2023 International Graduate School on Control
Course M16

Control of Soft and Articulated Elastic Robots

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Robots with Flexible Links:
Modeling and Control

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SAPIENZA
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Outline

- motivations for considering distributed link flexibility
  - few examples of robots with flexible links …
- dynamic modeling of flexible link robots
  - single flexible link (in the domain of linearity)
  - multiple flexible links (nonlinear dynamics, in the planar case)
- formulation of control problems
  - structural control properties in the linear and nonlinear case
- control design for regulation tasks
- control design for trajectory tracking tasks
  - joint-space trajectory
  - end-effector trajectory
- conclusions and basic references

Motivation

Link flexibility in robot manipulators

- distributed link deformation in robot manipulators arises when
  - very long and slender arms are needed by the specific application
  - lightweight materials are used to save weight/costs (without additional care)
- `link rigidity’ is always an ideal assumption which may fail …
  - for larger payload-to-weight ratios
  - in high-speed motion tasks or for large exchanged forces with the environment
  - when the control bandwidth is increased
- flexible structures in motion are present in different applications
  - manipulators in space, underwater, underground, automated cranes, …
- neglecting link flexibility in control design
  - limits static (steady-state errors) and dynamic (vibrations, poor tracking) performance
  - stability problems due to non-colocation between input commands and typical outputs to be controlled (non-minimum phase systems)

Robots with link flexibility

Space applications

- **SSRMS** (Space Shuttle Remote Manipulation System) and **Canadarm 2**

video

- **Tohoku** cooperating 6R flexible arms capturing a rolling satellite at

video

Tohoku University
(Prof. Masaru Uchiyama)

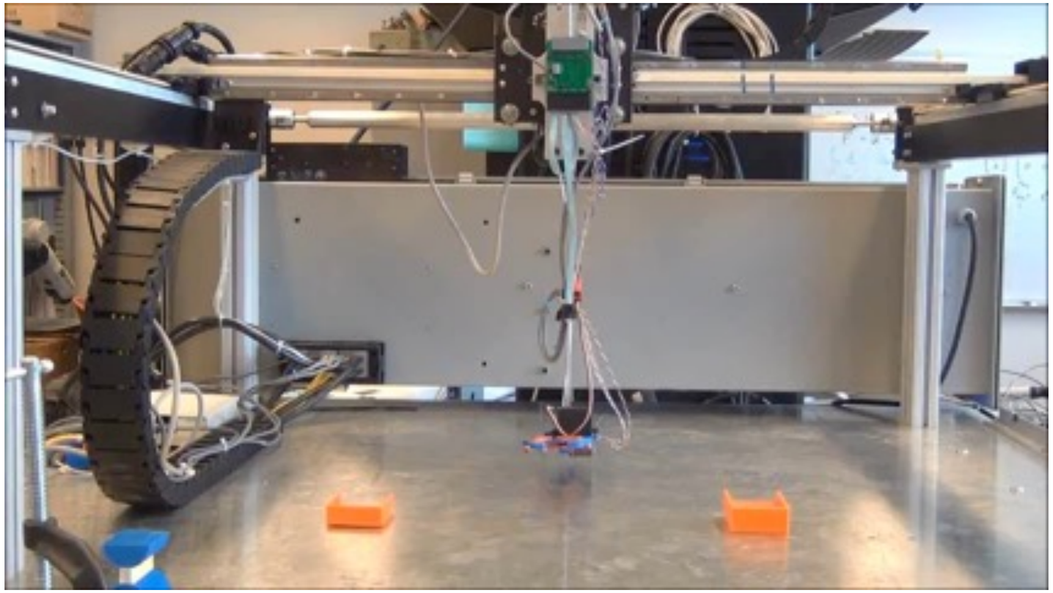
Robots with **link flexibility**

Underground applications

- **Sam II**, long flexible arm with macro-micro concept for remote exploration and manipulation of nuclear waste sites

video

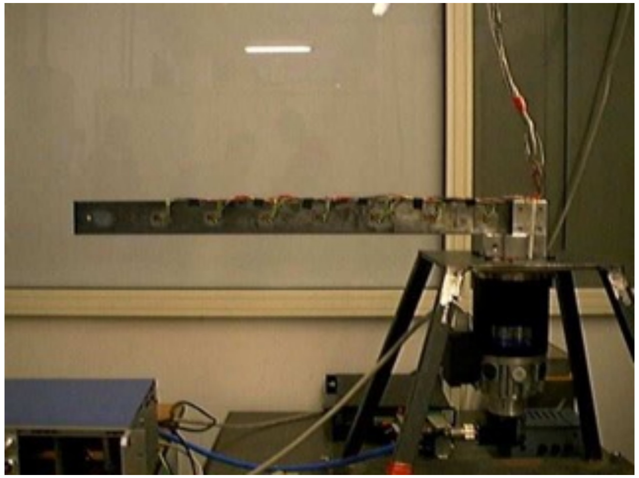
Georgia Tech
(Prof. Wayne Book)



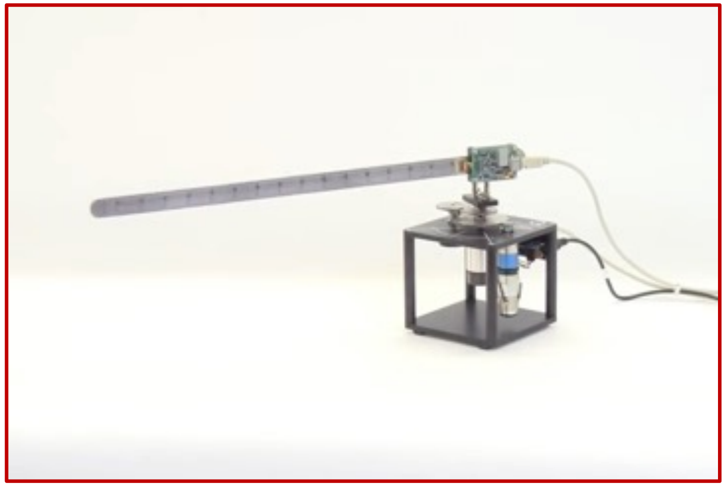
response of joint-level PID to external disturbance

Robots with flexible links

One-link prototypes



DMA - Sapienza harmonic steel beam (0.5 kg),
Direct-Drive DC motor, encoder, 7 strain gauges



video

QUANSER Rotary Flexible Link:
with strain feedback

video

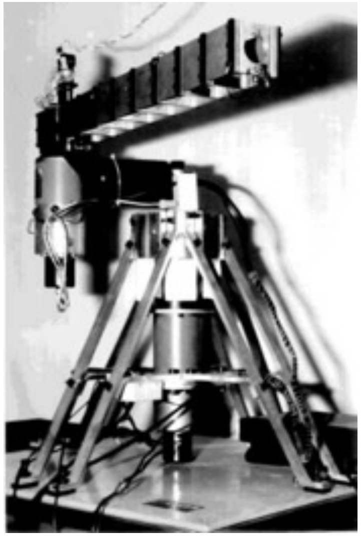
CUNY Brooklyn:
vision-driven + strain feedback

video

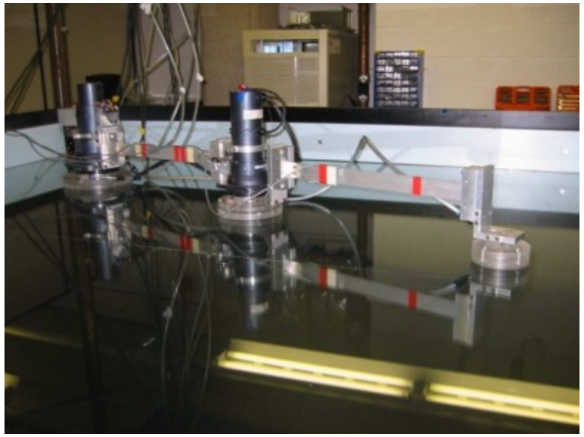
IS Técnico Lisbon: with two
piezoelectric sensing/actuation pairs

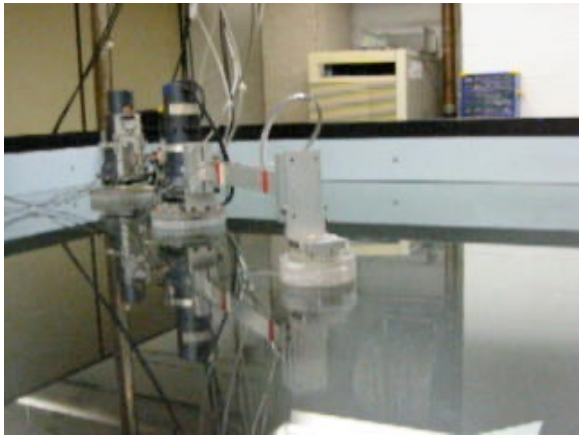
Robots with flexible links

Planar two-link prototypes

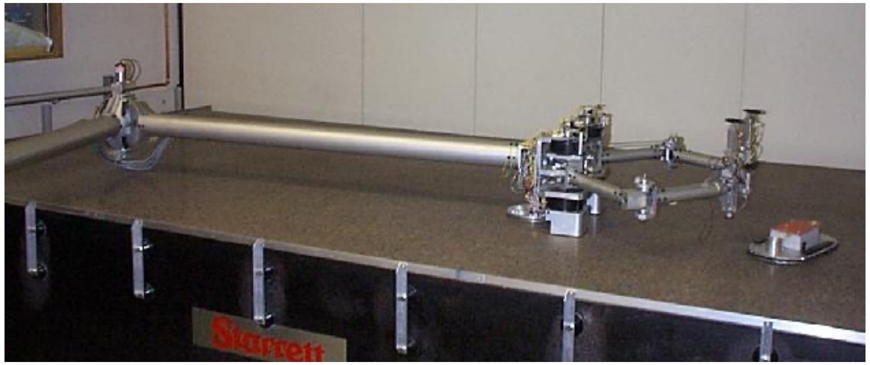


DIS/DIAG FLEXARM - Sapienza
planar two-link with
flexible forearm (0.7 m, 1.8 kg),
Direct-Drive DC motors,
encoders, on-board optical
sensor measuring deformation
at three points





video

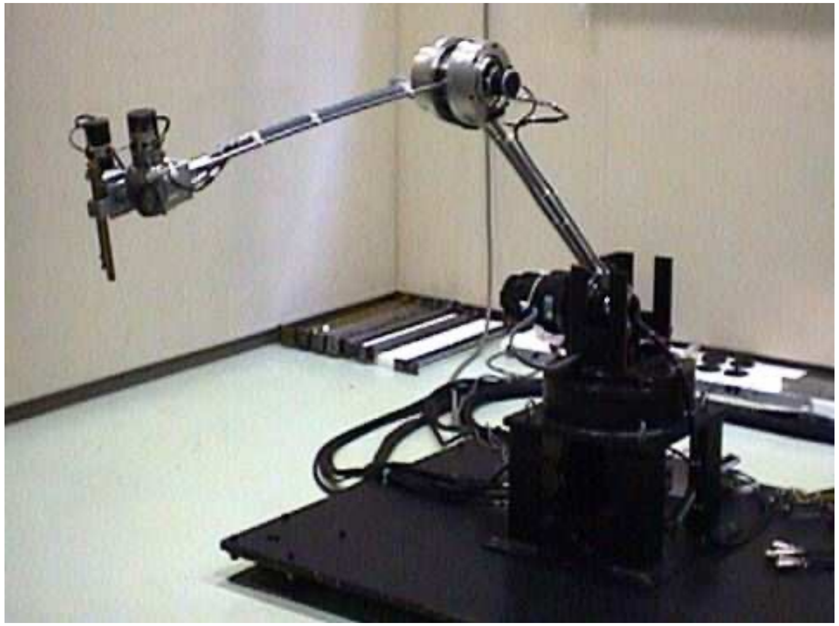


ARL Stanford two-link macro flexible arm,
with mini manipulator at the end
Stanford University
(Profs. Stephen Rock and Robert Cannon Jr.)

WATFLEX planar arm with two flexible links
(each with 2 strain gauges), encoders and tachos,
overviewing CCD camera, moving on air bearings
University of Waterloo (Prof. John McPhee)

Robots with flexible links

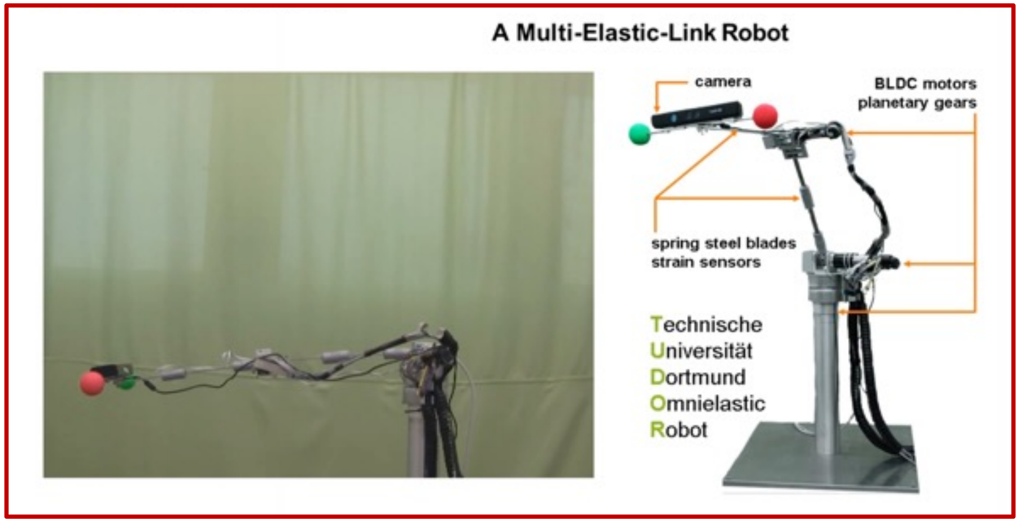
Spatial multi-link prototypes



Kyoto spatial 3R flexible arm
Kyoto University (Prof. Tsuneo Yoshikawa)

RST – TUDOR spatial 3R flexible arm
Technical University Dortmund
(Dr. Jorn Malzahn and Prof. Torsten Bertram)

video

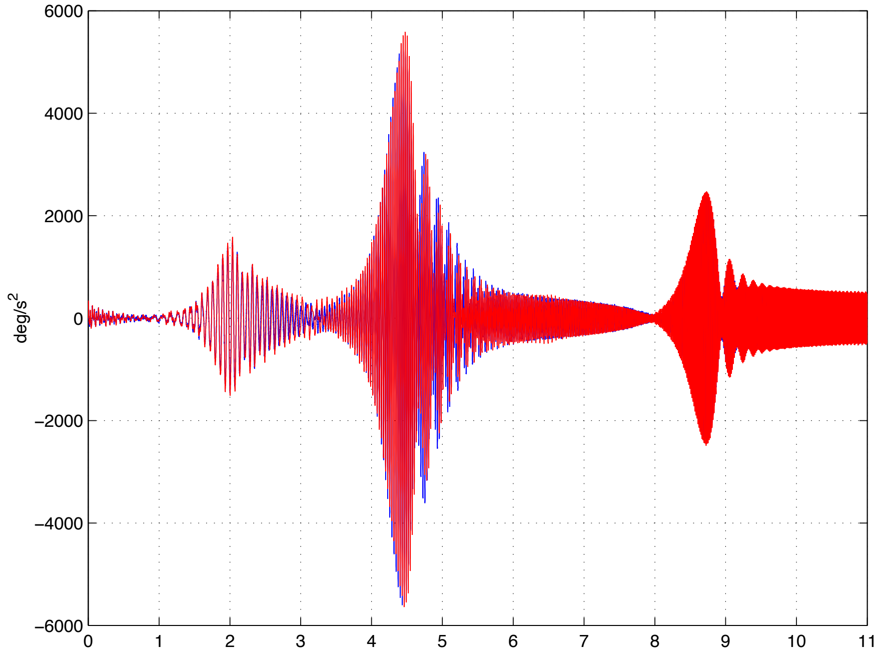


video

Frequency identification

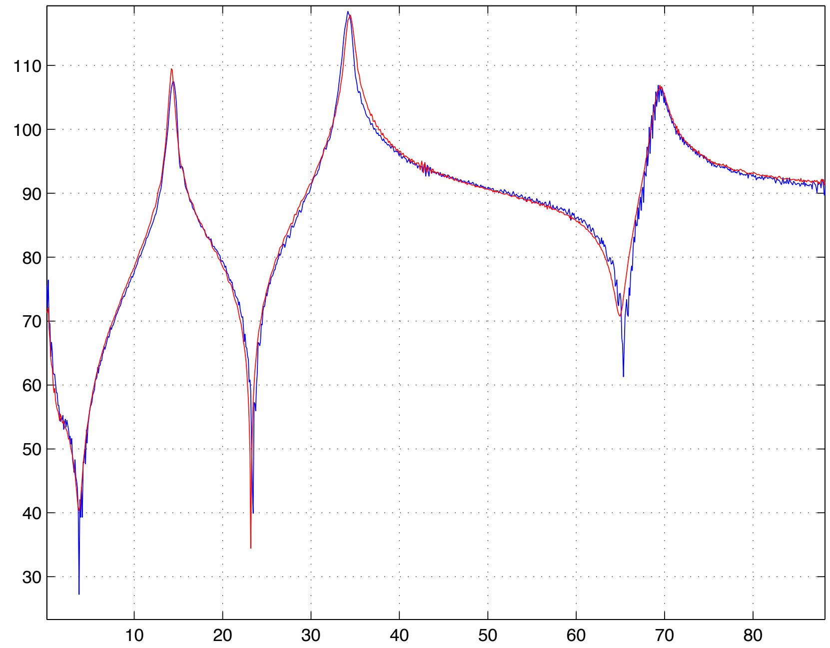
Single flexible link (DMA – Sapienza)

