Bachelor’s degree in Bioinformatics

# *Differential Equations*

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# What is a Differential Equation

A **differential equation** is an equation that contains an **unknown function** and **one or more of its derivatives**. Such equations arise in a variety of situations where we need to find a function, not just a single value.

The **order** of the differential equation is the order of the highest derivative appearing in the equation.

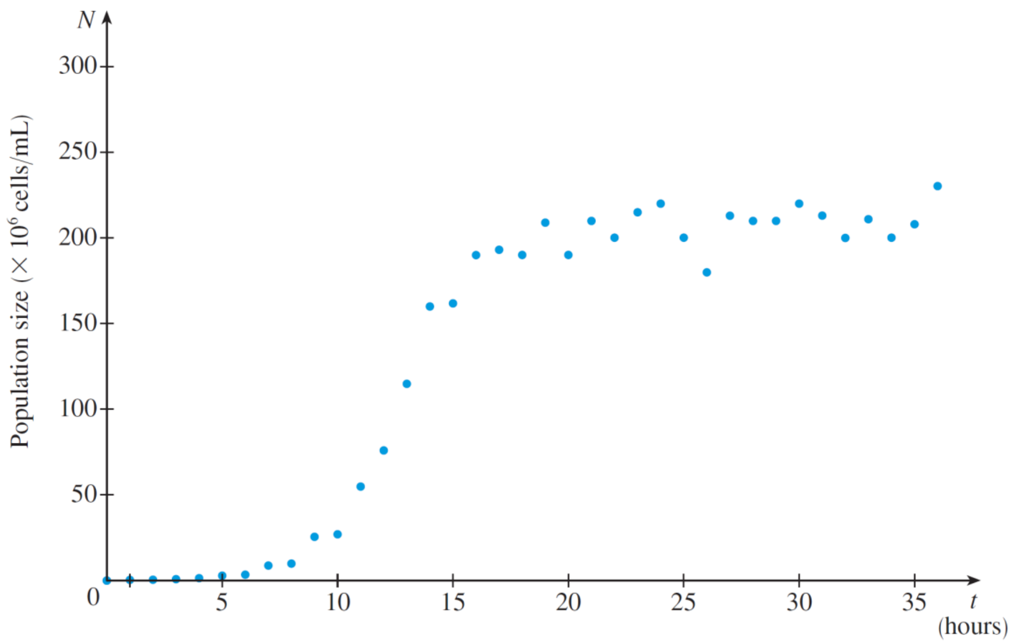
For example, $y'(t) + 2y(t) = 3$ is a **first-order** differential equation, whereas $5y''(t) - y'(t) = y(t)$ is a **second-order** differential equation.

The **solution** of a differential equation is a function that, when substituted into the equation, produces an equality.

# Models of Population Growth

One of the most common example is in models of population growth. Consider the growth of a population of yeast.

| Time (h) | Pop. size ( $\times 10^6$/mL) | Time (h) | Pop. size ( $\times 10^6$/mL) |
|---|---|---|---|
| 0 | 0.200 | 19 | 209 |
| 1 | 0.330 | 20 | 190 |
| 2 | 0.500 | 21 | 210 |
| 3 | 1.10 | 22 | 200 |
| 4 | 1.40 | 23 | 215 |
| 5 | 3.10 | 24 | 220 |
| 6 | 3.50 | 25 | 200 |
| 7 | 9.00 | 26 | 180 |
| 8 | 10.0 | 27 | 213 |
| 9 | 25.4 | 28 | 210 |
| 10 | 27.0 | 29 | 210 |
| 11 | 55.0 | 30 | 220 |
| 12 | 76.0 | 31 | 213 |
| 13 | 115 | 32 | 200 |
| 14 | 160 | 33 | 211 |
| 15 | 162 | 34 | 200 |
| 16 | 190 | 35 | 208 |
| 17 | 193 | 36 | 230 |
| 18 | 190 | | |



# We need to find a Function

Although the population was measured at one-hour intervals, the yeast themselves are replicating in a way that is nearly **continuous**.

