Gentzen Type Sequent Calculus for Propositional Dynamic Logic.

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ABSTRACT: In the first part of the paper we propose a Gentzen-type cut-free sequent calculus $G_0(PDL)$ for PDL and prove that it is sound and incomplete. Adding to the sequent calculus *cut* rule we obtain sequent calculus $G_0(PDL) + cut$ that is sound and incomplete. But in this calculus all the formulae, which are provable without the *induction* axiom in Hilbert-style axiomatization of PDL, are provable in $G_0(PDL) + cut$. In the second part we propose a Gentzen-type cut-free sequent calculus $G_1(PDL)$ for extended PDL and prove that it is sound for PDL formulae. The completeness of the calculus is still an open question. This calculus deals with a so-called extended PDL and i-sequents. Then we add to the calculus $G_1(PDL)$ *cut* rule and obtain sequent calculus $G_1(PDL)+cut$ that is sound and complete for PDL formulae. At the end we consider the calculus $G_2(PDL)$ that is sound and complete for PDL and can serve as a basis in construction of an automated theorem prover for PDL.

1 Introduction

Propositional Dynamic Logic (PDL) [2] [4] is a polymodal program logic with program constructs for sequential composition ;, non-deterministic choice $\cup$, non-deterministic iteration $\star$ and test ?. The syntax of PDL is constructed from a countable alphabet of (program) symbols and a countable alphabet of (propositional) variables. The syntax consists of programs (structured program schemata) and (logical) formulae which are defined by mutual induction as usual [4]. The semantics of PDL is defined in Kripke structures $M =< D, I >$ where program symbols are interpreted as binary relations over a domain D and propositional variables - as unary predicates on D in the usual manner [4]. We suppose that basic propositional operations in all variants of PDL are negation $\neg$, conjunction $\wedge$ and disjunction $\vee$. The implication $\rightarrow$ and equivalence $\leftrightarrow$ are usual abbreviations only.

For PDL some proof formalization methods such as tableux proof method and Hilbert-type axiomatizations are known [3] [4]. Gentzen type sequent calculus with cut for PDL without iteration has been developed in [6]. Gentzen type cut-free sequent calculus for PDL without iteration has been developed in [1] quite recently. In the first part of the paper we propose a Gentzen-type cut-free sequent calculus $G_0(PDL)$ for PDL and prove that it is sound and incomplete. Adding to the sequent calculus *cut* rule we obtain sequent calculus $G_0(PDL)+cut$ that is sound and incomplete. But in this calculus all the formulae, which are provable without the *induction* axiom in Hilbert-style axiomatization of PDL, are provable in $G_0(PDL)+$ cut.

In the second part of the paper we propose a Gentzen-type cut-free sequent calculus $G_1(PDL)$ for extended PDL and prove that it is sound for PDL formulae. The completeness of the calculus is still an open question. This calculus deals with a so-called extended PDL and i-sequents. The extended PDL has one additional program construct for v-times iteration, where v is a natural value variable. An i-sequent is a sequent with induction assumptions. Then we add to the calculus $G_1(PDL)$ cut rule and obtain sequent calculus $G_1(PDL) + cut$ that is sound and complete for PDL formulae.

2 Syntax and Semantics of PDL

In the section we propose an equivalent definition of the semantics of PDL. Let we have two disjoint sets of symbols: Φ_0, the set of propositional variables, and P_0, the set of atomic programs. We constuct inductively the set Φ of compound formulae and the set P of compound programs as follows:

- $\Phi_0 \subset \Phi$; $P_0 \subset P$;
- if $\phi, \psi \in \Phi$ then $(\phi \wedge \psi), \neg\phi \in \Phi$;
- if $\phi \in \Phi, \{p.q\} \subset P$ then $(p; q), (p \cup q), \phi?, p\star \in P$
- if $\phi \in \Phi, p \in P$ then $< p > \phi, [p]\phi \in \Phi$

We abbreviate $[p]\phi \rightleftharpoons \neg < p > \neg\phi$, $(\phi \vee \psi) \rightleftharpoons \neg(\neg\phi \wedge \neg\psi)$, $(\phi \to \psi) \rightleftharpoons (\neg\phi \vee \psi)$.