- experimental tests and dynamic model validation



[sec]

frequency sweep joint acceleration signal
plant vs. model



[Hz]

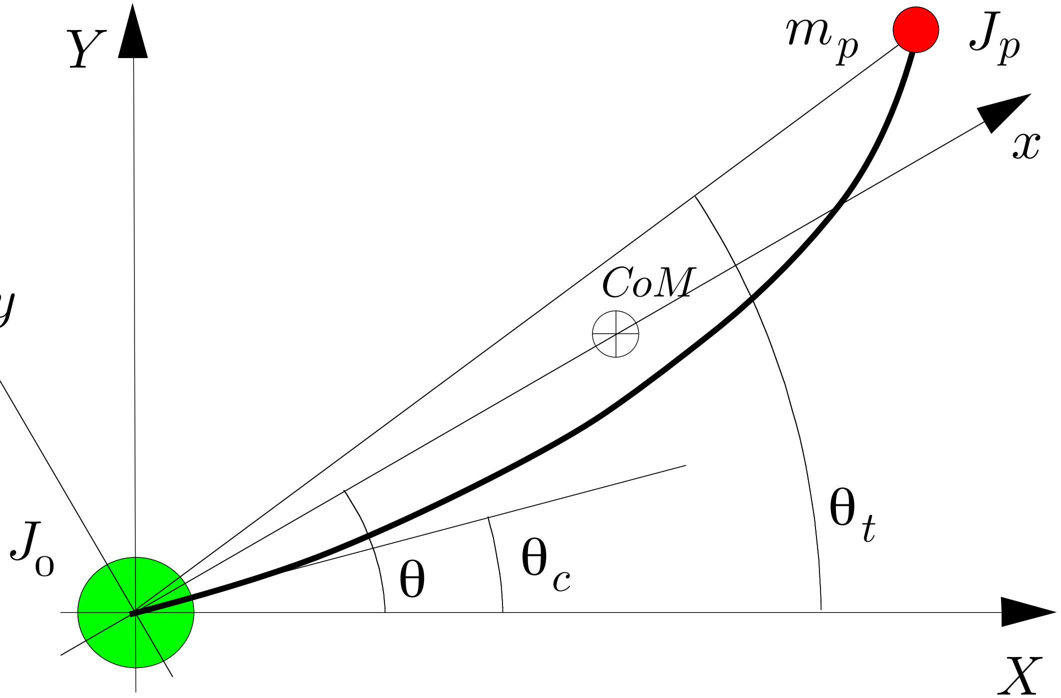
joint acceleration frequency response
plant vs. model

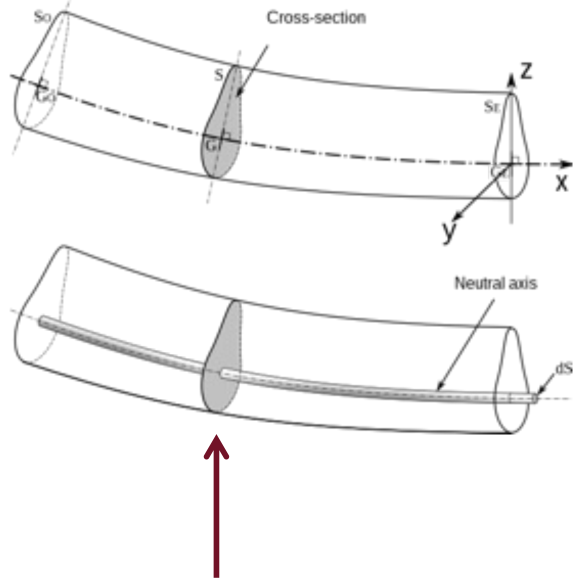
matching (within 1%) of resonances at
$f_1 = 14.4$, $f_2 = 34.2$, and $f_3 = 69.3$ Hz

Dynamic modeling

Single flexible link

- one-link flexible arm modeled as a Euler-Bernoulli beam in rotation





- length ℓ, uniform density ρ, Young modulus · cross-section inertia EI
- actuator inertia J_0, payload mass m_p and inertia J_p
- frames: (X, Y) absolute; (x, y) moving with instantaneous CoM

Dynamic modeling

Assumptions and definitions

- Euler-Bernoulli theory applies to slender arm design
  - length $\gg$ section dimensions
- beam undergoes small deformations of the pure bending type
  - restricted to the horizontal plane of motion (no gravity)
  - no torsion nor compression
- bending deformation $w(x,t)$, with $x \in [0,\ell]$ is directed along y-axis
  - no shear
- neglect isoperimetric constraint & rotational inertia of beam sections
  - $\rightarrow$ 'extension' of beam neutral axis negligible; $\rightarrow$ Timoshenko theory
- definition of relevant angular variables
  - position $\theta(t)$ of the CoM (not measurable, but very convenient)
  - position $\theta_c(t)$ of the tangent to the link base (measured by motor encoder)
  - position $\theta_t(t)$ of a line pointing to the beam tip (measurable in several ways)

Dynamic modeling

Basic steps

- build the Lagrangian from kinetic and potential energy of the arm
- using Hamilton principle and calculus of variations, the bending deformation and the angle satisfy the linear differential equations

$$EIw''''(x,t) + \rho\left(\ddot{w}(x,t) + x\ddot{\theta}(t)\right) = 0 \qquad \tau(t) - J\ddot{\theta}(t) = 0$$

i.e., a PDE (for the beam) and an ODE (for the rigid motion), with

$$J = J_0 + (\rho\ell^3)/3 + J_p + m_p\ell^2 \qquad \tau = \text{torque input}$$

- geometric/dynamic boundary conditions (b.c.'s) associated to PDE

$$w(0,t) = 0 \qquad \text{(no deformation at base } x = 0\text{)}$$

$$EIw''(0,t) = J_0\left(\ddot{\theta}(t) + \ddot{w}'(0,t)\right) - \tau(t) \qquad \text{(balance of moments at base)}$$

$$EIw''(\ell,t) = -J_p\left(\ddot{\theta}(t) + \ddot{w}'(\ell,t)\right) \qquad \text{(balance of moments at tip)}$$

$$EIw'''(\ell,t) = m_p\left(\ell\ddot{\theta}(t) + \ddot{w}(\ell,t)\right) \qquad \text{(balance of shear forces at tip)}$$

Dynamic modeling

Solving the PDE and ODE

- in free evolution ($\tau(t) \equiv 0 \Rightarrow \ddot{\theta}(t) \equiv 0$), PDE is solved by separation of variables

$$w(x,t) = \phi(x)\delta(t) \qquad \Rightarrow \qquad \frac{EI}{\rho}\frac{\phi''''(x)}{\phi(x)} = -\frac{\ddot{\delta}(t)}{\delta(t)} = \omega^2$$

for a positive constant ω^2 (self-adjoint problem) to be determined

- time solution

$$\ddot{\delta}(t) = -\omega^2\delta(t) \qquad \Rightarrow \qquad \delta(t) = c_1 \sin\omega t + c_2 \cos\omega t$$

with c_1, c_2 depending on the initial conditions $\delta(0)$ and $\dot{\delta}(0)$

- space solution

$$\phi''''(x) = \beta^4\phi(x) \qquad \beta^4 = \frac{\rho\omega^2}{EI}$$

$$\Rightarrow \quad \phi(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

with A, B, C, D given by the geometric/dynamic b.c.'s on $w(x,t)$

Dynamic modeling

Solving the PDE and ODE

- from $w(x,t) = \phi(x)\delta(t)$ and $\ddot{\delta}(t) = -\omega^2\delta(t)$, and holding the b.c.'s for any $\delta(t)$, these are rewritten in terms of $\phi(x)$ only

$$\begin{aligned} \phi(0) &= 0 \\ EI\phi''(0) + J_0\,\omega^2\phi'(0) &= 0 \\ EI\phi''(\ell) - J_p\,\omega^2\phi'(\ell) &= 0 \\ EI\phi'''(\ell) + m_p\,\omega^2\phi(\ell) &= 0 \end{aligned}$$

- using the general solution for $\phi(x)$, a system of linear homogeneous equations follows

$$\left[\quad \mathcal{A}(EI,\rho,\ell,J_0,m_p,m_p,\beta) \quad\right]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \qquad (\blacksquare)$$

to exclude the trivial solution, the determinant of matrix $\mathcal{A}$ should be set to zero (eigenvalue problem)

Dynamic modeling

Characteristic equation

- $\det \mathcal{A}(\beta) = 0$ at infinite (but countable!) real, positive, increasing roots $\beta = \beta_i \ (i = 1,2,\dots)$ of a transcendental characteristic equation

$$(c\,sh - s\,ch) - \frac{2\,m_p}{\varrho}\beta\,s\,sh - \frac{m_p}{\varrho^2}\beta^4 (J_0 + J_p)(c\,sh - s\,ch) - \frac{2\,J_p}{\varrho}\beta^3\,c\,ch$$
$$- \frac{2\,J_0}{\varrho}\beta^3 (1 + c\,ch) + \frac{J_0\,J_p}{\varrho^2}\beta^6 (c\,sh + s\,ch) - \frac{m_p J_0\,J_p}{\varrho^3}\beta^7 (1 - c\,ch) = 0$$

where $s = \sin\beta$, $c = \cos\beta$, $sh = \sinh\beta$, $ch = \cosh\beta$

- this is an exact result that includes common physical approximations
  - pinned-free model: $J_0 = m_p = J_p = 0 \quad \Rightarrow \quad c\,sh - s\,ch = 0$
  - clamped-free model: $J_0 \to \infty,\ m_p = J_p = 0 \Rightarrow \quad 1 + c\,ch = 0$

cantilever beam characteristic equation

Dynamic modeling

Eigenvalues (frequencies) and eigenvectors (modes)

- associated to each root $\beta_i > 0$ of the characteristic equation we have
  - an eigenfrequency $\omega_i = \sqrt{EI\beta_i^4/\rho}$ characterizing a resonance (system vibration)
  - an eigenmode $\phi_i(x)$ —a spatial shape of the deformed arm (defined up to a constant)
  - a deflection time variable $\delta_i(t)$ (oscillatory) weighting the shape
- a finite-dimensional approximation of the distributed bending deformation is obtained by truncation

$$w(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\,\delta_i(t) \approx \sum_{i=1}^{n_e} \phi_i(x)\,\delta_i(t)$$

where n_e is the (arbitrary) number of orthogonal modes included

- a proper normalization of the eigenmodes is chosen (an integral of $\phi_i(x)$ and $\phi_i'(x)$ equals 1 — or equals the total link mass m ...)

Dynamic model

Equations of motion of a single flexible link

- add motor torque τ (performing work on the rhs of the E-L equations)
- the final dynamic model is simple (after a quite complex analysis…)

$$
\begin{aligned}
J\ddot{\theta} &= \tau \\
\ddot{\delta}_i + \omega_i^2 \delta_i &= \phi_i'(0)\tau \qquad i = 1,2,\dots,n_e
\end{aligned}
$$

- notable properties
  - rigid body motion $\theta(t)$ and each vibratory deflection $\delta_i(t)$ are dynamically decoupled when the system is in free evolution ($\tau(t) \equiv 0$)
  - each mode is excited by an input $\tau(t)$, with a weight that depends on $\phi_i'(0)$ —the tangent at the link base to the i-th deformation mode shape
  - arm stiffness is summarized by the (squared) eigenfrequencies ω_i^2
  - each vibration mode is persistent during free evolution, if it is initially excited by $\delta_i(0) \neq 0$ (absence of damping in the modeling process)

Dynamic model

Addition of dissipative effects

- modal damping can be easily included in the dynamic model

$$J\ddot{\theta} = \tau$$
$$\ddot{\delta}_i + 2\zeta_i\omega_i\dot{\delta}_i + \omega_i^2\delta_i = \phi_i'(0)\tau \qquad i = 1,2,\dots,n_e$$

with damping coefficients $\zeta_i \in [0,1)$

- its matrix version, with coordinates $q = \left(\theta\ \delta_1 \dots \delta_{n_e}\right)^T \in \mathbb{R}^{n_e+1}$, shows the classical mass-spring-damper form

$$M\ddot{q} + D\dot{q} + Kq = B\tau$$

with

$$M = \begin{pmatrix} J & \\ & I_{n_e} \end{pmatrix} \quad D = \begin{pmatrix} 0 & \\ & 2Z\Omega \end{pmatrix} \quad K = \begin{pmatrix} 0 & \\ & \Omega^2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ \Phi' \end{pmatrix}$$

$$\Omega = \operatorname{diag}\{\omega_1 \dots \omega_{n_e}\} \quad Z = \operatorname{diag}\{\zeta_1 \dots \zeta_{n_e}\} \quad \Phi' = \operatorname{diag}\{\phi_1'(0) \dots \phi_{n_e}'(0)\}$$

Dynamic model

Change of coordinates

- with a different (but equivalent) choice of generalized coordinates, the input τ appears in just one equation

$$(\theta, \delta) = (\theta, \delta_1, \dots, \delta_{n_e})$$

$$\Downarrow$$

$$(\theta_c, \delta) = (\theta + \delta^T \Phi', \delta) = \left(\theta + \sum_{i=1}^{n_e} \phi_i'(0)\, \delta_i, \delta_1, \dots, \delta_{n_e}\right)$$

↑ clamped angle at beam base

leading to

$$\begin{pmatrix} J & -J\Phi'^T \\ -J\Phi' & I_{n_e} + J^2 \Phi' \Phi'^T \end{pmatrix} \begin{pmatrix} \ddot{\theta}_c \\ \ddot{\delta} \end{pmatrix} + \begin{pmatrix} F_v & \\ & 2Z\Omega \end{pmatrix} \begin{pmatrix} \dot{\theta}_c \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & \\ & \Omega^2 \end{pmatrix} \begin{pmatrix} \theta_c \\ \delta \end{pmatrix} = \begin{pmatrix} \tau \\ \mathbf{0} \end{pmatrix}$$

with diagonal damping matrix D (including motor viscous friction F_v), same stiffness K matrix, but full inertia matrix M

Choice of system output

Different angles to be controlled

- joint level (clamped angle)

$$y = \theta_c = \theta + \sum_{i=1}^{n_e} \phi_i'(0)\,\delta_i \qquad \lim_{x\to 0} \frac{\phi_i(x)}{x} = \phi_i'(0)!$$

always minimum phase: no zeros in right-hand side of complex plane

- tip level (angle pointing to the tip)

$$y = \theta_t = \theta + \sum_{i=1}^{n_e} \frac{\phi_i(\ell)}{\ell}\,\delta_i$$

is typically non-minimum phase (at least for no tip payload)

- angular output at a point $x \in [0, \ell]$ along the flexible beam

$$y = \theta_x = \theta + \sum_{i=1}^{n_e} \frac{\phi_i(x)}{x}\,\delta_i$$

various cases: may also have no zeros!