How can we model such processes? Let's assume that each individual yeast cell produces offspring at a constant rate $\beta$

Thus the total **birth rate** at time $t$ is $\beta N(t)$, where $N(t)$ is the number of yeast cells present at time $t$

Likewise, suppose the total **loss rate** of yeast cells through death at time $t$ is $\mu N(t)$, where $\mu$ is a constant death rate per individual.

With the preceding assumptions, we see that the **rate of change** of the number of yeast cells at time $t$ is the total birth rate minus the total death rate $\beta N(t) - \mu N(t)$

# We need to find a Function

And since the rate of change of *N*(*t*), the number of yeast cells, can also be written as *dN*(*t*)/*dt*, we can write

$$\frac{dN(t)}{dt} = \beta N(t) - \mu N(t)$$

Now if we define a constant *r* *as* $r = \beta - \mu$

we can write $$\frac{dN(t)}{dt} = rN(t)$$

Which is a **differential equation** with unknown function *N*(*t*). The quantity *r* is called the **per capita growth rate**. It is the rate of growth of the population *per individual*.
Since *dN*/*dt* is the rate of growth of the population, the rate of growth *per individual* is *dN*/*dt* divided by *N*(*t*).

# The Logistic Differential Equation

This was an example of a more general model for population growth called the *logistic differential equation*.

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N$$

where we also suppose that the per capita growth rate decreases linearly as the population size increases. The positive constant $K$ is called *carrying capacity*; it is the population size at which crowding and resource depletion cause the per capita growth rate to be zero.

# Solution of a Differential Equations

The **order** of the differential equation is the order of the highest derivative appearing in the equation.

Take for example the 1st order differential equation $dy/dt = 2 + y(t)$
In this case, the **unknown function** is $y(t) = e^t - 2$
We can verify it by substituting the function into the equation: left side and right side become

$$\frac{dy}{dt} = \frac{d}{dt}(e^t - 2) = e^t \qquad\qquad 2 + (e^t - 2) = e^t$$

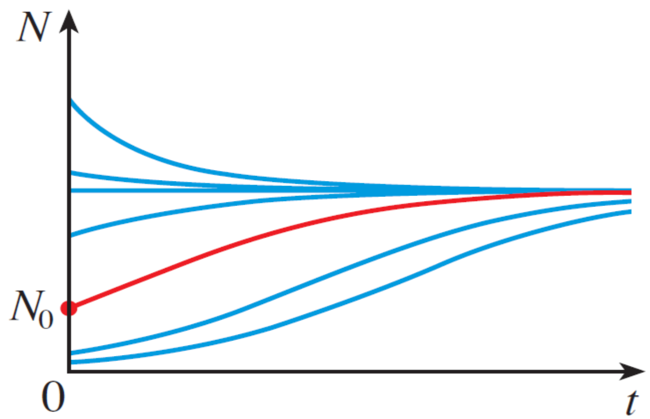
The right and left sides evaluate to the same expression, demonstrating that the function $y(t) = e^t - 2$ is indeed a solution.

# Solution of a Differential Equations

Typically, there are several solutions to a differential equation. We often need to find the particular solution that satisfies an additional condition of the form $y(t_0) = y_0$ , called an **initial condition**.

The problem of finding a solution satisfying an initial condition is called an **initial-value problem**. Graphically, when we impose an initial condition, we look at the family of solution curves and pick the one passing by the point $(t_0, y_0)$

For example, this is the family of solutions of the logistic equation. The one satisfying an initial condition $N(0) = N_0$ is in red.



# Types of Differential Equations

We consider three types of first-order differential equations: **Pure-time**, **Autonomous**, and **Non-autonomous** differential eq.

**Pure-time differential equations** involve the derivative of the function but not the function itself.

$$\frac{dy}{dt} = f(t)$$

For example, if the rate of change of population size *y* depends on time only, and not on the population, we have a differential equation of this form.

We can obtain the solution $y(t)$ by simply finding the **antiderivative** of $f(t)$. Although we refer to such equations as *pure-time differential equations*, the independent variable may also be another measure.