The semantics of PDL is defined relative to a given Kripke-structure $\mathcal{M} =< \mathcal{W}, \rho, \pi >$, where W is a set of worlds, $\rho : \Phi_0 \to 2^W$ and $\pi : P_0 \to 2^{W \times W}$ are interpretations of propositional formulae and atomic programs, respectively. We define the truth relation $\models \subset W \times \Phi$ between worlds and formulae of PDL as follows:

- If $\phi \in \Phi_0$ then $w \models \phi \rightleftharpoons w \in \rho(\phi)$;
- $w \models \phi \wedge \psi \rightleftharpoons w \models \phi \; and \; w \models \psi$;
- $w \models < p; q > \phi \rightleftharpoons w \models < p >< q > \phi$;
- $w \models < p \cup q > \phi \rightleftharpoons w \models < p > \phi$ and $w \models < q > \phi$;
- $w \models < p\star > \phi \rightleftharpoons$ there is some $n \in N$ such that $w \models < p^n > \phi$;
- $w \models < \phi? > \psi \rightleftharpoons w \models \phi$ and $w \models \psi$;
- If $p \in P_0$ then $w \models < p > \phi \rightleftharpoons$ there is some $v \in W$ such that $(w, v) \in \pi(p)$ and $v \models \phi$.

3 Gentzen-type Sequent Calculus for PDL $G_0(PDL)$

In this section we propose a naive Gentzen-type sequent calculus $G_0(PDL)$ for PDL and prove that it is sound and, unfortunately, incomplete.

A sequent is an ordered pair $\Gamma \vdash \Delta$ where $\Gamma = \{A_1, \ldots, A_n\}, \Delta = \{B_1, \ldots, B_m\}$ are finite sets of formulae. Let $M =< D, I >$ be a Kripke structure. The sequent $\Gamma \vdash \Delta$ is said to be valid in a state $s \in D$ (written $s \models (\Gamma \vdash \Delta)$) iff (for all $A \in \Gamma$)$(s \models A) \Longrightarrow$

(there is some $B \in \Delta$) ($s \models B$). Thus the meaning of a sequent $\Gamma \vdash \Delta$ is equivalent to the meaning of the formula $A_1 \wedge \ldots \wedge A_n \rightarrow B_1 \vee \ldots \vee B_m$.

A sequent $\Gamma \vdash \Delta$ is said to be valid (written $\models (\Gamma \vdash \Delta)$ or $\Gamma \models \Delta$) if it is valid in all Kripke structures in all states. Let us remark that a formula A is valid iff the sequent $\emptyset \vdash A$ is valid.

Gentzen type sequent calculus $G_0(PDL)$ for PDL is a set of axioms and a set of rules. A sequent Θ is said to be provable (written $\vdash_{G(PDL)} \Theta$) if there is a proof of the sequent. Below we will use the following notation:
$\Gamma^{[\alpha]} = \{A | [\alpha]A \in \Gamma\}$, $\Gamma^{<\alpha>} = \{A | <\alpha> A \in \Gamma\}$,
$<\alpha>\Gamma = \{<\alpha> A | <\alpha> A \in \Gamma\}$, $[\alpha]\Gamma = \{[\alpha]A | [\alpha]A \in \Gamma\}$,
Γ_0 = the subset of all propositional formulae of Γ,
where Γ is a set of formulae and α is a program.

