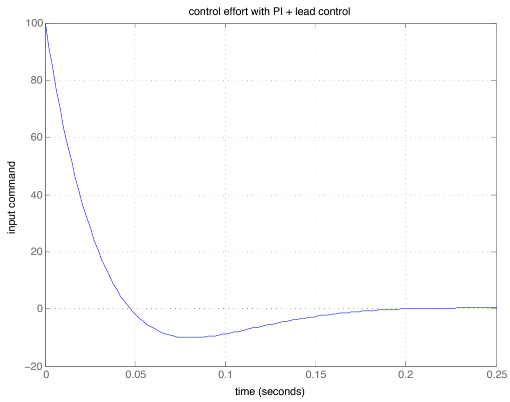

Figure 8: Control effort during the step response of Fig. 7.

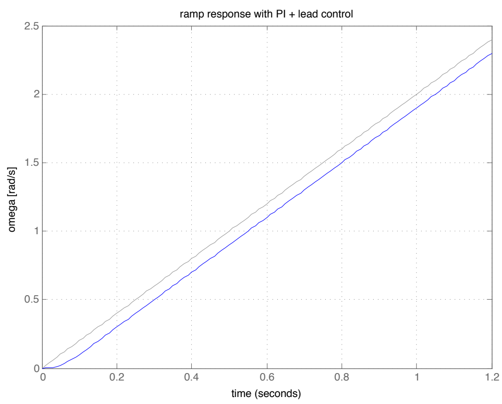

Figure 9: Response to a ramp input $\omega_d(t) = 2t$ for the modified closed-loop system.

Exercise 2

We start by deriving the input-output transfer function between the torque τ and the angular speed ω_m of the body with inertia M. Applying Laplace transform to the system equations gives

$$\left(Js^2 + Bs + K\right)\theta(s) = K\theta_m(s) + \tau(s) \tag{18}$$

$$\left(Ms^2 + Ds + K\right)\theta_m(s) = K\theta(s). \tag{19}$$

Solving (19) for $\theta(s)$ and replacing in (18) yields

$$\frac{\left(Js^2 + Bs + K\right)\left(Ms^2 + Ds + K\right)}{K}\,\theta_m(s) = K\theta_m(s) + \tau(s),$$

and thus

$$P'(s) = \frac{\theta_m(s)}{\tau(s)} = \frac{K}{(JM)s^4 + (BM + DJ)s^3 + (K(J + M) + BD)s^2 + K(B + D)s\,[+K^2 - K^2]}. \tag{20}$$

Equation (20) considers the angular position as output. Noting the presence of a pole in $s = 0$, the transfer function of our interest is obtained as

$$P(s) = \frac{\omega_m(s)}{\tau(s)} = sP'(s) = \frac{K}{(JM)s^3 + (BM + DJ)s^2 + (K(J + M) + BD)s + K(B + D)}, \tag{21}$$

i.e., a system with third-order dynamics (but no integral action) and without zeros. Using Routh criterion, it is easy to verify that the open-loop system (21) is asymptotically stable for all physically valid parameters $J > 0$, $M > 0$, $K > 0$, and with $B \geq 0$, $D \geq 0$, but not both simultaneously zero (or anyway too small). In fact, the stated assumption is that B and D are both strictly positive.

The request of output regulation (i.e., zero error at steady state) for any desired constant reference velocity ω_d asks for the inclusion of an integrator in the controller (type 1 control system). As a feedback law of minimum complexity, we will show next that a PI controller

$$PI(s) = K_P + \frac{K_I}{s} = K_P\,\frac{s + (1/T_I)}{s}, \qquad \text{with } T_I = K_I/K_P, \tag{22}$$

where $K_P > 0$ and $K_I > 0$ (and thus, with a positive integration time $T_I > 0$) is able to achieve also asymptotic stability of the closed-loop system. Our analysis will be mainly qualitative, based on a simple root locus. If desired, the actual conditions for asymptotic stability in terms of the original parametric data of the plant can be derived from the Routh criterion.