Transfer functions

Joint and tip level

- torque $\tau \longmapsto$ clamped joint angle θ_c

$$P_c(s) = \frac{\theta_c(s)}{\tau(s)} = \frac{1}{Js^2} + \sum_{i=1}^{n_e} \frac{\phi_i'(0)^2}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

$$= \frac{\frac{1}{J}\prod_{i=1}^{n_e}\left(s^2 + 2\zeta_i\omega_i s + \omega_i^2\right) + s^2 \sum_{i=1}^{n_e} \phi_i'(0)^2 \prod_{j\neq i}^{n_e}\left(s^2 + 2\zeta_j\omega_j s + \omega_j^2\right)}{s^2 \prod_{i=1}^{n_e}\left(s^2 + 2\zeta_i\omega_i s + \omega_i^2\right)}$$

- torque $\tau \longmapsto$ tip angle θ_t

$$P_t(s) = \frac{\theta_t(s)}{\tau(s)} = \frac{1}{Js^2} + \sum_{i=1}^{n_e} \frac{\phi_i'(0)\,\phi_i(\ell)/\ell}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

$$= \frac{\frac{1}{J}\prod_{i=1}^{n_e}\left(s^2 + 2\zeta_i\omega_i s + \omega_i^2\right) + s^2 \sum_{i=1}^{n_e}\left(\phi_i'(0)\frac{\phi_i(\ell)}{\ell}\right) \prod_{j\neq i}^{n_e}\left(s^2 + 2\zeta_j\omega_j s + \omega_j^2\right)}{s^2 \prod_{i=1}^{n_e}\left(s^2 + 2\zeta_i\omega_i s + \omega_i^2\right)}$$

A numerical example

A simple MATLAB code is available …

- physical data of the flexible arm –without payload ($m_p = J_p = 0$)

$$J_0 = 0.002 \left[\frac{Nm}{s^2}\right], \ell = 1\ [m], \rho = 0.5 \left[\frac{kg}{m}\right], EI = 1\ [Nm^2]$$

- by considering up to $n_e = 5$ modes (and no damping), we obtain

$$\Omega^2 = \text{diag}\ \{421.585, 3122.603, 10273.194, 31562.286, 82049.350\}$$

$$\omega_i^2 = (2\pi f_i)^2 \quad \Rightarrow \quad \text{e.g., } f_1 = \sqrt{421.585}/2\pi = 3.2678\ [\text{Hz}]$$

$$\Phi'^T = [\,7.8259 \quad 14.6803 \quad 12.1284 \quad 6.4761 \quad 3.7648\,]$$

$$\Phi_\ell^T = [\,-2.6954 \quad 2.3268 \quad -2.4970 \quad 2.7380 \quad -2.7982\,]$$

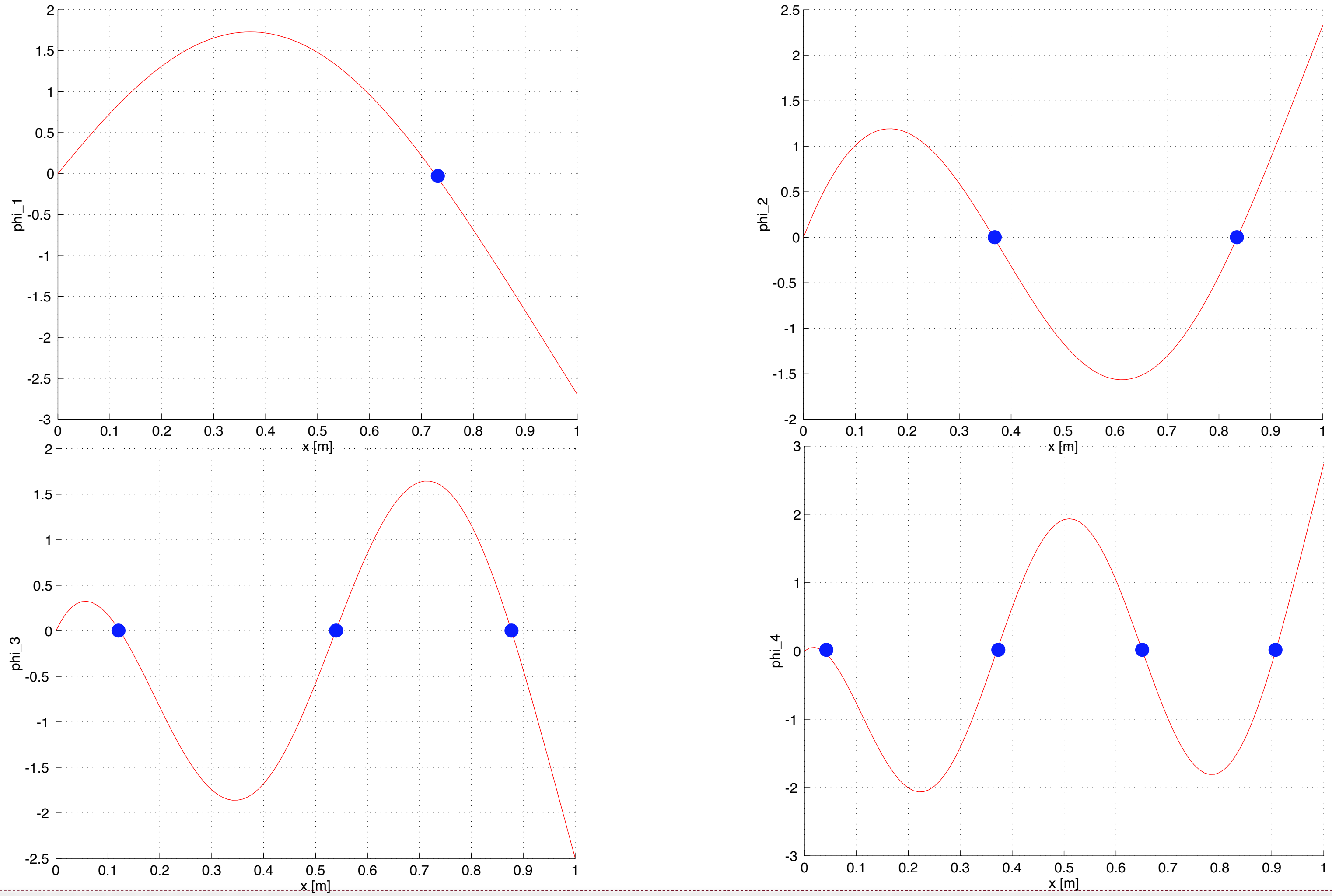
… note the alternating signs in the sequence of $\phi_i(\ell)$'s

Mode shapes

Shapes of spatial dynamic deformations of the flexible arm

- first four bending mode shapes (normalized to 1) at resonant frequencies

$$f_1 = 3.2678,\ f_2 = 8.8936,\ \ f_3 = 16.1314,\ \ f_4 = 28.2751\ \text{[Hz]}$$



nodes w.r.t. the neutral axis = modal index i

Pole-zero patterns

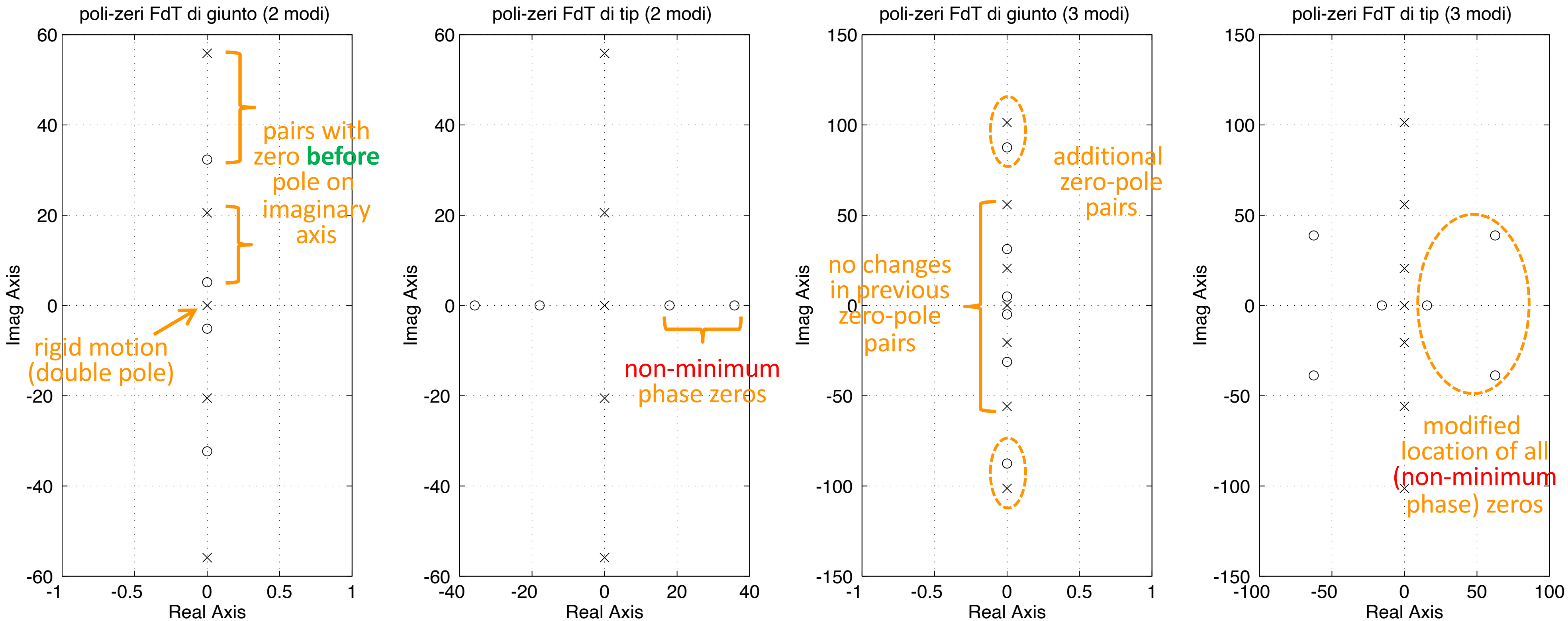
Joint and tip transfer functions (no modal damping)

- first two modes

  $\omega_1 = 20.5325, \ \omega_2 = 55.8801$ [rad/s]

- adding the third mode

  $\omega_3 = 101.3565$ [rad/s]



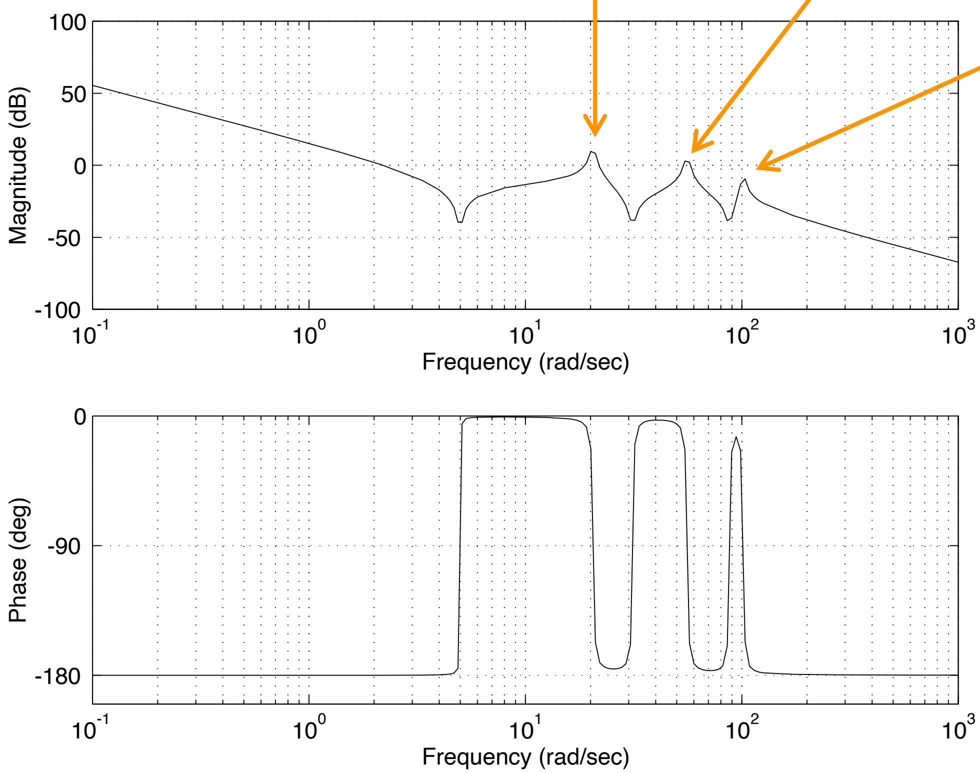
clamped joint output | tip output | clamped joint output | tip output

Frequency responses

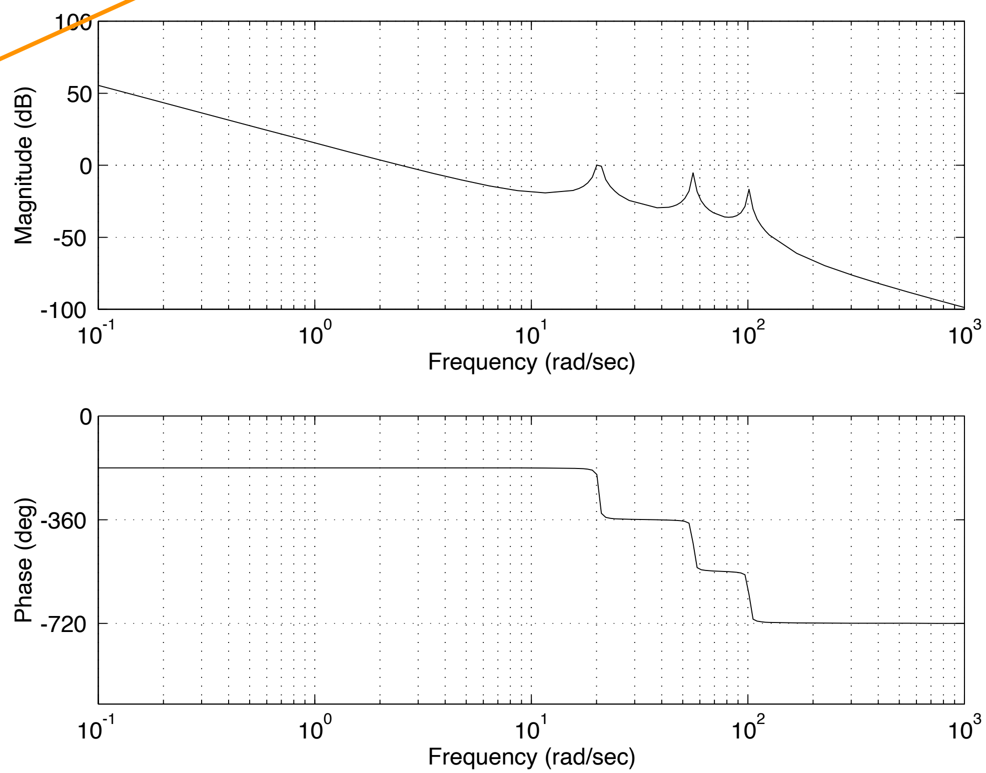
Bode plots with the first three modes of the flexible arm

$$\omega_1 = 20.5325,\ \omega_2 = 55.8801,\ \omega_3 = 101.3565\ [\text{rad/s}]$$

clamped joint output



tip output



mag: multiple anti-resonance/resonance patterns (similar as the single pattern for an elastic joint)
phase: nominally, there is always a stability margin

mag: pure resonances (no effect of specular zeros), with multiple 0dB crossing if gain is increased
phase: phase lag increases when adding modes …

Control-oriented remarks

Single flexible link

- in the pole-zero patterns of $P_c(s)$, zeros always precede and alternate with poles on the imaginary axis $\Rightarrow$ input-output passivity property!
- the zero patterns of $P_t(s)$ are always symmetric w.r.t. the imaginary axis $\Rightarrow$ non-minimum phase property $\Rightarrow$ no (direct) system inversion is feasible!
  - similar properties can be seen also from the frequency responses (Bode plots)
- modal damping does not modify the non-minimum phase nature of $P_t(s)$
  - it destroys the perfect symmetries in the zero-pole patterns of $P_c(s)$ or $P_t(s)$, but the open-loop system remains anyway asymptotically stable
- when `moving' the output along the link ($P_x(s)$), zeros migrate on the imaginary axis and different phenomena occur
  - total pole-zero cancellation when pointing at CoM (vibrations become unobservable from the rigid motion variable θ)
  - for a special $x^* \in (0, \ell)$, all zeros vanish together at infinity: $P_{x^*}(s)$ has then maximum relative degree equal to $2(n_e + 1)$
  - beyond x^* (e.g., for $x = \ell$, at the tip), all pairs of zeros reappear in $\mathbb{R}^+/\mathbb{R}^-$