Gentzen-type sequent calculus $G_0(PDL)$
Axioms: $\Gamma, A \vdash \Delta, A$, where Γ, Δ are sets of formula, A is a formula.
Inference rules:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}(\wedge \vdash) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}(\vdash \wedge)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}(\neg \vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}(\vdash \neg)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}(\vee \vdash) \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}(\vdash \vee)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma, A \rightarrow B \vdash \Delta}(\rightarrow \vdash) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}(\vdash \rightarrow)$$

$$\frac{\Gamma \vdash \Delta, <\alpha><\alpha\star> A, A}{\Gamma \vdash \Delta, <\alpha\star> A}(\vdash <\star>) \qquad \frac{\Gamma, [\alpha][\alpha\star]A, A \vdash \Delta}{\Gamma, [\alpha\star]A \vdash \Delta}([\star] \vdash)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, <A?> B \vdash \Delta}(<?>\vdash) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, <A?> B}(\vdash <?>)$$

$$\frac{\Gamma \vdash \Delta, <\alpha> A, <\beta> A}{\Gamma \vdash \Delta, <\alpha \cup \beta> A}(\vdash <\cup>) \qquad \frac{\Gamma, [\alpha]A, [\beta]A \vdash \Delta}{\Gamma, [\alpha \cup \beta]A \vdash \Delta}([\cup] \vdash)$$

$$\frac{\Gamma, <\alpha><\beta> A \vdash \Delta}{\Gamma, <\alpha;\beta> A \vdash \Delta}(<;>\vdash) \qquad \frac{\Gamma \vdash \Delta, <\alpha><\beta> A}{\Gamma \vdash \Delta, <\alpha;\beta> A}(\vdash <;>)$$

$$\frac{\Gamma, [\alpha][\beta]A \vdash \Delta}{\Gamma, [\alpha;\beta]A \vdash \Delta}([;] \vdash) \qquad \frac{\Gamma \vdash \Delta, [\alpha][\beta]A}{\Gamma \vdash \Delta, [\alpha;\beta]A}(\vdash [;])$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash \Delta, B}{\Gamma, [B?]A \vdash \Delta}([?] \vdash) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, [A?]B}(\vdash [?])$$

$$\frac{\Gamma^{[\alpha]}, A \vdash \Delta^{<\alpha>}}{\Gamma, <\alpha> A \vdash \Delta}(<>\vdash) \qquad \frac{\Gamma^{[\alpha]} \vdash \Delta^{<\alpha>}, A}{\Gamma \vdash \Delta, [\alpha]A}(\vdash [\,])$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, < \alpha >< \alpha\star > A \vdash \Delta}{\Gamma, < \alpha\star > A \vdash \Delta}([\star]\vdash)$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, [\alpha][\alpha\star]A}{\Gamma \vdash \Delta, [\alpha\star]A}(\vdash)[\star]$$

$$\frac{\Gamma, < \alpha > A \vdash \Delta \qquad \Gamma, < \beta > A \vdash \Delta}{\Gamma, < \alpha \cup \beta > A \vdash \Delta}([\star]\vdash)$$

$$\frac{\Gamma \vdash \Delta, [\alpha]A \qquad \Gamma \vdash \Delta, [\beta]A}{\Gamma \vdash \Delta, [\alpha \cup \beta]A}([\star]\vdash)$$

where Γ, Δ are sets of formulae, α, β are programs, A, B are formulae.

Lemma 1.
1. The *induction axiom* $< \alpha\star > A \rightarrow A\vee < \alpha\star > (\neg A\wedge < \alpha > A)$ can not be proved in $G_0(PDL)$.
2. All the axioms (except the *induction axiom*) of the axiomatization for PDL proposed by Segerberg (see [4]) are provable in $G_0(PDL)$.

Theorem 1 [Soundness and incompleteness theorem for $G_0(PDL)$].
1. The sequent calculus $G_0(PDL)$ is sound, i.e. all provable sequents are valid.
2. The sequent calculus $G_0(PDL)$ is incomplete, i.e. there is a valid sequent that is not provable in $G_0(PDL)$.

Proof. 1. Axioms are valid sequents. For all inference rules if all premises of the rule are valid sequents then the conclussion sequent of the rule is valid. From it follows soundness of $G_0(PDL)$
2. It follows from lemma 1. □

Let us add to the sequent calculus *cut* rule $\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}(cut)$
and denote the extended calculus as $G_0(PDL) + cut$.

Theorem 2 [Soundness and incompleteness theorem for $G_0(PDL) + cut$].
1. The sequent calculus $G_0(PDL) + cut$ is sound.
2. The sequent calculus $G_0(PDL)$ is incomplete.
3. All the formulae of PDL that can be proved in Hilbert-style axiomatization of PDL without using the *induction axiom* are provable in $G_0(PDL) + cut$.

