Exercise 1. Consider the following UML class diagram.

1. Express it in FOL.
2. Express it in DL-Lite$_A$, highlighting parts that are not expressible.
3. Given the ABox $A = \{ A(a) \}$ and the conjunctive query $q(x) \leftarrow Rac(x, y), Rbd(y, z), A(z)$, return the certain answer by exploiting the DL-Lite$_A$ rewriting algorithm.

Solution

\[
\begin{align*}
q(x) & \leftarrow Rac(x, y), Rbd(y, z), A(z) & B \subseteq A \\
q(x) & \leftarrow Rac(x, y), Rbd(y, z), B(z) & D \subseteq B \\
q(x) & \leftarrow Rac(x, y), Rbd(y, z), D(z) & \exists Rbd^{-} \subseteq D \\
q(x) & \leftarrow Rac(x, y), Rbd(y, z), Rbd(w, z) & w = y, z = z \\
q(x) & \leftarrow Rac(x, y), Rbd(y, z) & B \subseteq \exists Rbd \\
q(x) & \leftarrow Rac(x, y), B(y) & C \subseteq B \\
q(x) & \leftarrow Rac(x, y), C(y) & \exists Rac^{-} \subseteq C \\
q(x) & \leftarrow Rac(x, y), Rac(v, y) & v = x, y = y \\
q(x) & \leftarrow Rac(x, y) & A \subseteq \exists Rac \\
q(x) & \leftarrow A(x) & \implies x = a
\end{align*}
\]

Exercise 2. Model check the Mu-Calculus formula $\nu X.\mu Y.(a \lor \langle \text{next} \rangle X) \land [\text{next}] Y$ and the CTL formula $EG(\neg a \supset AXAFa)$ (showing its translation in Mu-Calculus) against the following transition system:

Solution

We denote by $X_i$ a set of states and by $X_i$ a new proposition such that $[X_i] = X_i$. Similarly for $Y_i$ and $Y_i$. We need to compute $[[\nu X.\mu Y.(a \lor \langle \text{next} \rangle X) \land [\text{next}] Y]]$
\[ \mathcal{X}_0 = \{1, 2, 3, 4\} \]

\[ \mathcal{X}_1 = [\mu Y. (a \lor \langle next \rangle X) \land [next] Y] \]

\[ \mathcal{Y}_{10} = \emptyset \]

\[ \mathcal{Y}_{11} = [(a \lor \langle next \rangle X) \land [next] Y_0] = \{s1, s2, s3, s4\} \cap PreA(\emptyset) = \emptyset \]

\[ \mathcal{X}_1 = \emptyset \]

\[ \mathcal{X}_2 = [\mu Y. (a \lor \langle next \rangle X) \land [next] Y] \]

\[ \mathcal{Y}_{20} = \emptyset \]

\[ \mathcal{Y}_{21} = [(a \lor \langle next \rangle X) \land [next] Y_0] = \{s1, s2, s3, s4\} \cap PreA(\emptyset) = \emptyset \]

\[ \mathcal{X}_2 = \emptyset \]

So the TS does not satisfy the formula, since its initial state s1 is not in \([\nu X. \mu Y. (a \lor \langle next \rangle X) \land [next] Y]\).

Notice that the fact that we are looking for a least fixpoint of a variable (Y) in a next operator [next]Y that occurs in AND with a complex expression \(\mu Y. (a \lor \langle next \rangle X) \land [next] Y\) trivializes the computation of the fixpoint to the empty set.

Checking the CTL formula and translating it into mu-calculus is left as an exercise.

**Exercise 3.** Consider the following predicates \texttt{Employee(x)} saying that x is an employee, \texttt{Manages(x, y)} saying that x manages y, and \texttt{MSc(x)} saying that x is a person with master degree. Express in FOL the following boolean queries (stating which ones are CQs):

1. There exists an employee with master degree that manages someone with the master degree.
2. There exists an employee with master degree that manages at least two people with the master degree.
3. There exists an employee that manages someone with the master degree and someone without the master degree.
4. There exists an employee that manages only people with master degree.
5. There exists an employee that manages all the people with master degree.

**Solution**

1. There exists an employee with master degree that manages someone with the master degree

   \[ \exists x. \text{Employee}(x) \land \text{MSc}(x) \land \text{Manages}(x, y) \land \text{MSc}(y) \] (CQ)

2. There exists an employee with master degree that manages at least two people with the master degree

   \[ \exists x. \text{Employee}(x) \land \text{MSc}(x) \land \text{Manages}(x, y) \land \text{MSc}(y) \land \text{Manages}(x, z) \land \text{MSc}(z) \land y \neq z \]

3. There exists an employee that manages someone with the master degree and someone without the master degree

   \[ \exists x. \text{Employee}(x) \land \text{Manages}(x, y) \land \text{MSc}(y) \land \text{Manages}(x, z) \land \neg \text{MSc}(z) \]

4. There exists an employee that manages only people with master degree.

   \[ \exists x. \text{Employee}(x) \land (\forall y. \text{Manages}(x, y) \supset \text{MSc}(y)) \]

5. There exists an employee that manages all the people with master degree.

   \[ \exists x. \text{Employee}(x) \land (\forall y. \text{Msc}(y) \supset \text{Manages}(x, y)) \]
Exercise 4. Compute the certain answers to the CQ \( q(x) \leftarrow Employee(x), Manages(x, y) \) over the incomplete database (naive tables):

\[
\begin{array}{|c|c|}
\hline
\text{Employee} & \text{Manages} \\
\hline
\text{name} & \text{mgr} & \text{mgd} \\
\hline
\text{Smith} & \text{Smith} \\
\text{null} & \text{null} \\
\text{Brown} & \text{null} \\
\hline
\end{array}
\]

Solution

- Evaluate \( q \) over the database as it was a complete database
- Filter out all answers where null appears (certain answers are constituted by tuples of constants in \( Cons \))

Answer: \{Smith, Brown\}

Exercise 5. Compute the weakest precondition for getting \( \{x = y\} \) by executing the following program:

\[
x := y + 1; \\
\text{if } (y > 0) \text{ then} \\
\quad x := x + y \\
\text{else } x := y + 100; \\
\text{else } x := y + 100; \\
x := x + y;
\]

Solution

\[
\{\{y > 0 \land y + 1 + y = 0\} \mid (y = -100)\} = \{ (y > 0 \land y = -0.5 \mid y = -100\} = \{\text{false} \mid y = -100\}
\]

\[
x := y + 1;
\]

\[
\{\{y > 0 \land x + y = 0\} \mid (y =< 0 \land y=-100)\} = \{(y > 0 \land x + y = 0) \mid (y = -100)\}
\]

\[
\text{if } (y > 0) \text{ then} \\
\quad \{x + y = 0\}
\]

\[
\quad x := x + y
\]

\[
\quad \{x=0\}
\]

\[
\quad (y + 200 = 0) = (y = -100)
\]

\[
\text{else } x := y + 100;
\]

\[
\quad \{x=0\}
\]

\[
\quad \{x+y = y\} = \{x=0\}
\]

\[
x := x + y;
\]

\[
\{x=y\}
\]