Dynamic modeling

Robots with multiple flexible links

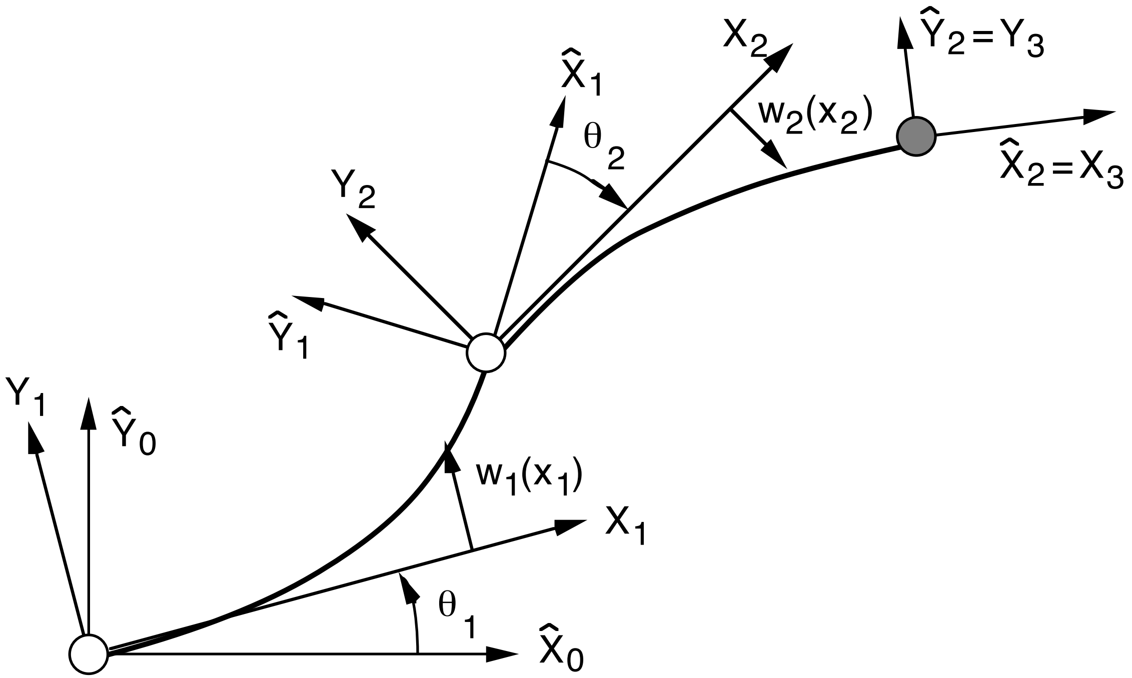
- a convenient kinematic description should be adopted, both for rigid body motion and flexible deformation
- differential relationships for computing kinetic and potential energy, within a Lagrangian approach
- use recursive procedures for open chains of flexible links, as in rigid case
- modeling results from the single link case can be embedded (with caution on boundary conditions) in the description of each flexible link of the robot
- to limit complexity, we sketch here only the planar case
  - robots with N flexible links
  - under small bending deformations limited to the plane of motion
  - possibly including gravity

video

QUANSER 2 DOF Serial Flexible Link with strain feedback

Kinematics

Planar robots with multiple flexible links



here, $N = 2$

for link i

- rigid motion by clamped angle $\theta_i(t)$; lateral bending $w_i(x_i, t)$, $x_i \in [0, \ell_i]$
- position vectors and (rigid/flexible) rotation matrices ($w'_{ie} = \left.\frac{\partial w_i}{\partial x_i}\right|_{x_i = \ell_i}$)

$$ {}^i p_i(x_i) = \begin{pmatrix} x_i \\ w_i(x_i) \end{pmatrix} \qquad A_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \qquad E_i = \begin{pmatrix} 1 & -w'_{ie} \\ w'_{ie} & 1 \end{pmatrix} $$

$$ {}^i r_{i+1} = {}^i p_i(\ell_i) $$

Kinematics

Planar robots with multiple flexible links

- recursive equations for absolute quantities in base frame $(\hat{X}_0, \hat{Y}_0)$

$$p_i = r_i + W_i\ {}^i p_i \qquad r_{i+1} = r_i + W_i\ {}^i r_{i+1} \qquad W_i = W_{i-1}\, E_{i-1}\, A_i$$

- differential kinematics
  - absolute angular velocity of frame (X_i, Y_i)

$$\dot{\alpha}_i = \sum_{j=1}^{i} \dot{\theta}_j + \sum_{k=1}^{i-1} \dot{w}'_{ke}$$

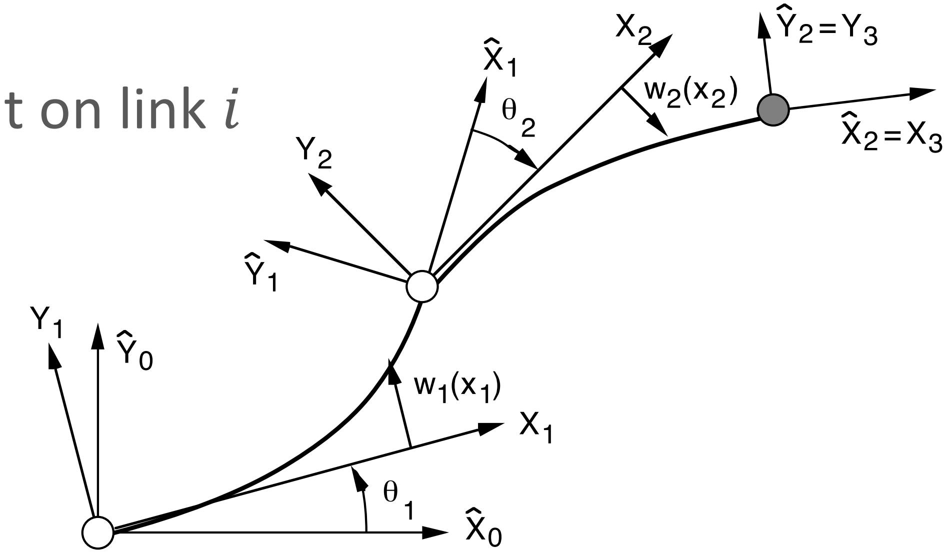
  - absolute angular velocity of a point on link i

$$\dot{p}_i = \dot{r}_i + \dot{W}_i\ {}^i p_i + W_i\ {}^i \dot{p}_i$$

with

$${}^i\dot{p}_i = \begin{pmatrix} 0 \\ \dot{w}_i(x_i) \end{pmatrix}$$

link extension is neglected



Kinetic and potential energy

Planar robots with multiple flexible links

$$T = \sum_{i=1}^{N} T_{hi} + \sum_{i=1}^{N} T_{\ell i} + T_p$$

- kinetic energy of hub *i*

$$T_{hi} = \frac{1}{2} m_{hi} \dot{r}_i^T \dot{r}_i + \frac{1}{2} J_{hi} \dot{\alpha}_i^{\,2}$$

- kinetic energy of link *i*

$$T_{\ell i} = \frac{1}{2} \int_0^{\ell_i} \rho_i(x_i) \dot{p}_i^T(x_i) \dot{p}_i(x_i)\, dx_i$$

- kinetic energy of payload

$$T_p = \frac{1}{2} m_p \dot{r}_{N+1}^T \dot{r}_{N+1} + \frac{1}{2} J_p (\dot{\alpha}_N + \dot{w}'_{Ne})^2$$

$$U = \sum_{i=1}^{N} U_{ghi} + \sum_{i=1}^{N} U_{g\ell i} + U_{gp} + \sum_{i=1}^{N} U_{ei}$$

- gravitational energy of hub *i*

$$U_{ghi} = -m_{hi} g_0^T r_i$$

- gravitational energy of link *i*

$$U_{g\ell i} = -g_0^T \int_0^{\ell_i} \rho_i(x_i)\, p_i(x_i) dx_i$$

- gravitational energy of payload

$$U_{gp} = -m_p g_0^T r_{N+1}$$

- elastic energy of link *i*

$$U_{ei} = \frac{1}{2} \int_0^{\ell_i} (EI)_i(x_i) \left(\frac{d^2 w_i(x_i)}{dx_i^2} \right)^2 dx_i$$

Euler-Lagrange equations

Planar robots with multiple flexible links

- introduce any finite-dimensional discretization for $w_i(x_i,t)$

$$w_i(x_i,t) = \sum_{j=1}^{n_{ei}} \varphi_{ij}(x_i)\delta_{ij}(t) \qquad i = 1, \dots, N$$

- the Lagrangian is given in terms of $N + M$ generalized coordinates, with $M = \sum_{i=1}^{N} n_{ei}$ (flexible variables)

$$L = T - U = L(\{\theta_i(t)\}, \{\delta_{ij}(t)\}, \{\dot{\theta}_i(t)\}, \{\dot{\delta}_{ij}(t)\})$$

and satisfies to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = \tau_i \qquad i = 1, \dots, N$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_{ij}}\right) - \frac{\partial L}{\partial \delta_{ij}} = \mathbf{0} \qquad j = 1, \dots, n_{ei} \quad i = 1, \dots, N$$

being τ_i the torque delivered by the actuator at joint i

Dynamic model

Planar robots with multiple flexible links

- the general dynamic model (with modal damping) is then given by

$$\begin{pmatrix} M_{\theta\theta}(\theta,\delta) & M_{\theta\delta}(\theta,\delta) \\ M_{\theta\delta}^T(\theta,\delta) & M_{\delta\delta}(\theta,\delta) \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\delta} \end{pmatrix} + \begin{pmatrix} c_\theta(\theta,\delta,\dot{\theta},\dot{\delta}) \\ c_\delta(\theta,\delta,\dot{\theta},\dot{\delta}) \end{pmatrix}$$

$$+ \begin{pmatrix} g_\theta(\theta,\delta) \\ g_\delta(\theta,\delta) \end{pmatrix} + \begin{pmatrix} 0 \\ D\dot{\delta} + K\delta \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

  with blocks of suitable sizes (e.g., $M_{\theta\delta}$ in the inertia matrix is $N \times M$)

- … or in the more compact form

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) + \begin{pmatrix} 0 \\ D\dot{\delta} + K\delta \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

  being $q = (\theta,\delta) \in \mathbb{R}^{N+M}$

- as in the rigid case, the vector of centrifugal/Coriolis terms can be factorized using the Christoffel symbols

$$c(q,\dot{q}) = S(q,\dot{q})\dot{q} = \begin{pmatrix} S_{\theta\theta}(q,\dot{q}) & S_{\theta\delta}(q,\dot{q}) \\ S_{\delta\theta}(q,\dot{q}) & S_{\delta\delta}(q,\dot{q}) \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\delta} \end{pmatrix}$$

Model properties

Planar robots with multiple flexible links

- matrix $\dot{M} - 2S$ is skew-symmetric —also blockwise, e.g., $\dot{M}_{\delta\delta} - 2S_{\delta\delta}$
- the dynamics of flexible robots can be expressed in terms of a set of dynamic coefficients $a \in \mathbb{R}^p$ that summarize the mechanical (rigid + flexible) properties of the links

$$Y(\theta, \delta, \dot{\theta}, \dot{\delta}, \ddot{\theta}, \ddot{\delta})\, a = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

  - a linear parametrization is useful for the experimental identification of a
- possible choices of the assumed modes — i.e., the basis functions $\varphi_{ij}(x_i)$ for describing the bending deformation shapes of the links
  - admissible functions satisfy only geometric b.c.'s
  - comparison functions (Finite Elements, Ritz-Kantorovich expansion) satisfy also natural b.c.'s
  - orthonormal eigenfunctions (links models as Euler-Bernoulli beams) lead to simplifications in inertia submatrix $M_{\delta\delta}$ (block diagonal, constant)

Some model simplifications

Planar robots with multiple flexible links

- a common approximation evaluates the total kinetic energy in the undeformed arm configuration, i.e., with deflections $\delta = 0$
  - $\Rightarrow M = M(\theta)$, and thus $c = c(\theta, \dot{\theta}, \dot{\delta})$
  - $\Rightarrow c_\delta$ loses its quadratic dependence on $\dot{\delta}$
- moreover, if $M_{\delta\delta}$ is constant
  - $\Rightarrow c_\delta$ becomes a quadratic function of $\dot{\theta}$ only
  - $\Rightarrow c_\theta$ loses its quadratic dependence on $\dot{\delta}$
- if also $M_{\theta\delta}$ is constant
  - $\Rightarrow c_\delta \equiv 0$
  - $\Rightarrow c_\theta$ becomes a quadratic function of $\dot{\theta}$ only
- assumption of small deformations of each link implies $g_\delta = g_\delta(\theta)$

Control problems

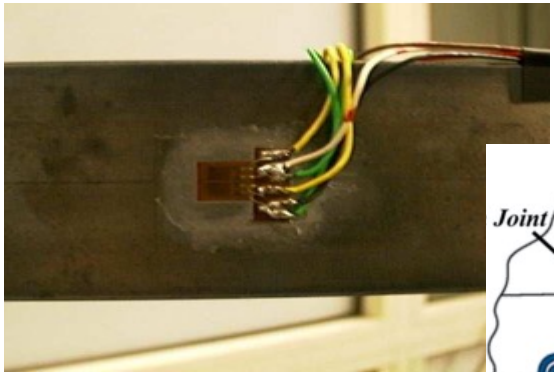
Formulation of objectives and operative conditions

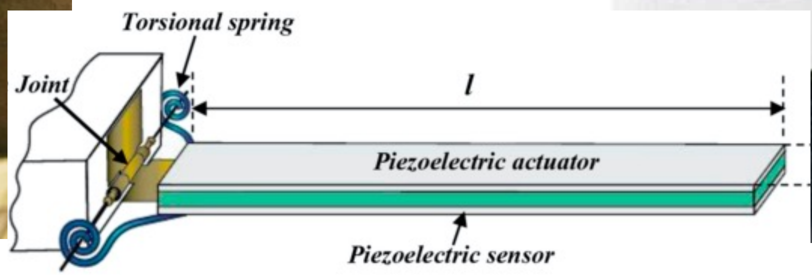
- regulation to an equilibrium configuration $(\theta, \delta, \dot{\theta}, \dot{\delta}) = (\theta_d, \delta_d, 0, 0)$
  - only a desired joint position θ_d is given, δ_d is to be determined
  - may use full or partial state feedback, depending on available sensors
  - θ_d may come from the kineto-static inversion of a desired Cartesian pose/position r_d, although no closed-form inverse solution exists
  - direct kinematics of flexible link robots is in fact a function of all the rigid and flexible variables: $r = \mathrm{kin}(\theta, \delta)$
- asymptotic tracking of a joint trajectory $\theta_d(t)$ —*the easy case*
- asymptotic tracking of an end-effector trajectory $r_d(t)$ —*more difficult*
  - in both cases, we assume that the full state is measurable
  - tracking control laws will stiffen the flexible arm at the chosen output
- rest-to-rest motion in given time T (not just a trajectory planning task!)