4 Gentzen-type I-sequent Calculus for PDL $G_1(PDL)$

Let us extend the syntax of PDL with a countable alphabet of natural value variables and the set of PDL programs by means of a new program construct: for any natural value variable v and any program α let α^v be a program too. We obtain the so-called extended PDL (written EPDL).

The semantics of $EPDL$ is defined in a Kripke-structure $\mathcal{M} =< W, \rho, \pi, E >$, where W is a set of worlds, $\rho : \Phi_0 \rightarrow 2^W$ and $\pi : P_0 \rightarrow 2^{W\times W}$ are interpretations of propositional formulae and atomic programs, respectively, $E : Var \mapsto \aleph$ is an evaluation of natural value variables. We interpret a formula $<\alpha^v>\phi$ by $<\alpha^{E(v)}>\phi$. In the extended PDL we will consider the formulae without double instances of any natural value variable.

A sequent is an ordered pair $\Gamma\vdash\Delta$ where Γ,Δ are finite sets of formulae. If S is a finite set of sequents then we will denote by $Var(S)$ a set of all natural value variables from S and for any evaluation $E : Var(S)\rightarrow Nat$ we will denote by $E(S)$ the substitution of the values of the variables into S.

The sequent with induction assumptions (in short i-sequent) is an ordered pair $\Sigma\|\Pi$, where Π is a sequent and Σ is a finite set of sequents (the induction assumptions of the sequent Π).

Let $\mathcal{M}=<W,\rho,\pi,E>$ be a Kripke structure. The sequent $\Gamma\vdash\Delta$ is said to be valid in a state $s\in W$ (written $s\models(\Gamma\vdash\Delta)$) iff
(for all $A\in\Gamma$) $(s\models A)$ implies (there is some $B\in\Delta$) ($s\models B$). A sequent $\Gamma\vdash\Delta$ is said to be valid (written $\models(\Gamma\vdash\Delta)$) if it is valid in all Kripke structures in all states.

An i-sequent $\Sigma\|\ \Gamma\vdash\Delta$ is said to be valid in a Kripke structure $\mathcal{M}=<W,\rho,\pi,E>$ iff for the evaluation $E : Var\rightarrow Nat$, where $Var=\{v_1,\ldots,v_n\}=Var(\ \Sigma\|\ \Gamma\vdash\Delta\ )$, the following holds:
$(\forall i_1\leq E(v_1))\ldots(\forall i_n\leq E(v_n))$ for all $(\ \Gamma'\vdash\Delta'\ )\in\Sigma$ $(\models(\ \Gamma'\vdash\Delta'\ )[v_1/i_1,\ldots,v_n/i_n])$ implies $\models E(\ \Gamma\vdash\Delta\ )$. Two i-sequents are said to be equivalent iff they are the same up to renaming of natural value variables.

A i-sequent $\Sigma\|\ \Gamma\vdash\Delta$ is said to be valid (written $\models\ \Sigma\|\ \Gamma\vdash\Delta$) if it is valid in all Kripke structures.

Lemma 2.
The following are equivalent:
1) $\models\ <\alpha\star>\phi,\Gamma\vdash\Delta$
2) for all $v\in\aleph\ \models\ <\alpha^v>\phi,\Gamma\vdash\Delta$
3) $\models\ \phi,\Gamma\vdash\Delta$ and for all $v\in aleph\models\ <\alpha><\alpha^v>\phi,\Gamma\vdash\Delta$
4) $\models\ \emptyset\|\ \phi,\Gamma\vdash\Delta$ and $\models\ \{<\alpha^v>\phi,\Gamma\vdash\Delta\}\|\ <\alpha><\alpha^v>\phi,\Gamma\vdash\Delta$

Let us remark that a formula A is valid iff the sequent $\emptyset\vdash A$ is valid and a sequent $\Gamma\vdash\Delta$ is valid iff the i-sequent $\emptyset\|\ \Gamma\vdash\Delta$ is valid.