To proceed, consider only the loop transfer function

$$\begin{aligned} F(s) = PI(s)P(s) &= \frac{KK_P(s + (1/T_I))}{JMs^4 + (BM + DJ)s^3 + (K(J + M) + BD)s^2 + K(B + D)s} \\ &= \frac{KK_P}{JM}\,\frac{s + (1/T_I)}{s\left(s^3 + \dfrac{BM + DJ}{JM}s^2 + \dfrac{K(J + M) + BD}{JM}s + \dfrac{K(B + D)}{JM}\right)} \\ &= K'\,\frac{s + (1/T_I)}{s(s + a)(s^2 + bs + c)}. \end{aligned} \tag{23}$$

In the last expression, we have $a > 0$, $b > 0$, and $c > 0$, since the plant has been assumed asymptotically stable. Moreover, apart from the pole in $s = 0$ introduced by the PI controller, at least one pole should be real and negative, i.e., in $-a$. On the other hand, the two roots of $s^2 + bs + c = 0$ can be either complex conjugate or both real, possibly also coincident (in any case, always with negative real part). As a consequence, we can design the zero of the PI controller so as to cancel one real pole of the plant, or

$$a\,T_I = a\,\frac{K_I}{K_P} = 1 \quad \Rightarrow \quad F(s) = \frac{K'}{s(s^2 + bs + c)} \quad \Rightarrow \quad W(s) = \frac{F(s)}{1 + F(s)} = \frac{K'}{s(s^2 + bs + c) + K'}. \tag{24}$$

The root locus for $K' > 0$ takes one of the two possible forms shown in Fig. 10, depending on the presence or not of a pair of complex conjugate poles in $F(s)$. The overall behavior is anyway similar, with an upper bound K'_{max} for K' that preserves asymptotic stability of the closes-loop system.

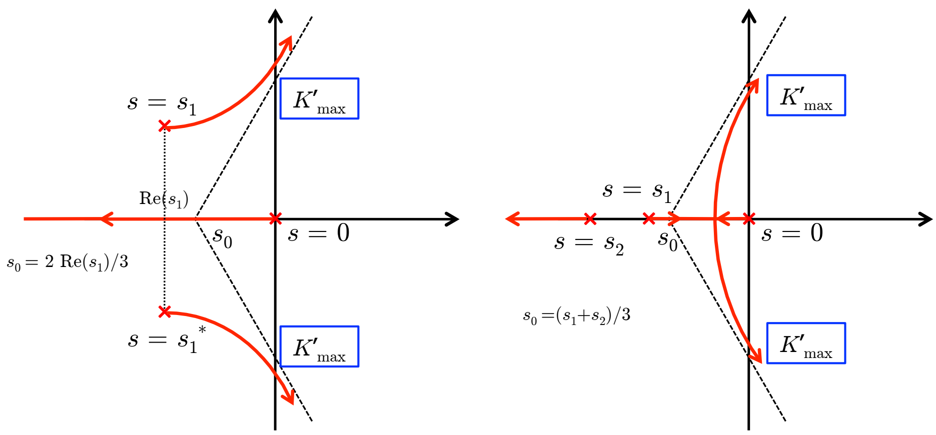

Figure 10: The two possible root loci of interest for $K' > 0$, depending on whether $F(s)$ in (24) has one real and two complex poles [left] or three real poles [right].

The actual upper limit $K_{P,max}$ for K_P can be found (still in symbolic form) from the Routh table associated to the denominator of $W(s)$ in (24):

$$\begin{array}{c|cc} 3 & 1 & c \\ 2 & b & K' \\ 1 & \dfrac{bc - K'}{b} & \\ 0 & K' & \end{array}$$

Being $b > 0$ and $c > 0$, the necessary and sufficient conditions for asymptotic stability are

$$0 < K' < K'_{max} = b\,c \qquad \Rightarrow \qquad K_{P,max} = \frac{JM\,b\,c}{K}. \tag{25}$$

This implies also an upper limit on the actual gain $K_I = K_P T_I$ of $PI(s)$. Indeed, when increasing the proportional gain K_P up to its limit, also the integral gain K_I should be increased similarly, so that their ratio (i.e., T_I) remains constant and equal to $1/a$ (to preserve the cancelation of a pole of the plant by the zero of the PI controller).