# Example of Pure-time Eq.

Solve the following differential equation for function $n(x)$

$$\frac{dn}{dx} = 1 - 2e^{-x}$$ with condition $n(10) = 20$

First we find $n(x)$ by integrating both sides with respect to $x$

$$\int \frac{dn}{dx}\, dx = \int (1 - 2e^{-x})\, dx$$

We find the family of solutions $n(x) = x + 2e^{-x} + C$

Now we need to choose the one satisfying $n(10) = 20$

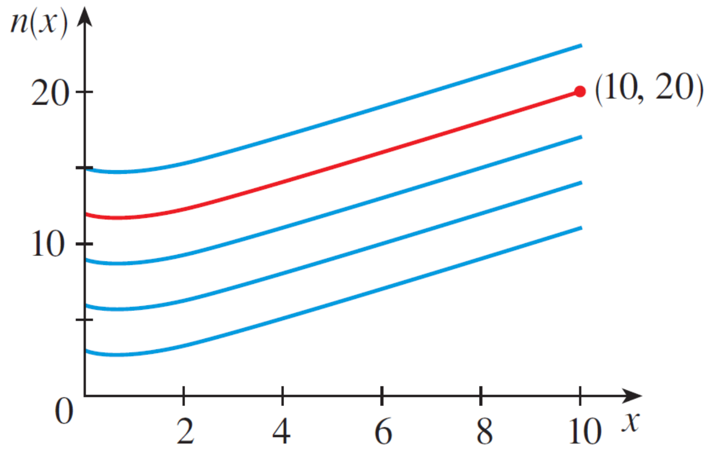
$$n(10) = 10 + 2e^{-10} + C = 20$$

This gives $C = 10 - 2e^{-10}$

# Example of Pure-time Eq.

Therefore the solution is the one in red:

$n(x) = x + 2e^{-x} + 10 - 2e^{-10}$



# Autonomous Differential Equations

**Autonomous differential equations** arise when the equation involves both the derivative of the function and the function itself, but when there is no explicit dependence on the independent variable of the function. Such equations have the general form

$$\frac{dy}{dt} = g(y)$$

In practice, it describes a system that is time-invariant: the behavior depends only on the current state, and the time axis can be shifted without effects (attention: this does not mean that the behavior is constant).
The system is generally self-contained, without external inputs.

# Example of Autonomous Eq.

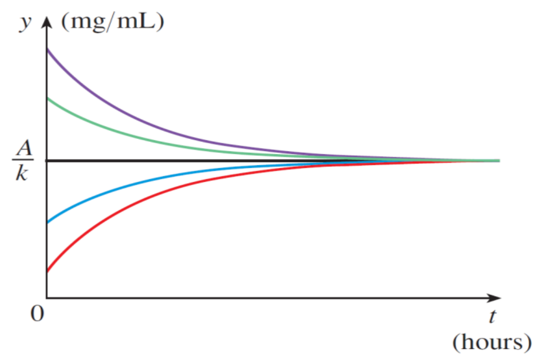
A drug is metabolized at a rate depending on the current concentration of the drug in the blood, denoted by $y(t)$ (and not only depending on time, like in the previous case) Hence, the **outflow** of drug through metabolism is $ky$ where $k$ is a positive constant of proportionality.

The drug is administered through a constant intravenous supply (this does not depend on the drug concentration in blood). The **inflow** of drug through IV supply is $A$ (a positive constant).

Thus, the total rate of change of concentration resulting from both processes (that is, $dy/dt$) is the family of solutions to the equation.

$$\frac{dy}{dt} = A - ky$$

And the drug concentration tends to a limiting value of $A/k$ as time passes, regardless of the initial concentration.



# Non-autonomous Differential Eq.

**Non-autonomous differential equations** are a combination of pure-time and autonomous differential equations. They arise when the equation involves the function and its derivative, and the independent variable appears explicitly as well.

In practice, the system behavior changes over time: there is some external input that modifies its behavior.

Example: a drug is administered to a patient intravenously at a time-varying rate of $A(t) = 1 + \sin t$, and is metabolized at a rate of $y(t)$, where $y(t)$ is the concentration at time $t$. Thus we now have the following non-autonomous differential equation

$$\frac{dy}{dt} = 1 + \sin t - y$$

# Example of Non-autonomous Eq.

Sample equation: $$\frac{dy}{dt} = 1 + \sin t - y$$

We can verify that this family of functions satisfy the equation.

$$y(t) = Ce^{-t} + \tfrac{1}{2}(2 - \cos t + \sin t)$$

We substitute this solution in left and right sides, and we obtain that the equation is satisfied.