Gentzen type i-sequent calculus $G_1(PDL)$ for the extended PDL is a set of axioms and a set of rules. An i-sequent Θ is said to be provable (written $\vdash_{G(PDL)}\Theta$) if there is a proof of the i-sequent. Below we will use the following notation:
$\Gamma^{[\alpha]}=\{A|[\alpha]A\in\Gamma\}$, $\Gamma^{<\alpha>}=\{A|<\alpha>A\in\Gamma\}$,
$<\alpha>\Gamma=\{<\alpha>A|<\alpha>A\in\Gamma\}$, $[\alpha]\Gamma=\{[\alpha]A|[\alpha]A\in\Gamma\}$,
Γ_0 = the subset of all propositional formulae of Γ,
where Γ is a set of formulae and α is a program. For Gentzen-type cut-free axiomatization G(PDL) of the extended PDL see below. The rules of this system fall into three parts: modal rules $((<>\vdash),(\vdash[\ ]))$, induction rules $((<*>\vdash),(\vdash[*]))$ and all the other rules are simplifying ones.

Theorem 2.
Gentzen-type i-sequent calculus $G_1(PDL)$ is sound, i.e all provable i-sequents are valid.
Proof. All the axioms are valid i-sequents and for each inference rule if all premises of the rule are valid i-sequents then the conclusion is a valid i-sequent.$\Box$

Corollary 1.
1. For any PDL sequent $\Gamma\vdash\Delta$ the following holds:
$\vdash_{G(PDL)}\ \emptyset\|\ \Gamma\vdash\Delta$ implies $\models\ \Gamma\vdash\Delta$.

2. For any PDL formula A the following holds: $\vdash_{G(PDL)} \quad \emptyset \| \ \emptyset \vdash A \quad$ implies $\models A$.

Let us extend $G_1(PDL)$ with the *cut* rule $\dfrac{\Sigma \| \ \Gamma, A \vdash \Delta \qquad \Sigma \| \ \Gamma \vdash \Delta, A}{\Sigma \| \ \Gamma \vdash \Delta}(cut)$

and denote the extended calculus as $G_1(PDL) + cut$.

Theorem 4 [Soundness and completeness theorem].
1. Gentzen-type i-sequent calculus $G_1(PDL) + cut$ is sound, i.e all provable i-sequents are valid.
2. Gentzen-type i-sequent calculus $G_1(PDL) + cut$ is complete, i.e all valid i-sequents are provable.

Proof.
1. The proof is tedious and ommited.
2. We show by induction on the length of the proof that all the formulae which are provable in Hilbert-style axiomatization $H(PDL)$ for PDL are provable in $G_1(PDL) + cut$ too. Let A be a formula of $H(PDL)$ and $A_1, \ldots, A_n$ be a sequence of formulae such that $A_n = A$, and for each other formula from the sequence we have either A_i is an axiom or it can be proved from the formulae A_j and A_k (where $j < i, k < i$) by using *modus ponens* or *modal generalization* rules.
All the axioms (including the *induction axiom*) which are provable in $H(PDL)$ are provable in $G_1(PDL) + cut$. Therefore, if A_i is an axiom then $\vdash_{G_1(PDL)+cut} A$
Let A_i be a formula that proved in $H(PDL)$ by using *modus ponens* rule from A_j and $A_k = A_j \rightarrow A_i$ where $j < i, k < i$. Then by induction hypothesis we have that A_j and A_k are provable in $G_1(PDL) + cut$, i.e. $\vdash_{G_1(PDL)+cut} A_j$ and $\vdash_{G_1(PDL)+cut} A_j \rightarrow A_i$. From it foolows that $A_j \vdash_{G_1(PDL)+cut} A_i$ and hence by using *cut* rule we have $\vdash_{G_1(PDL)+cut} A_i$.
Let $A_i = [\alpha]A_j$ (where $j < i$) be a formula proved by using *modal generalization* rule in $H(PDL)$ from the formula A_j. Then by the induction hypothesis we have that $\vdash_{G_1(PDL)+cut} A_j$ and using modal rule ($\vdash [\,]$) of $G_1(PDL)$ we obtain $\vdash_{G_1(PDL)+cut} [\alpha]A_j$.$\square$

Let us extend $G_1(PDL)$ with the rule:

$$\frac{\Sigma \| \ \Gamma, A \vdash \Delta \qquad \Sigma \| \ \Gamma, < \alpha >< \alpha^v > A \vdash \Delta}{\Sigma \| \ \Gamma, < \alpha^v > A \vdash \Delta}(< v > \vdash)$$

and denote the extended calculus as $G_2(PDL)$.