For the generation of a feedforward torque $\tau_{\text{ffw}}(t)$ associated to a desired (smooth) trajectory $\omega_d(t)$

for the chosen system output, consider again the differential equations

$$J\dot{\omega} = \tau - B\omega + K(\theta_m - \theta) \tag{26}$$

$$M\dot{\omega}_m = -D\omega_m + K(\theta - \theta_m). \tag{27}$$

We first isolate θ from eq. (27),

$$\theta = \theta_m + K^{-1}\left(M\dot{\omega}_m + D\omega_m\right), \tag{28}$$

and then differentiate (28) twice

$$\omega = \omega_m + K^{-1}\left(M\ddot{\omega}_m + D\dot{\omega}_m\right), \tag{29}$$

$$\dot{\omega} = \dot{\omega}_m + K^{-1}\left(M\dddot{\omega}_m + D\ddot{\omega}_m\right). \tag{30}$$

We isolate next τ from eq. (26),

$$\tau = J\dot{\omega} + B\omega + K(\theta - \theta_m), \tag{31}$$

and substitute then in (31) both (29) and (30), as well as $K(\theta - \theta_m)$ from (27). This yields

$$\tau = J\dot{\omega}_m + JK^{-1}\left(M\dddot{\omega}_m + D\ddot{\omega}_m\right) + B\omega_m + BK^{-1}\left(M\ddot{\omega}_m + D\dot{\omega}_m\right) + \left(M\dot{\omega}_m + D\omega_m\right). \tag{32}$$

Substituting in (32) $\omega_m = \omega_d(t)$, together with its three time derivatives $\dot{\omega}_m = \dot{\omega}_d(t)$, $\ddot{\omega}_m = \ddot{\omega}_d(t)$, and $\dddot{\omega}_m = \dddot{\omega}_d(t)$, and reorganizing terms leads to the nominal command

$$\tau_{\text{ffw}}(t) = (B + D)\,\omega_d(t) + \left(J + M + K^{-1}BD\right)\dot{\omega}_d(t) + K^{-1}\left(JD + MB\right)\ddot{\omega}_d(t) + K^{-1}JM\dddot{\omega}_d(t), \tag{33}$$

where we have assumed that all estimated parameters are the true ones ($\hat{J} = J$, $\hat{M} = M$, $\hat{B} = B$, $\hat{D} = D$, and $\hat{K} = K$). From (33), the required smoothness of the desired trajectory is $\omega_d(t) \in C^3$.

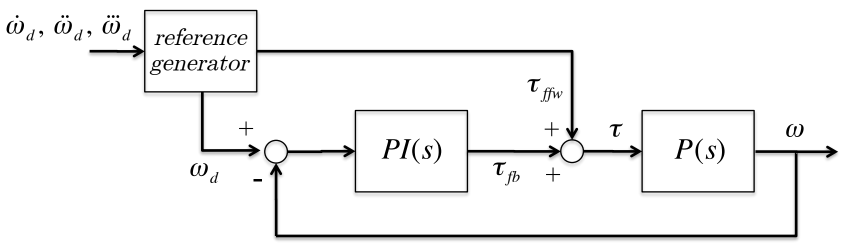

Figure 11: Combined feedback/feedforward control scheme for the given two-mass/spring system.

Indeed, the same result could have been obtained in a simpler fashion by just inverting the plant (21) in the Laplace domain, $\tau_{\text{ffw}}(s) = P^{-1}(s)\omega_d(s)$. However, the time domain analysis helps in finding also the correct initial conditions which guarantee that the feedforward command $\tau_{\text{ffw}}(t)$ achieves perfect tracking of $\omega_d(t)$ right from the initial instant $t = 0$, For this, the initial state of the system has to be matched with the initial value of the trajectory and its first few derivatives. In particular, the following conditions should hold:

$$\begin{aligned} \theta(0) - \theta_m(0) &= K^{-1}\left(M\dot{\omega}_d(0) + D\omega_d(0)\right), \\ \omega(0) &= \omega_d(0) + K^{-1}\left(M\ddot{\omega}_d(0) + D\dot{\omega}_d(0)\right), \\ \omega_m(0) &= \omega_d(0). \end{aligned} \tag{34}$$

Note that only the relative angular position of the two bodies with inertia J and M is constrained at $t = 0$. Being the desired trajectory specified at the velocity level, the absolute angular position of one of the two bodies is completely free. The block diagram of the combined feedback/feedforward control scheme is drawn in Fig. 11.

Exercise 3

Figure 12 shows the requested control scheme. Splitting the single feedback path in two successive loops, and resolving the internal one first, leads to a final closed-loop transfer function

$$W(s) = \frac{\omega(s)}{\omega_d(s)} = \frac{K_I}{s\big((Js+B)(1+\tau_A s)+K_P\big)+K_I} = \frac{K_I}{J\tau_A s^3 + (J+B\tau_A)s^2 + (B+K_P)s + K_I}, \tag{35}$$

which is characterized by the absence of zeros.