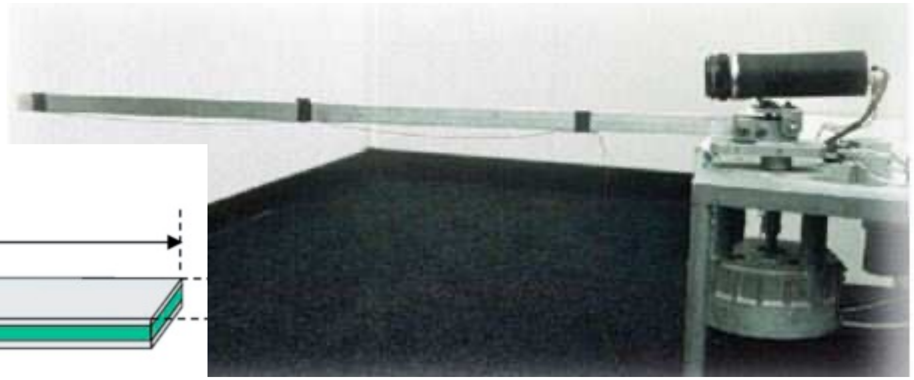
Sensing requirements

For full or partial state feedback

- full state feedback requires sensing of
  - joint/motor position and velocity variables θ (encoders) and $\dot{\theta}$ (tacho)
  - deflection variables δ and deflection rates $\dot{\delta}$ (no direct sensor available)
- at least an encoder on motor axis + online numerical differentiation
- different sensors can measure the link deflection δ (or deformation related quantities), each with pros and cons
  - strain gauges, accelerometers, optical sensors, video camera (on board or fixed in workspace), piezoelectric actuation/sensing devices, ….
- use of state observers, especially in linear case (separation principle)







Regulation with joint PD + feedforward

Partial state feedback solution

- consider the control law

$$\tau = K_P(\theta_d - \theta) - K_D\dot{\theta} + g_\theta(\theta_d, \delta_d)$$

with symmetric (diagonal) $K_P > 0$, $K_D > 0$, and link deflection at steady state corresponding to θ_d given by

$$\delta_d = -K^{-1} g_\delta(\theta_d)$$

Theorem

If

$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha \qquad \text{and} \qquad \lambda_{min}\begin{pmatrix} K_P & 0 \\ 0 & K \end{pmatrix} > \alpha > 0$$

then the desired closed-loop equilibrium state $(\theta_d, \delta_d, 0, 0)$ is globally asymptotically stable ◀

Regulation with joint PD + feedforward

Sketch of analysis

- Lyapunov-based proof, using LaSalle (as in the flexible joint case*)
- determination of lower bound α
  - in view of small link deformations

$$U_e = \frac{1}{2}\delta^T K \delta \leq U_{e,max} \quad \Rightarrow \quad \|\delta\| \leq \sqrt{\frac{2U_{e,max}}{\lambda_{max}(K)}} < \infty$$

  - bound on the gradient of the gravitational term

$$\left\|\frac{\partial g}{\partial q}\right\| \leq \alpha_0 + \alpha_1\|\delta\| \leq \alpha_0 + \alpha_1\sqrt{\frac{2U_{e,max}}{\lambda_{max}(K)}} = \alpha$$

- in the absence of gravity, a pure PD law on the motor position error
- for a desired tip pose r_d, compute θ_d solving via iterative techniques

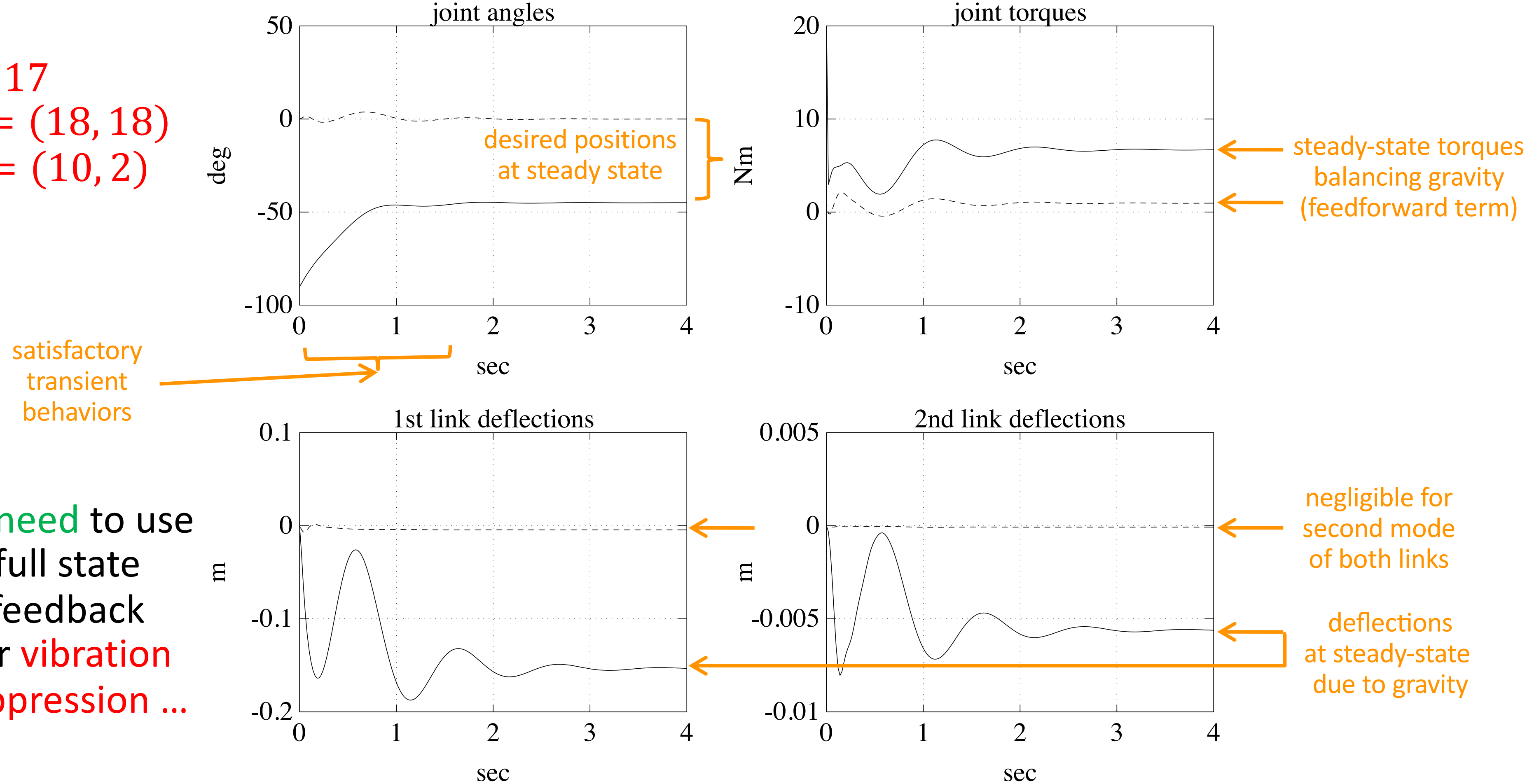
$$\mathrm{kin}\big(\theta, -K^{-1}g_\delta(\theta)\big) = r_d$$

Regulation with joint PD + feedforward

Numerical results

- a planar two-link flexible robot **with** gravity (in vertical plane), with two bending modes for each link at $f_{11} = 1.4,\ f_{12} = 5.1$ and $f_{21} = 5.3,\ f_{22} = 32.4$ [Hz]
- at rest from the downward vertical $\theta(0) = (-90°, 0°)$ to $\theta_d = (-45°, 0°)$

$\alpha \cong 17$
$K_P = (18, 18)$
$K_D = (10, 2)$



no need to use full state feedback for vibration suppression …

Joint trajectory tracking

Control design approach

- assume that
  - the dynamic model of the (planar) robot with flexible links is available
  - the system state is fully measurable
- given a desired joint trajectory $\theta_d(t) \in C^2$, we proceed by system inversion from the joint position output
- a nonlinear static state feedback is obtained that exactly linearizes and decouples the input-output behavior, leaving an unobservable internal (nonlinear) dynamics
- exponential stabilization of the output tracking error is performed on the linear side of the problem
- stability/boundedness of the internal dynamics should be enforced

Joint trajectory tracking

System input-output inversion

- from second set of M equations in dynamic model, solve (globally) for

$$\ddot{\delta} = -M_{\delta\delta}^{-1}\left(c_\delta + g_\delta + K\delta + D\dot{\delta} + M_{\theta\delta}^T\ddot{\theta}\right)$$

- plug it in the first set of N equations $\Rightarrow$ effects of flexible dynamics on rigid dynamics

$$\left(M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^T\right)\ddot{\theta} + c_\theta + g_\theta - M_{\theta\delta}M_{\delta\delta}^{-1}\left(c_\delta + g_\delta + K\delta + D\dot{\delta}\right) = \tau$$

- the matrix weighting $\ddot{\theta}$ has always full rank (as Schur complement of an invertible matrix)

$$\begin{pmatrix} M_{\theta\theta} & M_{\theta\delta} \\ M_{\theta\delta}^T & M_{\delta\delta} \end{pmatrix}\begin{pmatrix} I & 0 \\ -M_{\delta\delta}^{-1}M_{\theta\delta}^T & I \end{pmatrix} = \begin{pmatrix} M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^T & M_{\theta\delta} \\ 0 & M_{\delta\delta} \end{pmatrix}$$

- $\ddot{\theta}$ depends on τ in a nonsingular way, and thus the output θ has uniform vector relative degree $\{2, 2, \ldots, 2\}$

Joint trajectory tracking

Input-output decoupling and exact linearization

- define the nonlinear control law

$$\tau = \left(M_{\theta\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}M_{\theta\delta}^{T}\right) a + c_{\theta} + g_{\theta} - M_{\theta\delta}M_{\delta\delta}^{-1}\left(c_{\delta} + g_{\delta} + K\delta + D\dot{\delta}\right)$$

in which the only inversion needed is of the simpler inertia block $M_{\delta\delta}$

- the closed-loop system is

$$\ddot{\theta} = a$$
$$\ddot{\delta} = -M_{\delta\delta}^{-1}\left(M_{\theta\delta}^{T}\, a + c_{\delta} + g_{\delta} + D\dot{\delta} + K\delta\right)$$

- for exponentially stabilizing the output tracking error $e = \theta_d - \theta$, set

$$a = \ddot{\theta}_d + K_D\left(\dot{\theta}_d - \dot{\theta}\right) + K_P(\theta_d - \theta)$$

with (diagonal) $K_P > 0$, $K_D > 0$

Joint trajectory tracking

Analysis of the internal dynamics

- zero dynamics: when the output $\theta(t) \equiv 0$ (or is a constant)

$$\ddot{\delta} = -M_{\delta\delta}^{-1}\left(c_\delta + g_\delta + D\dot{\delta} + K\delta\right)$$

  has an asymptotically stable equilibrium at $\delta_e = -K^{-1} g_\delta(0)$

  - shown via Lyapunov argument (the entire closed-loop system is stable)

- clamped dynamics: when the output $\theta(t) \equiv \theta_d(t)$

$$\ddot{\delta} = -A_2(t)\dot{\delta} - A_1(t)\delta + f_\delta(t)$$

  where (in the simpler case of inertia matrix independent from δ)

$$f_\delta(t) = -M_{\delta\delta}^{-1}(\theta_d)\left(M_{\theta\delta}^T(\theta_d)\ddot{\theta}_d + c_\delta\left(\theta_d, \dot{\theta}_d\right) + g_\delta(\theta_d)\right)$$

$$A_1(t) = M_{\delta\delta}^{-1}(\theta_d)K$$

$$A_2(t) = M_{\delta\delta}^{-1}(\theta_d)D$$

  all time-varying functions are bounded ⇒ closed-loop stability holds

Joint trajectory tracking

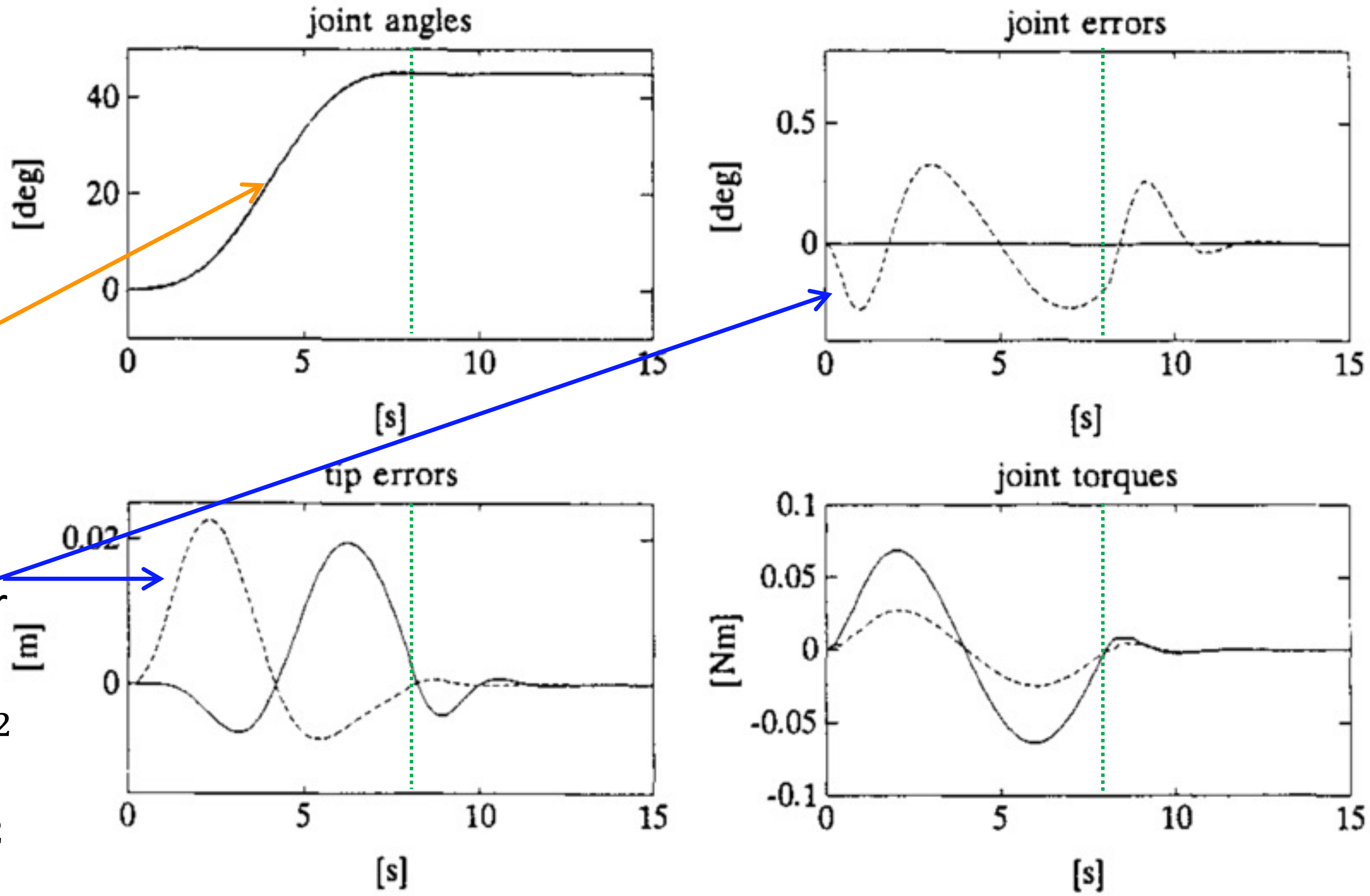
Numerical results

- a planar two-link flexible robot **without** gravity (in horizontal plane), with two modes for each link at $f_{11} = 0.48,\ f_{12} = 1.8$ and $f_{21} = 2.18,\ f_{22} = 15.9$ [Hz]
- rest-to-rest sinusoidal trajectory: $\theta_d(0) = (0°, 0°)$ to $\theta_d(T) = (45°, 45°)$ in $T = 8$ s

$K_P = (1, 4)$
$K_D = (2, 4)$
low gains, but "stiffer" at joint 2

perfect tracking on both joints

non-minimum phase behavior of 2nd link tip: error $y_{t2} - \theta_{d2}$ is opposite to error $\theta_2 - \theta_{d2}$



less than 0.3° error at second joint

moderate control torque efforts

—— = joint 1
- - - = joint 2

Joint trajectory tracking

Final remarks

- input-output linearization as nonlinear/MIMO counterpart of inverting $P_c(s) = \theta_c(s)/\tau(s)$ with minimum phase zeros (stable zero dynamics)
- the 'stiffer' is the tracking of a desired trajectory at the joint level, the less vibrational energy is dissipated in the rest of the flexible arm!
- a nominal feedforward is computed by integration of flexible dynamics

$$\ddot{\delta} = -M_{\delta\delta}^{-1}(\theta_d, \delta)\left(c_\delta\left(\theta_d, \delta, \dot{\theta}_d, \dot{\delta}\right) + g_\delta(\theta_d) + D\dot{\delta} + K\delta + M_{\theta\delta}^T(\theta_d, \delta)\ddot{\theta}_d\right)$$

  starting from $\delta_d(0) = \delta_0, \dot{\delta}_d(0) = \dot{\delta}_0$ (typically, both $= 0$) $\Rightarrow$ nominal (bounded) evolutions $\left(\delta_d(t), \dot{\delta}_d(t)\right)$ associated to the output $\theta_d(t)$
- use of $\left(\theta_d(t), \delta_d(t), \dot{\theta}_d(t), \dot{\delta}_d(t)\right)$ in the inversion control law (without nonlinear feedback) yields $\tau_d(t)$ and a simple local tracking controller

$$\tau = \tau_d(t) + K_D\left(\dot{\theta}_d(t) - \dot{\theta}\right) + K_P\left(\theta_d(t) - \theta\right)$$