Theorem 4 [Soundness and completeness theorem].
1. Gentzen-type i-sequent calculus $G_2(PDL)$ is sound and complete, i.e. the sets of valid i-sequents and provable i-sequents coincide

Proof.
1. The proof is tedious and ommited.$\square$

Gentzen-type i-sequent calculus $G_1(PDL)$

Axioms are i-sequents of two types:

(Usual) $\quad \Sigma \| \ \Gamma, A \vdash \Delta, A$
(Induction) $\quad \Sigma \cup \{\Gamma' \vdash \Delta'\} \| \ \Gamma \vdash \Delta$
$\qquad$ if $\quad \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta$

where Γ, Δ are sets of formula, Σ is a set of sequents, A is a formula.

Inference rules:

$$\frac{\Sigma\|\ \Gamma, A, B \vdash \Delta}{\Sigma\|\ \Gamma, A \wedge B \vdash \Delta}(\wedge \vdash) \qquad \frac{\Sigma\|\ \Gamma \vdash \Delta, A \qquad \Sigma\|\ \Gamma \vdash \Delta, B}{\Sigma\|\ \Gamma \vdash \Delta, A \wedge B}(\vdash \wedge)$$

$$\frac{\Sigma\|\ \Gamma \vdash A, \Delta}{\Sigma\|\ \Gamma, \neg A \vdash \Delta}(\neg \vdash) \qquad \frac{\Sigma\|\ \Gamma, A \vdash \Delta}{\Sigma\|\ \Gamma \vdash \neg A, \Delta}(\vdash \neg)$$

$$\frac{\Sigma\|\ \Gamma, A \vdash \Delta \qquad \Sigma\|\ \Gamma, B \vdash \Delta}{\Sigma\|\ \Gamma, A \vee B \vdash \Delta}(\vee \vdash) \qquad \frac{\Sigma\|\ \Gamma \vdash \Delta, A, B}{\Sigma\|\ \Gamma \vdash \Delta, A \vee B}(\vdash \vee)$$

$$\frac{\Sigma\|\ \Gamma, A \vdash \Delta \qquad \Sigma\|\ \Gamma \vdash \Delta, B}{\Sigma\|\ \Gamma, A \rightarrow B \vdash \Delta}(\rightarrow\vdash) \qquad \frac{\Sigma\|\ \Gamma, A \vdash \Delta, B}{\Sigma\|\ \Gamma \vdash \Delta, A \rightarrow B}(\vdash\rightarrow)$$

$$\frac{\Sigma\|\ \Gamma \vdash \Delta, <\alpha><\alpha\star> A, A}{\Sigma\|\ \Gamma \vdash \Delta, <\alpha\star> A}(\vdash<\star>) \qquad \frac{\Sigma\|\ \Gamma, [\alpha][\alpha\star]A, A \vdash \Delta}{\Sigma\|\ \Gamma, [\alpha\star]A \vdash \Delta}([\star]\vdash)$$

$$\frac{\Sigma\|\ \Gamma, A, B \vdash \Delta}{\Sigma\|\ \Gamma, <A?> B \vdash \Delta}(<?>\vdash) \qquad \frac{\Sigma\|\ \Gamma \vdash \Delta, A \qquad \Sigma\|\ \Gamma \vdash \Delta, B}{\Sigma\|\ \Gamma \vdash \Delta, <A?> B}(\vdash<?>)$$