Left side: $$-Ce^{-t} + \tfrac{1}{2}(\sin t + \cos t).$$

Right side: $$1 + \sin t - y = 1 + \sin t - Ce^{-t} - \tfrac{1}{2}(2 - \cos t + \sin t)$$

$$= -Ce^{-t} + \tfrac{1}{2}(\sin t + \cos t)$$

# Phase Plots

**Phase plots** provide a way to visualize the dynamics of autonomous differential equations, to locate their equilibria, and to determine the stability properties of these equilibria. Consider the autonomous differential equation

$$\frac{dy}{dt} = g(y)$$

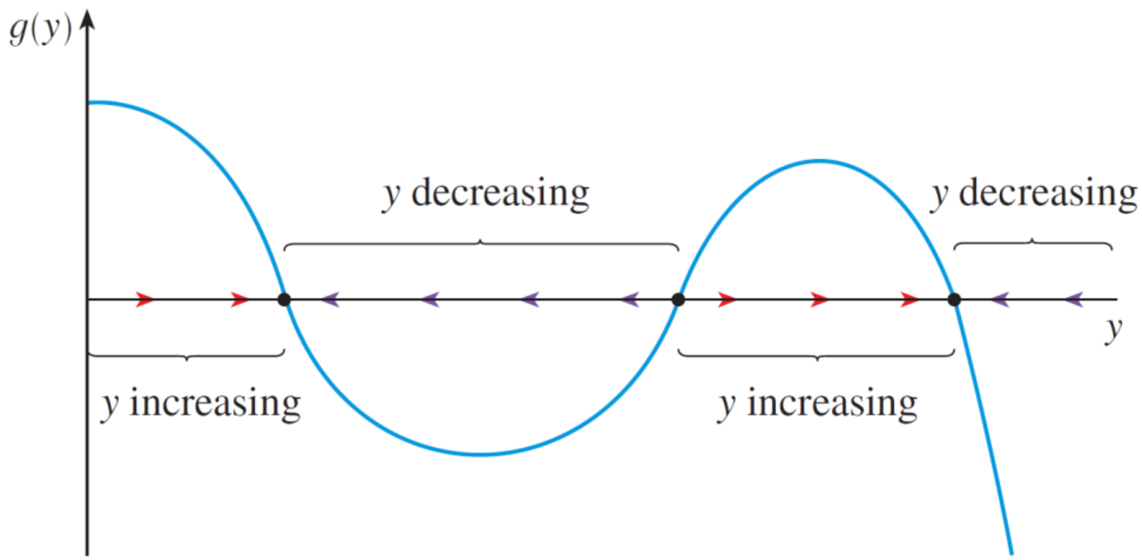
To construct a phase plot we graph the right side of the differential equation, $g(y)$, as a function of the dependent variable $y$.

Where this plot lies above the horizontal axis, $y'(t) > 0$ and so $y$ is increasing. Where it lies below the horizontal axis, $y'(t) < 0$ and so $y$ is decreasing.

# Phase Plots

Points where the plot crosses the axis correspond to values of the variable at which $y'(t) = 0$

We can use these considerations to place arrows on the horizontal axis indicating the direction of change in $y$

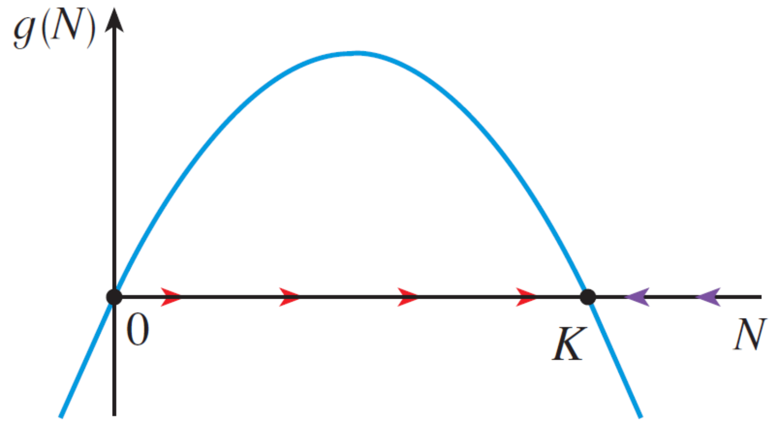


# Example – The logistic equation

Construct a phase plot for the logistic growth model
$dN/dt = r(1 - N/K)N$   assuming that $r > 0$

Solution:
We need to plot $g(N) = r(1 - N/K)N$ as a function of $N$. This is a parabola opening downward, crossing the horizontal axis at $N = 0$ and $N = K$.