End-effector trajectory tracking

Control design approaches

- accurate end-effector trajectory tracking is the `hardest’ control problem for robots with flexible links
- direct application of inversion control to the end-effector/tip output leads to closed-loop instability (viz. unboundedness of internal state)
  - linear (single-link) case: non-minimum phase tip transfer function
  - nonlinear (multilink) case: unstable zero dynamics in end-effector motion
- main ideas suggested in the literature
  - resort to tailored feedforward strategies (input shaping, flatness, non-causal bounded solutions for exact output trajectory reproduction)
  - use feedback for stabilization to a suitable state trajectory, avoiding cancelations (causal solutions for asymptotic output trajectory tracking)
- choice of smooth trajectories inducing smaller arm deflections is in any case of interest (but not sufficient)

Stable inversion of non-minimum phase system

Worked out SISO linear example for an exact and causal solution

- a plant with transfer function

$$P(s) = \frac{y(s)}{u(s)} = \frac{s-1}{s(s+2)}$$

- an equivalent minimal (reachable and observable) state-space realization

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad C(sI - A)^{-1}B = P(s)$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

or $\quad \dot{x}_1 = x_2 \qquad \dot{x}_2 = -2x_2 + u \qquad y = x_2 - x_1$

- desired output trajectory

$$y_d(t) = 1 - e^{-\alpha t} \qquad \alpha > 0 \qquad (y_d(0) = 0)$$

- we proceed first in the time domain and then in the Laplace domain

Stable inversion of non-minimum phase system

In the time domain

- differentiate the output as many times as needed (here, just once) to obtain u

$$y = x_2 - x_1 \qquad \dot{y} = \dot{x}_2 - \dot{x}_1 = -3x_2 + u$$

- the inversion-based control

$$u = 3x_2 + \dot{y}_d = u_d(x, \dot{y}_d) \qquad \Rightarrow \qquad \dot{y} = \dot{y}_d$$

guarantees, with $y(0) = x_1(0) - x_2(0) = y_d(0)$, that $y(t) = y_d(t), \forall t \geq 0$, provided the evolution of the internal state remains bounded

- the inverse system of our plant is

$$\dot{\xi} = A\xi + Bu_d(\xi, \dot{y}_d) \quad u = u_d(x, \dot{y}_d) \quad \text{with } \xi(0) = x(0)$$

or $$\dot{\xi}_1 = \xi_2 \qquad \dot{\xi}_2 = \xi_2 + \dot{y}_d \qquad u = 3\xi_2 + \dot{y}_d$$

which is clearly unstable: for a generic initial condition, its evolution is unbounded …

Stable inversion of non-minimum phase system

In the time domain

- for the desired output trajectory, the second state variable evolves as

$$\dot{x}_2 = x_2 + \dot{y}_d = x_2 + \alpha e^{-\alpha t}$$

- its solution is

$$x_2(t) = \left(x_2(0) + \frac{\alpha}{\alpha+1}\right) e^{t} - \left(\frac{\alpha}{\alpha+1}\right) e^{-\alpha t}$$

  and is bounded if and only if $x_2(0) = -\alpha/(\alpha+1)$

- from $y_d(0) = 0$, it also follows that $x_1(0) = x_2(0) = -\alpha/(\alpha+1)$
- with these initial conditions, the state evolution is bounded under inverse control

$$x_1(t) = \frac{1}{\alpha+1}\left(e^{-\alpha t} - (\alpha+1)\right) \qquad x_2(t) = -\left(\frac{\alpha}{\alpha+1}\right) e^{-\alpha t}$$

  and the exact trajectory tracking problem is solved by

$$u_d(t) = 3x_2(t) + \dot{y}_d(t) = \left(\frac{\alpha(\alpha-2)}{\alpha+1}\right) e^{-\alpha t}$$

Stable inversion of non-minimum phase system

In the Laplace domain

- invert the transfer function of the plant

$$\frac{u(s)}{y(s)} = P^{-1}(s) = \frac{s(s+2)}{s-1} = \frac{d_P(s)}{n_P(s)}$$

- compute in the transformed domain

$$u_d(s) = P^{-1}(s)y_d(s) = \frac{s+2}{s-1}\,\dot{y}_d(s)$$

- however, the transfer function is a `complete' representation of a plant only in the zero state ($x(0)=0$)

- we should take instead the initial conditions into account when using the Laplace transform of the state and output equations in time, i.e.,

$$sx_1(s) - x_1(t=0) = x_2(s) \qquad sx_2(s) - x_2(t=0) = -2x_2(s) + u(s)$$

$$y(s) = x_2(s) - x_1(s)$$

Stable inversion of non-minimum phase system

In the Laplace domain

- the complete (input + initial state)-output mapping in the Laplace domains is thus

$$y(s) = \frac{s-1}{s(s+2)}u(s) + \frac{\big(x_2(0) - x_1(0)\big)s - \big(2x_1(0) + x_2(0)\big)}{s(s+2)}$$

$$= P(s)u(s) + \frac{N(x(0),s)}{s(s+2)}$$

- inversion for a desired $y_d(s)$ is given by

$$u_d(s) = P^{-1}(s)\left(y_d(s) - \frac{N(x(0),s)}{d_P(s)}\right) = P^{-1}(s)\, y_d(s) - \frac{N(x(0),s)}{n_P(s)}$$

- the Laplace transform of the desired output trajectory $y_d(t)$ is

$$y_d(s) = \frac{1}{s} - \frac{1}{s+\alpha} \qquad \alpha > 0$$

Stable inversion of non-minimum phase system

In the Laplace domain

- expansion in partial fractions/residuals of $u_d(s)$ leads (with tedious passages) to

$$u_d(s) = \frac{s(s+2)}{s-1}\left(\frac{1}{s} - \frac{1}{s+\alpha}\right) - \frac{N(x(0),s)}{s-1}$$

$$= \frac{(s+2)}{s-1} - \frac{s(s+2)}{(s-1)(s+\alpha)} - \frac{N(x(0),s)}{s-1}$$

$$= 1 + \frac{3}{s-1} - \left(1 - \frac{(3-\alpha)s+\alpha}{(s-1)(s+\alpha)}\right) - \frac{N(x(0),s)}{s-1}$$

$$= \frac{3}{s-1} - \left(\frac{3/(\alpha+1)}{s-1} + \frac{\alpha(2-\alpha)/(\alpha+1)}{s+\alpha}\right) - \frac{N(x(0),s)}{s-1}$$

$$= \frac{3\alpha/(\alpha+1) - N(x(0),s)}{s-1} + \frac{\alpha(\alpha-2)/(\alpha+1)}{s+\alpha}$$

- to discard the presence of the unstable pole in $s = 1$ (i.e., of the unbounded exponential e^t in the time domain), it is necessary and sufficient that

$$N(x(0),s) = \frac{3\alpha}{\alpha+1} \quad \Leftrightarrow \quad x_2(0) - x_1(0) = 0 \quad 2x_1(0) + x_2(0) = -\frac{3\alpha}{\alpha+1}$$

which lead to the same initial conditions (and inversion command) already found

Inversion in the frequency domain

Non-causal exact reproduction of end-effector trajectories

- to get rid of initial conditions, the idea is to view the desired trajectory as part of a periodic profile ⇒ use Fourier transform (in linear domain)
- single-link flexible arm (with generic variables)

$$\begin{pmatrix} m_{\theta\theta} & m_{\delta\theta}^T \\ m_{\delta\theta} & m_{\delta\delta} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \theta \\ \delta \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

- tip position output

$$y(t) = \begin{pmatrix} 1 & c_e^T \end{pmatrix} \begin{pmatrix} \theta \\ \delta \end{pmatrix}$$

- dynamic model rewritten in terms of (y, δ)

$$\begin{pmatrix} m_{\theta\theta} & m_{\delta\theta}^T - m_{\theta\theta} c_e^T \\ m_{\delta\theta} & m_{\delta\delta} - m_{\delta\theta} c_e^T \end{pmatrix} \begin{pmatrix} \ddot{y} \\ \ddot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \dot{y} \\ \dot{\delta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} y \\ \delta \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

non-symmetric!

Inversion in the frequency domain

In the Fourier domain

- take bilateral Fourier transforms (at the acceleration level)

$$\ddot{Y}(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega t)\, \ddot{y}(t) dt \qquad \ddot{\Delta}(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega t)\, \ddot{\delta}(t) dt$$

$$T(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega t)\, \tau(t) dt$$

and obtain in the dynamic model

$$\begin{pmatrix} m_{\theta\theta} & m_{\delta\theta}^T - m_{\theta\theta} c_e^T \\ m_{\delta\theta} & m_{\delta\delta} - m_{\delta\theta} c_e^T + \frac{1}{j\omega} D - \frac{1}{\omega^2} K \end{pmatrix} \begin{pmatrix} \ddot{Y}(\omega) \\ \ddot{\Delta}(\omega) \end{pmatrix} = \begin{pmatrix} T(\omega) \\ 0 \end{pmatrix}$$

- solve for the accelerations and then for the torque, by `inversion' in frequency domain

$$\begin{pmatrix} \ddot{Y}(\omega) \\ \ddot{\Delta}(\omega) \end{pmatrix} = \begin{pmatrix} g_{11}(\omega) & g_{12}^T(\omega) \\ g_{21}(\omega) & G_{22}(\omega) \end{pmatrix} \begin{pmatrix} T(\omega) \\ 0 \end{pmatrix} \Rightarrow T(\omega) = \frac{1}{g_{11}(\omega)} \ddot{Y}(\omega) = r(\omega) \ddot{Y}(\omega)$$

Inversion in the frequency domain

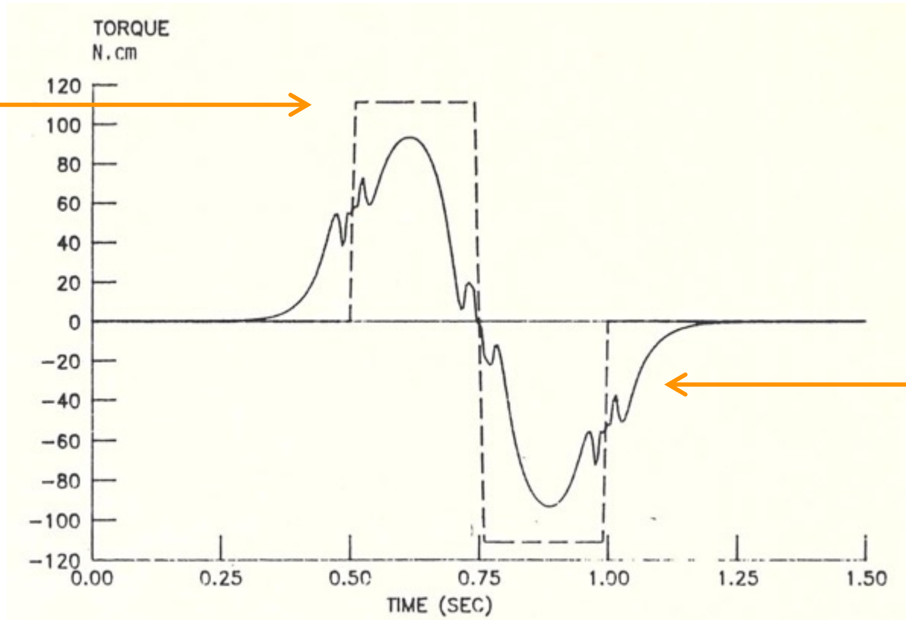
Computational procedure

- for a zero-mean $\ddot{y}_d(t)$, with $\ddot{y}_d(t) = 0$ for $t \leq -T/2$ and $t \geq T/2$, acceleration can be embedded in $(-\infty, +\infty)$ as a signal of period T
- $\ddot{y}_d(t) \longrightarrow \ddot{Y}_d(\omega) \longrightarrow T_d(\omega) \longrightarrow \tau_d(t)$: finite inverse Fourier transform

$$\tau_d(t) = \int_{-\infty}^{+\infty} r(t-\sigma)\ddot{y}_d(\sigma)\, d\sigma = \int_{-T/2}^{+T/2} r(t-\sigma)\ddot{y}_d(\sigma)\, d\sigma$$

expanding beyond the definition interval $[-T/2, T/2]$ (non-causal)

bang-bang acceleration profile $\ddot{y}_d(t)$ of $T = 0.5$ s for rest-to-rest motion of a single flexible link



input torque profile $\tau_d(t)$ lasts $T_d \approx 0.95$ s, starting before (and ending after) $\ddot{y}_d(t)$

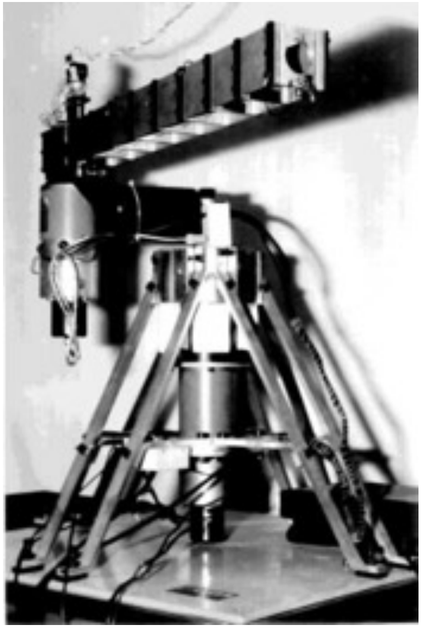
Inversion in the frequency domain

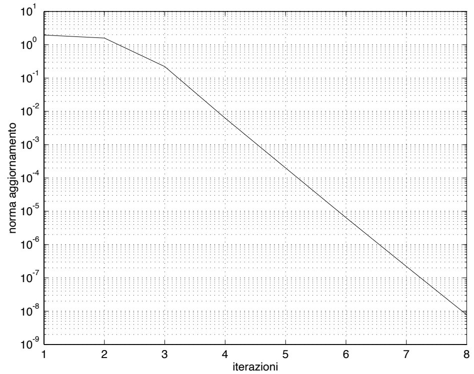
Remarks

- outside the given interval T of output motion, the input torque has
  - a pre-charging action, to bring the internal flexible state from rest to a suitable initial state at $t = -T/2$
  - a discharging action, to bring the internal flexible state from the final state at $t = T/2$ back to rest
- from the obtained initial state at $t = -T/2$ (unique for the given trajectory) inversion control gives a bounded internal evolution
- truncations (in time and/or in frequency) are inherent to the actual computations (FFT)
- the method was recast also in the time domain (stable/anti-stable dynamics) and extended to the (multilink flexible) nonlinear setting
  - by iterative linear approximations along the nominal trajectory (starting from the rigid body motion)