$$\frac{\Sigma\|\ \Gamma \vdash \Delta, <\alpha> A, <\beta> A}{\Sigma\|\ \Gamma \vdash \Delta, <\alpha \cup \beta> A}(\vdash<\cup>) \qquad \frac{\Sigma\|\ \Gamma, [\alpha]A, [\beta]A \vdash \Delta}{\Sigma\|\ \Gamma, [\alpha \cup \beta]A \vdash \Delta}([\cup]\vdash)$$

$$\frac{\Sigma\|\ \Gamma, <\alpha><\beta> A \vdash \Delta}{\Sigma\|\ \Gamma, <\alpha;\beta> A \vdash \Delta}(<;>\vdash) \qquad \frac{\Sigma\|\ \Gamma \vdash \Delta, <\alpha><\beta> A}{\Sigma\|\ \Gamma \vdash \Delta, <\alpha;\beta> A}(\vdash<;>)$$

$$\frac{\Sigma\|\ \Gamma, [\alpha][\beta]A \vdash \Delta}{\Sigma\|\ \Gamma, [\alpha;\beta]A \vdash \Delta}([;]\vdash) \qquad \frac{\Sigma\|\ \Gamma \vdash \Delta, [\alpha][\beta]A}{\Sigma\|\ \Gamma \vdash \Delta, [\alpha;\beta]A}(\vdash [;])$$

$$\frac{\Sigma\|\ \Gamma, A \vdash \Delta \qquad \Sigma\|\ \Gamma \vdash \Delta, B}{\Sigma\|\ \Gamma, [B?]A \vdash \Delta}([?]\vdash) \qquad \frac{\Sigma\|\ \Gamma, A \vdash \Delta, B}{\Sigma\|\ \Gamma \vdash \Delta, [A?]B}(\vdash [?])$$

$$\frac{\Sigma\|\ \Gamma^{[\alpha]}, A \vdash \Delta^{<\alpha>}}{\Sigma\|\ \Gamma, <\alpha> A \vdash \Delta}(<>\vdash) \qquad \frac{\Sigma\|\ \Gamma^{[\alpha]} \vdash \Delta^{<\alpha>}, A}{\Sigma\|\ \Gamma \vdash \Delta, [\alpha]A}(\vdash [\,])$$

$$\frac{\Sigma\|\ \Gamma, A \vdash \Delta \qquad \Sigma \cup \{\ \Gamma, <\alpha^v> A \vdash \Delta\ \}\|\ \Gamma, <\alpha><\alpha^v> A \vdash \Delta}{\Sigma\|\ \Gamma, <\alpha\star> A \vdash \Delta}(<\star>\vdash)$$

$$\frac{\Sigma\|\ \Gamma \vdash \Delta, A \qquad \Sigma \cup \{\ \Gamma \vdash \Delta, [\alpha^v]A\ \}\|\ \Gamma \vdash \Delta, [\alpha][\alpha^v]A}{\Sigma\|\ \Gamma \vdash \Delta, [\alpha\star]A}(\vdash [\star])$$

$$\frac{\Sigma\|\ \Gamma, <\alpha> A \vdash \Delta \qquad \Sigma\|\ \Gamma, <\beta> A \vdash \Delta}{\Sigma\|\ \Gamma, <\alpha \cup \beta> A \vdash \Delta}(<\cup>\vdash)$$

$$\frac{\Sigma\|\ \Gamma \vdash \Delta, [\alpha]A \qquad \Sigma\|\ \Gamma \vdash \Delta, [\beta]A}{\Sigma\|\ \Gamma \vdash \Delta, [\alpha \cup \beta]A}(\vdash [\cup])$$

where Γ, Δ are sets of formulae, Σ is a set of sequents, α, β are programs, A, B are formulae, v is a new natural value variable.

5 Conclussion

In the paper we proposed some Gentzen-type sequent calculi for PDL. There are still some open questions:
1. Is the i-sequent calculus $G_1(PDL)$ complete?
2. Can the *cut* rule of $G_1(PDL) + cut$ be eliminated?
3. Is there some other way of extending of $G_0(PDL)$ to obtain complete calculus?
These questions are good start points for future research.

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