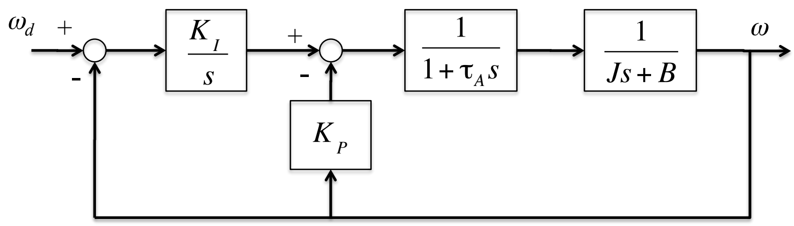

Figure 12: PI control scheme for the plant in Exercise 1, with proportional action relocated in the feedback path.

To study the asymptotic stability of $W(s)$, we construct the Routh table

3	|	$J\tau_A$	$B+K_P$
2	|	$J+B\tau_A$	K_I
1	|	$(J+B\tau_A)(B+K_P) - K_I J\tau_A$	
0	|	K_I	

Being $J>0$, $B>0$, and $\tau_A>0$, and assuming $K_P>0$ (although this is not strictly necessary), we have asymptotic stability when

$$K_P > 0, \qquad 0 < K_I < \frac{(J+B\tau_A)(B+K_P)}{J\tau_A} = K_{I,max}. \tag{36}$$

This means in particular that by arbitrarily increasing $K_P>0$, we can also arbitrarily increase the value of $K_I>0$ without losing asymptotic stability. Replacing now the original data from Exercise 1, we obtain

$$K_P > 0, \qquad 0 < K_I < 1.01 + \frac{K_P}{0.1}. \tag{37}$$

On the other hand, if we fix one gain and let the other vary, a graphical analysis can be performed with the help of a suitable root locus. The simpler case is to fix $K_P>0$ in the denominator of $W(s)$ and have $K_I>0$ vary. Rewriting

$$\text{den}\, W(s) = d_1(s) + K_1' n_1(s), \text{ with } d_1(s) = s\left(s^2 + \frac{J+B\tau_A}{J\tau_A}s + \frac{K_P+B}{J\tau_A}\right),\ K_1' = \frac{K_I}{J\tau_A},\ n_1(s) = 1, \tag{38}$$

a root locus can drawn as a function of $K_1'>0$ by considering as zeros of $F(s)$ the roots of $n_1(s)$ (i.e., no zeros) and as poles of $F(s)$ the three roots of $d_1(s)$. It is easy to see that this is exactly the situation already encountered in Exercise 2, with the $W(s)$ defined in (24). Therefore, one can use the same qualitative root loci of Fig. 10 to conclude that there will be asymptotic stability for increasing K_I until reaching an upper bound. This upper bound is indeed exactly the $K_{I,max}$ defined in (36).

When we fix instead $K_I > 0$ in the denominator of $W(s)$ and let $K_P > 0$ vary, we can rewrite

$$\text{den}\, W(s) = d_2(s) + K_2' n_2(s), \quad \text{with } d_2(s) = s^3 + \frac{J + B\tau_A}{J\tau_A} s^2 + \frac{B}{J\tau_A} s + \frac{K_I}{J\tau_A}, \; K_2' = \frac{K_P}{J\tau_A}, \; n_2(s) = s. \tag{39}$$

The associated root locus (taking in those cases the more general name of *root contour*) can be generated with the same rules of the first case. Figure 13 shows two situations in which two out of the three poles of $d_2(s)$ are complex conjugate (other situations may occur as well, and can be treated similarly). When analyzing the first column of the Routh table of $d_2(s)$, two subcases are possible:

$$0 < K_I < \frac{B(J + B\tau_A)}{J\tau_A} \quad \Longleftrightarrow \quad \text{all three roots have negative real parts;} \tag{40}$$

$$K_I \geq \frac{B(J + B\tau_A)}{J\tau_A} \quad \Longleftrightarrow \quad \begin{array}{l}\text{two (complex) roots have positive real parts,}\\ \text{the remaining one is real and negative.}\end{array} \tag{41}$$

In the first subcase (40), shown on the left of Fig. 13, K_2' and thus K_P can be arbitrarily increased without a loss of asymptotic stability. In the second subcase (41), shown on the right of Fig. 13, there is a lower bound $K'_{2,min}$ on K_2' (i.e., on K_P) that should be overcome before obtaining asymptotic stability. This analysis —especially the guarantee that the center s_0 of the asymptotic directions of the positive locus (which are vertical for a pole-zero excess = 2) is always negative— is indeed in accordance with the conditions for asymptotic stability stated in (36).