Inversion in the frequency domain

Application to the two-link FLEXARM



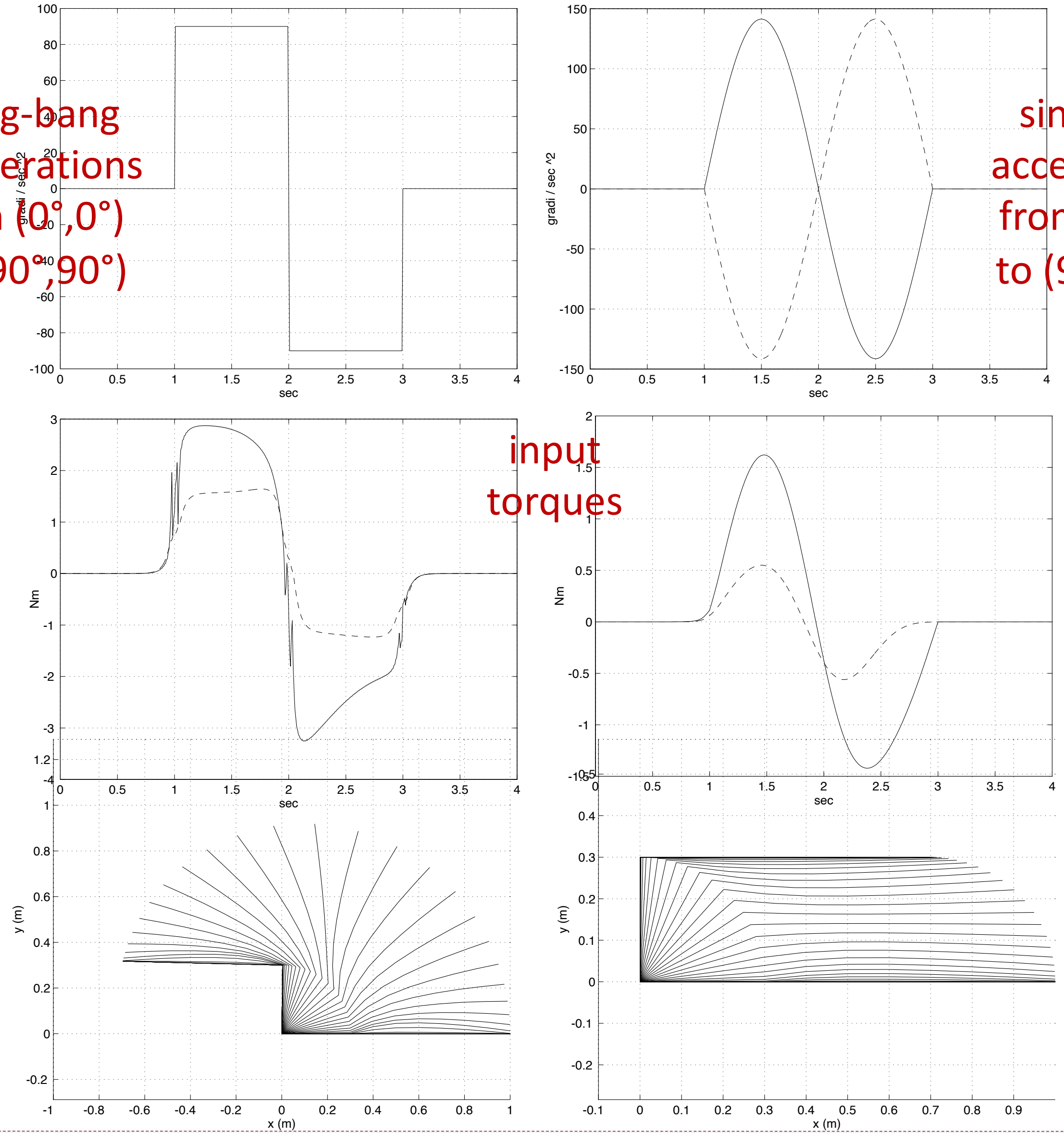


exponential convergence rate of iterations (linear in log scale)

bang-bang accelerations from (0°,0°) to (90°,90°)

sinusoidal accelerations from (0°,0°) to (90°,–90°)

input torques



End-effector trajectory tracking by state feedback

Based on the general regulation theory

- the end-effector trajectory tracking task in robots with flexible links is an instance of asymptotic output tracking problems ($e \rightarrow 0$) with internal state stability –including disturbances (regulator problem)
- well-established solution techniques in the linear case and, by now, also in the nonlinear case
- to avoid internal instability during output tracking, the idea is to compute a `natural' (and bounded!) state trajectory
  - that corresponds to the desired output trajectory
  - with the desired output trajectory (and the disturbances, if present) being generated by an autonomous dynamic system (exosystem)
  - stabilizing the system with a feedback on the state trajectory error
  - including in the control design also a feedforward that keeps the error to zero in nominal conditions

End-effector trajectory tracking by state feedback

Linear regulator problem

- let the state-output-error equations (with $x = (q, \dot{q})$) of the flexible arm be

$$\dot{x} = Ax + B\tau \quad y = Cx \qquad e = y - y_d$$

- a (smooth) desired output trajectory is assumed to be generated by the autonomous (anti-stable) exosystem (with state w)

$$\dot{w} = Sw \qquad y_d = -Qw$$

- when (A, B) is stabilizable, the problem has a solution ($\forall x(0), w(0)$) if and only if the regulator equations are solvable in matrices Π and Γ

$$\Pi S = A\Pi + B\Gamma \qquad C\Pi + Q = 0$$

- a state feedback + feedforward controller is then

$$\tau = F(x - \Pi w) + \Gamma w$$

  - with gain matrix F such that $A + BF$ is Hurwitz ($\mathrm{Re}(\lambda) < 0$)
  - $x_d(t) = \Pi w(t)$ is the desired state trajectory: $x_d(0)$ is the unique initial state giving a bounded state solution under inversion control!
  - from $x_d(0) = \Pi w(0)$, $\tau_d(t) = \Gamma w(t)$ will give exact trajectory tracking

Output regulation by state feedback

Reprise of worked out SISO linear example of a non-minimum phase system

- plant (with a zero in $s = 1$)

$$\dot{x} = Ax + Bu \quad y = Cx \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

- exosystem for the (class of) output trajectories $y_d(t) = 1 - e^{-\alpha t}, \alpha > 0$

$$\dot{w} = Sw = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix} w \quad \Rightarrow \quad w_1(t) = w_1(0),\ w_2(t) = w_2(0) e^{-\alpha t}$$

$$y_d = -Qw = \begin{pmatrix} 1 & -1 \end{pmatrix} w \quad \Rightarrow \quad y_d(t)|_{w(0)=(1,1)} = 1 - e^{-\alpha t}$$

- regulator equations for Π (2×2) and Γ (1×2)

$$\begin{pmatrix} 0 & -\alpha\pi_{12} \\ 0 & -\alpha\pi_{22} \end{pmatrix} = \begin{pmatrix} \pi_{21} & \pi_{22} \\ -2\pi_{21} & -2\pi_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \pi_{21} - \pi_{11} & \pi_{22} - \pi_{12} \end{pmatrix} + \begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

indeed, for $w(0) = (1,1)$ it is the same solution as before

- solution

$$\Pi = \begin{pmatrix} -1 & \frac{1}{\alpha+1} \\ 0 & -\frac{\alpha}{\alpha+1} \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0 & \frac{\alpha(\alpha-2)}{\alpha+1} \end{pmatrix} \quad \Rightarrow \quad x_d(t) = \begin{pmatrix} \frac{w_2(0)}{\alpha+1} e^{-\alpha t} - w_1(0) \\ -\frac{\alpha w_2(0)}{\alpha+1} e^{-\alpha t} \end{pmatrix}$$

stabilizing gains $F = \begin{pmatrix} F_1 < 0 & F_2 < 0 \end{pmatrix}$ $\qquad \tau_d(t) = w_2(0)(\alpha(\alpha-2)/(\alpha+1))e^{-\alpha t}$

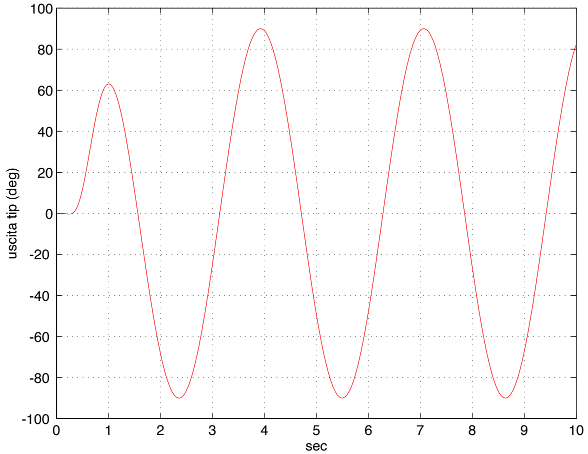
End-effector trajectory tracking by regulation

Numerical results

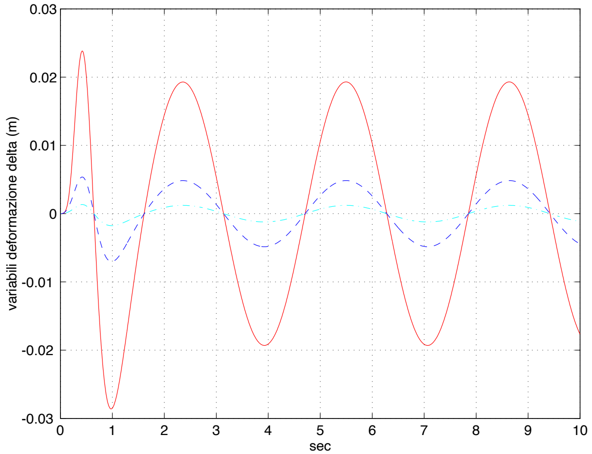
- a single-link flexible with three modes at $f_1 = 3.2,\ f_2 = 8.9$ and $f_3 = 16.1$ [Hz]
- sinusoidal tip trajectory: $y_{td}(t) = (\pi/2)\sin(2\pi t/3)$

gains F place all eigenvalues in -10

tip angle output

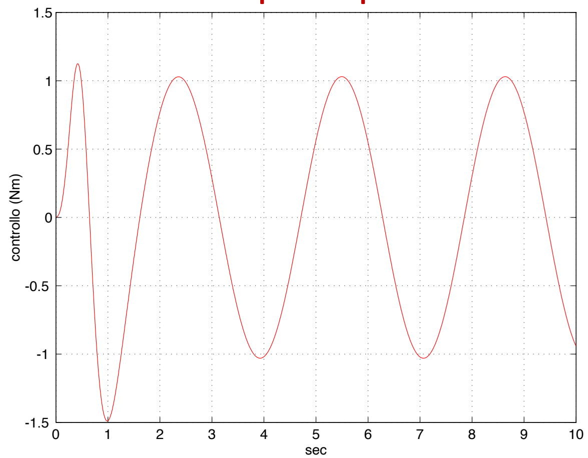


deflection variables

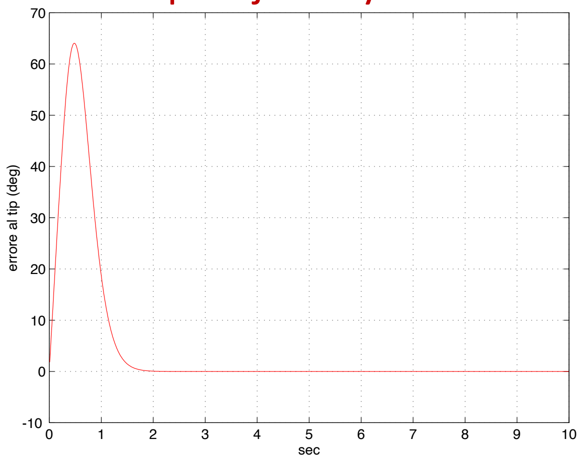


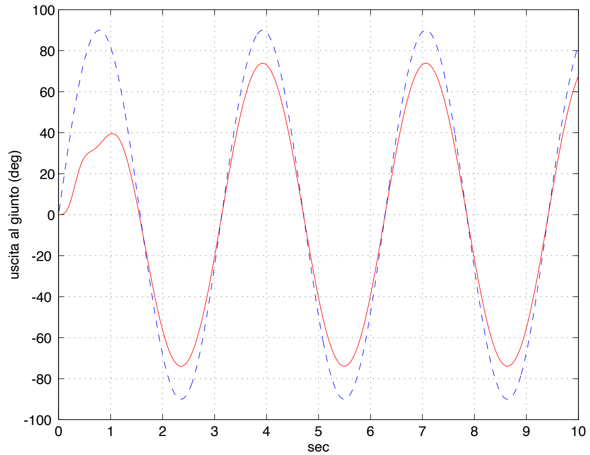
—— = δ_1
- - - = δ_2
- - - = δ_3

torque input



tip trajectory error





clamped joint angle (—) and desired tip trajectory (---)

End-effector trajectory tracking by nonlinear regulation

Experimental results on FLEXARM

- nonlinear version of the regulator equations …
- two modes of the flexible forearm at $f_1 = 4.7$ and $f_3 = 14.7$ [Hz]
- rest-to-rest 7th-order polynomial trajectory for (θ_1, y_{t2})

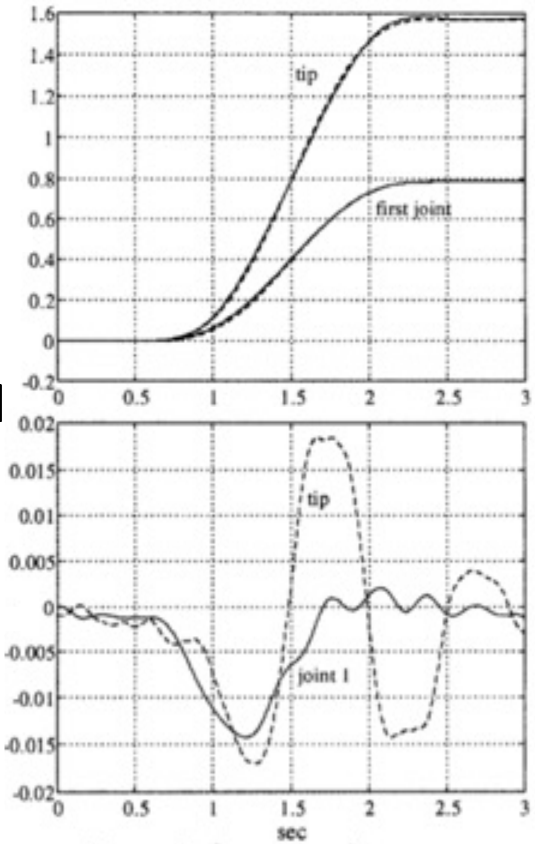
from $(0,0)$ to $(\pi/4, \pi/2)$ in $T = 2$ s

from $(0,0)$ to $(\pi/4, \pi/4)$ in $T = 1.5$ s

desired and actual output trajectories

[rad]

tracking errors



video

Rest-to-rest motion

Problem formulation and solution approach

- task: execute a rest-to-rest slew motion with a flexible link robot between two (undeformed) configurations in given time
- issue: fast transfers induce residual oscillations, extending the actual task completion time
- strategy: design suitable system outputs and plan their trajectories (and associated torque profiles) so to induce a complete absence of vibrations at the given final time
- idea: find outputs with maximum relative degree (no zero dynamics)
  - closed-form solution in the SISO linear case (absence of zeros)
  - direct extension to MIMO nonlinear case (flat outputs `to be found', meaning that the system is exactly linearizable by dynamic feedback …)
  - a feedforward torque command, that can be made more robust, e.g., by adding a PD action on errors w.r.t. the associated joint trajectories

Rest-to-rest motion

Algorithm for a single flexible link

$$J\ddot{\theta} = \tau \qquad \ddot{\delta}_i + \omega_i^2 \delta_i = \phi_i'(0)\tau \qquad i = 1,2,\dots,n_e$$

- choose a parametric output y, with yet unknown coefficients c_i's

$$y = \theta + \sum_{i=1}^{n_e} c_i \delta_i = \theta + c^T \delta$$

- impose input τ-independence of the successive (even) derivatives

$$\ddot{y} = \ddot{\theta} + \sum_{i=1}^{n_e} c_i \ddot{\delta}_i = \left(\frac{1}{J} + \sum_{i=1}^{n_e} c_i \phi_i'(0)\right)\tau - \sum_{i=1}^{n_e} c_i \omega_i^2 \delta_i \;\Rightarrow\; \sum c_i \phi_i'(0) = -\frac{1}{J}$$

$$y^{[4]} = \frac{d^4 y}{dt^4} = -\sum_{i=1}^{n_e} c_i \omega_i^2 \phi_i'(0)\,\tau + \sum_{i=1}^{n_e} c_i \omega_i^4 \delta_i \;\Rightarrow\; \sum c_i \omega_i^2 \phi_i'(0) = 0$$

$$y^{[6]} = \cdots$$

  and so on, until a set of n_e equations is obtained

  - the torque τ will appear in the $2(n_e + 1)$-th output derivative (the last one)

- solve for the coefficients $c = (c_1, \dots, c_{n_e})$

$$V \cdot \mathrm{diag}\{\phi_1'(0), \dots, \phi_{n_e}'(0)\}\, c = (-1/J \quad 0 \quad \dots \quad 0)^T$$

  with a Vandermonde matrix V generated by $\left(\omega_1^2, \dots, \omega_{n_e}^2\right)$