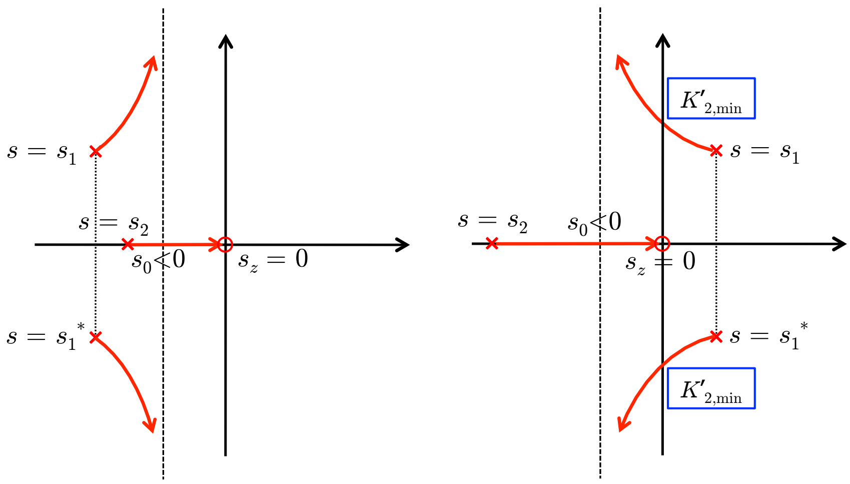

Figure 13: The two possible root contours for $K_2' > 0$, depending on whether the complex poles of $d_2(s)$ in (39) have their real part negative [left] or positive [right].

Exercise 4

We manipulate the block diagram in Fig. 2 in order to obtain the discrete-time transfer function of the closed-loop system. With reference to Fig. 14, we split the feedback path in two and resolve the inner loop gives first

$$W_1(z) = \frac{\dfrac{T_c}{J}\dfrac{1}{z-1}}{1 + K_P \cdot \dfrac{z+1}{2z} \cdot \dfrac{T_c}{J}\dfrac{1}{z-1}} = \frac{2T_c\, z}{2J\, z(z-1) + K_P T_c\,(z+1)}. \tag{42}$$

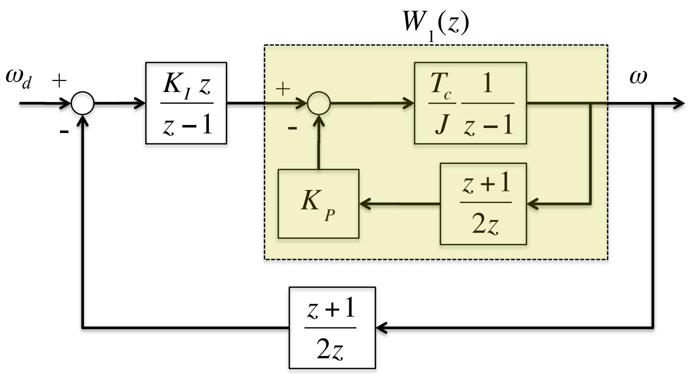

Figure 14: Unwrapping the two loops of Fig. 2.

Next, we resolve also the outer loop and obtain finally

$$\begin{aligned} W(z) &= \frac{W_1(z)\cdot\dfrac{K_I z}{z-1}}{1 + W_1(z)\cdot\dfrac{K_I z}{z-1}\cdot\dfrac{z+1}{2z}} = \frac{\dfrac{2T_c\, z}{2J\, z(z-1) + K_P T_c\,(z+1)}\cdot\dfrac{K_I z}{z-1}}{1 + \dfrac{2T_c\, z}{2J\, z(z-1) + K_P T_c\,(z+1)}\cdot\dfrac{K_I z}{z-1}\cdot\dfrac{z+1}{2z}} \\ &= \frac{2K_I T_c\, z^2}{2J\, z(z-1)^2 + K_P T_c\,(z+1)(z-1) + K_I T_c\,(z+1)}. \end{aligned} \tag{43}$$

In view of the request of a parametric study of the asymptotic stability by means of a root locus, we divide numerator and denominator by $2J$ (to make unitary the coefficient of the highest power at the denominator) and insert the numerical data of the problem ($J = 0.1$ kg m^2, $T_c = 0.005$ s, and $K_I = 1$). This leads to

$$W(z) = \frac{0.1\, z^2}{z(z^2 - 1.95z + 1.05) + 0.05K_P\,(z+1)(z-1)} = \frac{0.1\, z^2}{d(z) + K'\, n(z)}. \tag{44}$$

The three roots of $d(z) = 0$ are in $\{0, 0.9750 \pm j0.3152\}$ and the complex roots are outside the unit circle (on a circle of radius $r = 1.0247$). Figure 15 shows the root locus traced with MATLAB for $K' = 0.05K_P > 0$.

The system is asymptotically stable when all three roots belong to the unit circle. This happens in the gain interval

$$K' \in [0.0264, 0.9486] \qquad \Longleftrightarrow \qquad K_P \in [0.5272, 18.9728] = [K_{P,min}, K_{P,max}]\,. \tag{45}$$

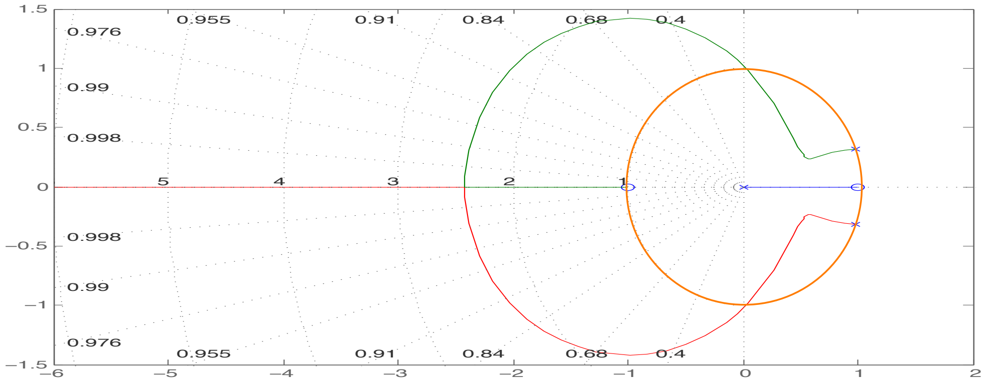

Figure 15: The root locus associated to $K' > 0$, $n(z)$, and $d(z)$, as defined in (44). The unit circle in the z-plane is also shown (in orange).

The location of the poles of $W(z)$ in (44) corresponding to the stability boundaries are

$$\begin{aligned} K_{P,min} = 0.5272 \qquad &\Rightarrow \qquad z = \{0.0264, 0.9486 \pm j0.3163\} \\ &\qquad\quad \text{with distance to the origin of the complex pair} = 0.0264; \\ K_{P,max} = 18.9728 \qquad &\Rightarrow \qquad z = \{0.9486, 0.0264 \pm j0.9996\} \\ &\qquad\quad \text{with distance to the origin of the complex pair} = 0.9486. \end{aligned} \tag{46}$$

Figure 16 shows the discrete-time step response of $W(z)$ in (44) obtained for $K_P = 10$, i.e., of

$$W(z) = \frac{0.1\, z^2}{z^3 - 1.45z^2 + 1.05z - 0.5}. \tag{47}$$

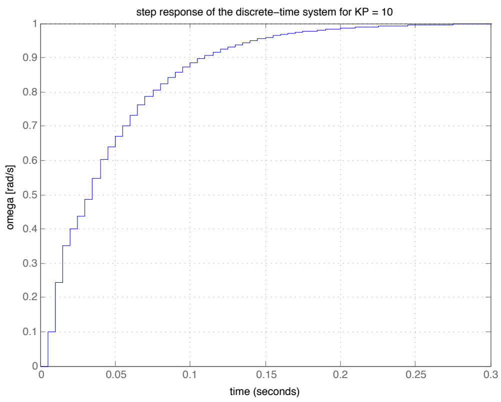

Figure 16: Discrete-time step response of $W(z)$ in (47) ($T_c = 0.005$ s).

* * * * *