Rest-to-rest motion

Algorithm for a single flexible link

- the torque $\tau_d(t)$ is found by inversion of the highest derivative, imposing

$$y^{[2(n_e+1)]} = y_d^{[2(n_e+1)]}$$

for a suitably planned trajectory $y_d(t), t \in [0, T]$ (the given transfer time)

- e.g., by solving the interpolation problem

$$y_d(0) = \theta_i \quad y_d(T) = \theta_f \quad y_d^{[i]}(0) = y_d^{[i]}(T) = 0 \quad i = 1, \dots, 2n_e + 1$$

for which a polynomial of degree $4n_e + 3$ will be sufficient

- in the Laplace domain, imposing no zeros to the transfer function leads to the closed-form expression

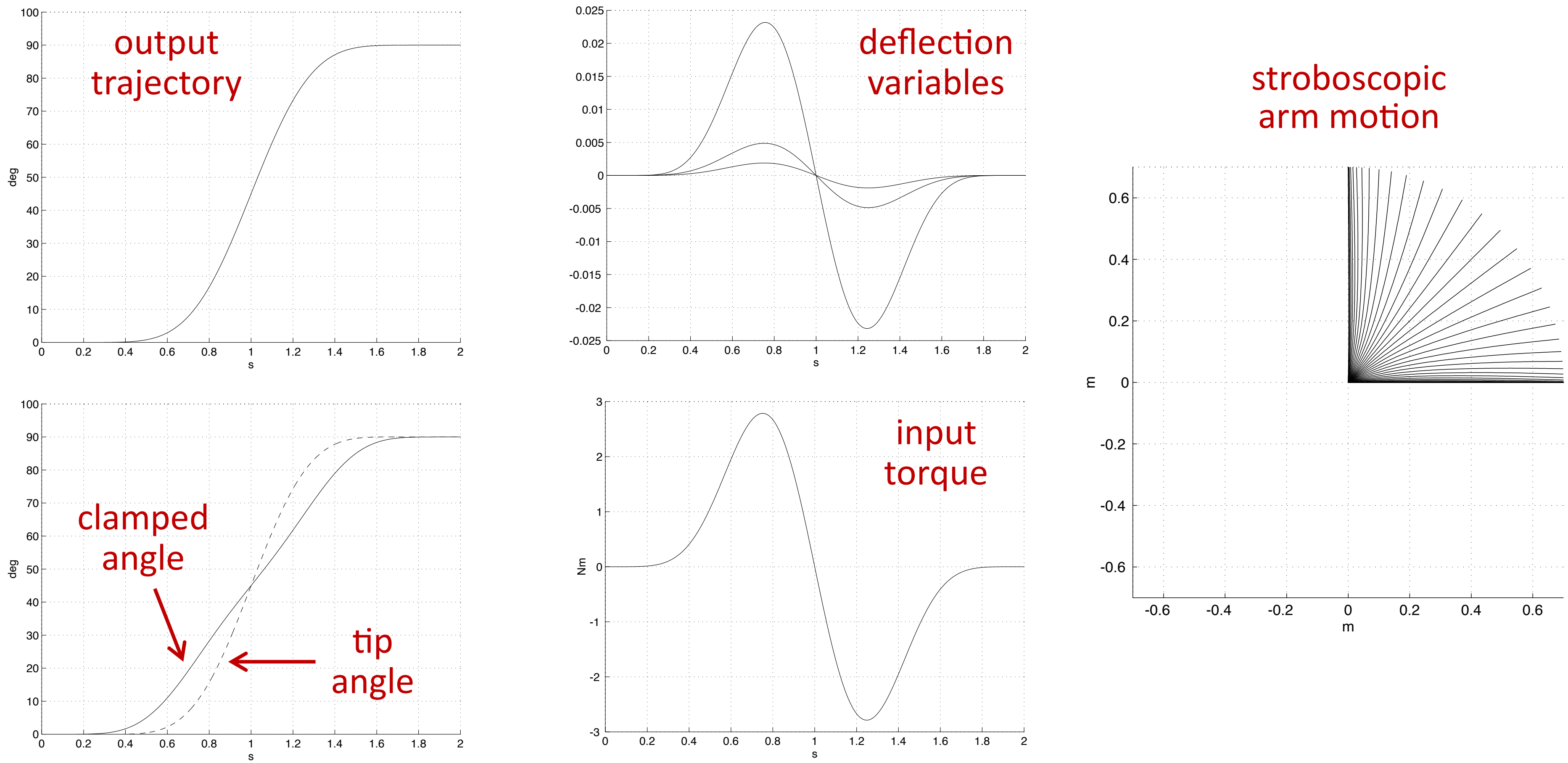
$$\tau_d(s) = \frac{J}{\prod_{i=1}^{n_e} \omega_i^2} \left(s^2 \prod_{i=1}^{n_e} \left(s^2 + \omega_i^2 \right) \right) y_d(s)$$

to be transformed back in time to yield $\tau_d(t)$

Rest-to-rest motion

Numerical results

- single flexible link with $n_e = 3$ modes at $f_1 = 4.05, f_2 = 12.34$ and $f_3 = 22.87$ [Hz]
- angular displacement of $\theta_f - \theta_i = 90°$ in $T = 2$ s
- 19-th degree polynomial (also with continuous torque derivatives)



Rest-to-rest motion

Remarks

- method applies to any linear model of a single-link flexible arm
  - output design is related to the controllability canonical form
- in the limit, design output is a specific point x^* on the physical beam: for a given n_e, $c_i = \phi_i(x^*_{n_e})/x^*_{n_e}$ while $\lim_{n_e\to\infty} x^*_{n_e} = x^*$
- modified output structure for modal damping in the dynamics

$$y = \theta + \sum_{i=1}^{n_e} c_i\delta_i + \gamma\dot{\theta} + \sum_{i=1}^{n_e} d_i\dot{\delta}_i$$

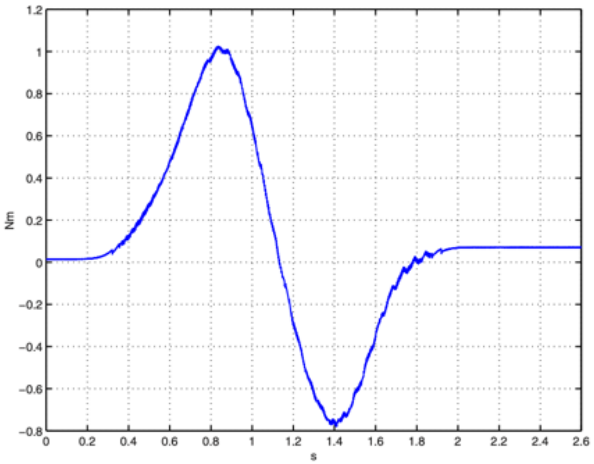
- for better torque/time performance, use smoothed bang-bang or bang-coast-bang torques (with polynomial interpolating phases)
- the planned feedforward command can be combined with an error feedback action, e.g., on the clamped joint reference (a by-product)

$$\tau = \tau_d(t) + K_P\big(\theta_{c,d}(t) - \theta_c\big) + K_D\big(\dot{\theta}_{c,d}(t) - \dot{\theta}_c\big)$$

Rest-to-rest motion

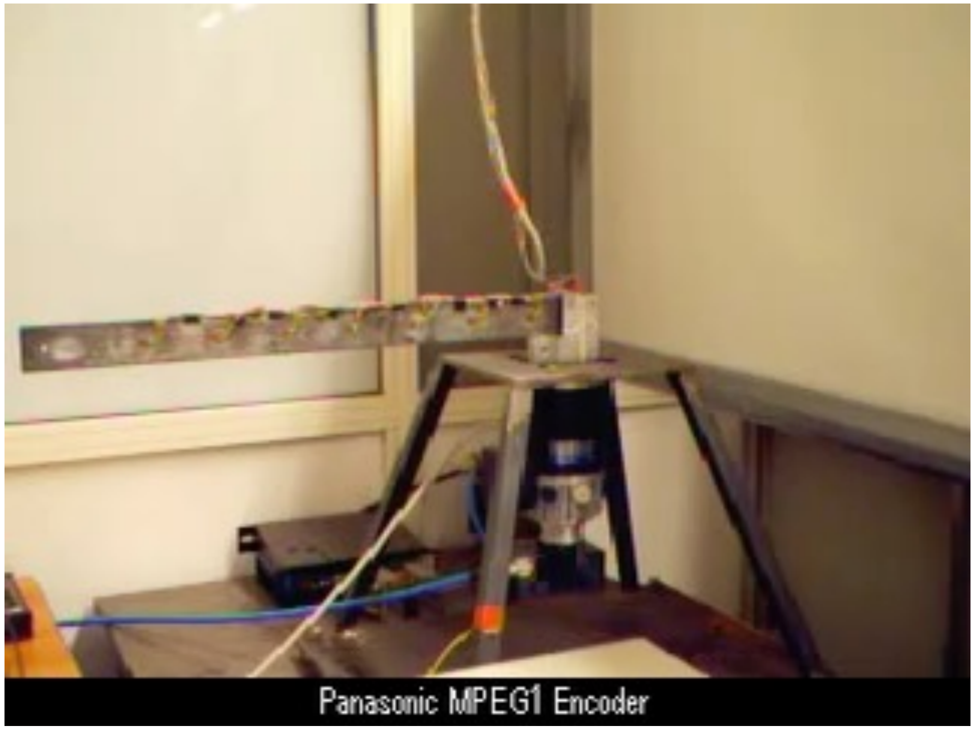
Experimental results on **DMA Sapienza** flexible link

- data: $\ell = 0.655$ m, $\rho = 0.7733$ kg/m, $EI = 6.22$ Nm^2
- three modes at $f_1 = 14.4, f_2 = 34.2$ and $f_3 = 69.3$ [Hz]
- rest-to-rest 19th-order polynomial trajectory for the design output



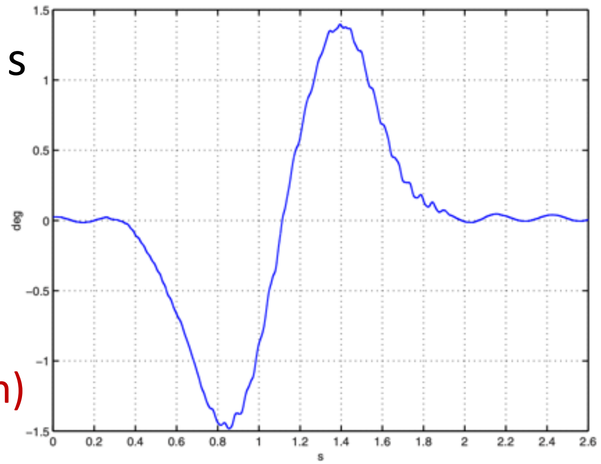
control torque (including PD)

slew of $\pi/2$ in $T = 1$ s



video

slew of π in $T = 2$ s

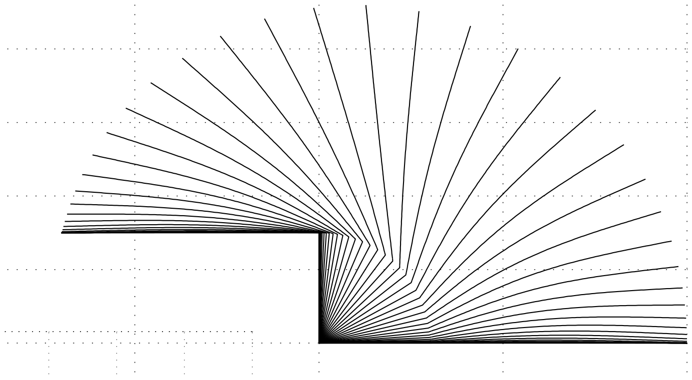


tip angle (deformation)

Rest-to-rest motion

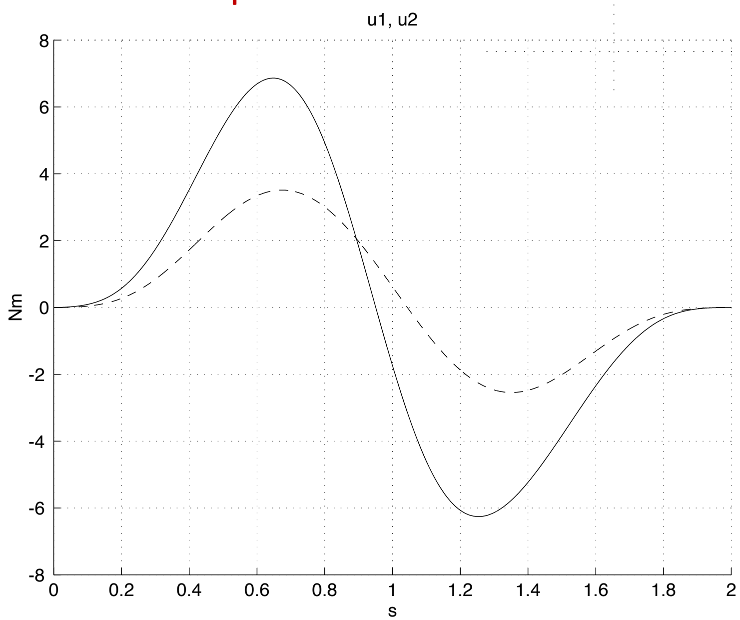
Numerical results on FLEXARM

- two flat outputs can be found (with relative degrees 4 + 4 after dynamic extension with 2 integrators), when only one mode is considered (state dimension = 6)
- rest-to-rest 11th-order polynomial trajectories for the two design outputs

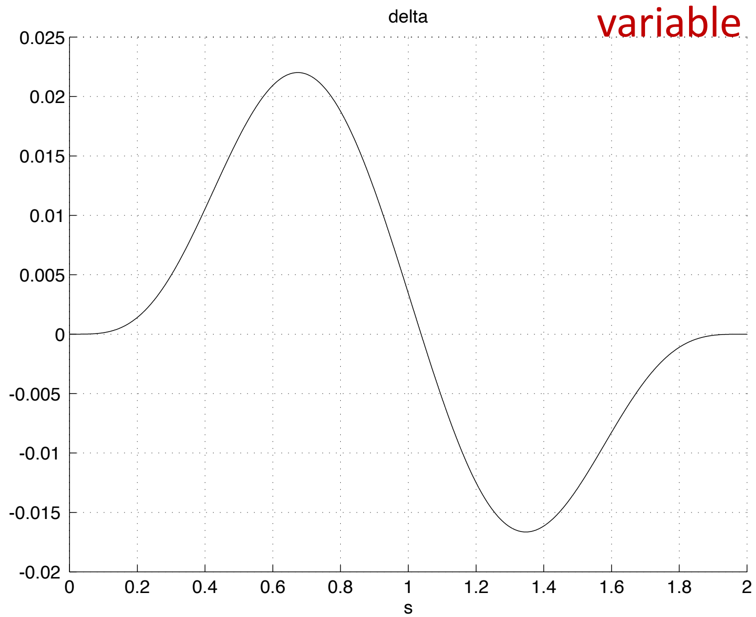


from $(0,0)$ to $(\pi/2, \pi/2)$
in $T = 2$ s

control torques



first deflection variable



Other issues

Many aspects have been left out!

- spill-over effects
  - when truncating infinite-dimensional models
- vibration damping
  - especially in regulation tasks
- strain feedback
  - direct use in the control design and analysis of the PDE equations
- handling model uncertainties and disturbances
  - model identification with link flexibility, robust and adaptive control
- state observers
  - reconstructing missing information from different sensor suites
- interaction with the environment
  - collision detection and reaction, control of the exchanged forces
- other control methods
  - singular perturbation approach, iterative learning, optimal control, …

Conclusions

… in short

- extra effort in dynamic modeling pays off
  - model-based controllers for accurate trajectory tracking
  - proof of stability for model-independent regulation controllers
- more classical control strategies tend to suppress vibrations wherever they arise
  - outcome of our analysis is that the controlled system should be brought to a vibratory behavior compatible with the given output task
- paradigm shift
  - intentional deformation and flexibility to be preserved, rather than handled as a parasitic effect to be eliminated by control
- robots with flexible links versus robots with flexible joints
  - although mechanically similar in a first approximation, they are intrinsically different from the control point of view

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