Now we can see that $N$ increases when taking on values between 0 and $K$ and decreases when taking on values greater than $K$: the population tends to $K$



# Example – The Allee effect

Some populations decline to extinction once their size is less than a critical value. For example, if the population size is too small, then individuals might have difficulty finding mates for reproduction.
This is referred to as an *Allee effect* after the American ecologist Warder Clyde Allee. This effect can be modelled by a simple extension of the logistic model, where $0 < a < K$

$$\frac{dN}{dt} = r(N - a)\left(1 - \frac{N}{K}\right)N$$

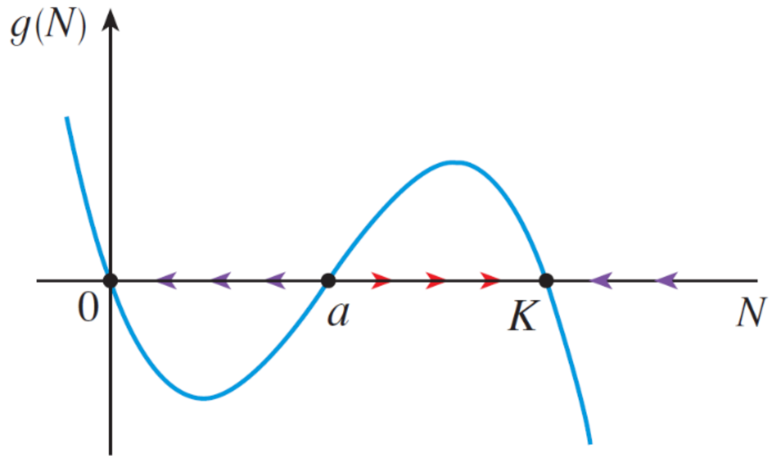
# Example – The Allee effect

Construct a phase plot assuming that $r > 0$

We plot $g(N) = r(N - a)(1 - N/K)N$ as a function of $N$. This is a cubic polynomial whose graph crosses the horizontal axis at $N = 0$, $N = a$, and $N = K$

The graph lies below the horizontal axis for values of $N$ between 0 and $a$ and for values of $N > K$

Therefore $N$ will approach 0 if it starts between 0 and $a$, whereas it will approach $K$ if it starts anywhere greater than $a$




(a) Phase plot for an Allee effect

# Equilibria and Stability

Values of the y at which no change occurs are called *equilibria*

**Definition** Consider the autonomous differential equation

**(1)**
$$\frac{dy}{dt} = g(y)$$

An **equilibrium** solution is a constant value of $y$ (denoted $\hat{y}$) such that $dy/dt = 0$ when $y = \hat{y}$.

Equilibria are found by determining values of $\hat{y}$ that satisfy $g(\hat{y})=0$. They correspond to places where the phase plot crosses the horizontal axis.

# Equilibria in the logistic eq.

Show that $N=0$ and $N=K$ are equilibria of the logistic growth model

Solution:

Substituting $N=0$ and $N=K$ into the equation $dN/dt = r(1 - N/K)N$ gives $dN/dt = 0$ in both cases.

This means that the derivative is 0, so no changes in time. In the yeast data, $N=K$ means steady number of yeast cells reached as the experiment progressed, $N=0$ corresponds to the absence of yeast.

# Equilibria and Stability

An equilibrium can be stable or unstable.

**Definition** An equilibrium $\hat{y}$ of differential equation (1) is **locally stable** if $y$ approaches the value $\hat{y}$ as $t \longrightarrow \infty$ for all initial values of $y$ sufficiently close to $\hat{y}$.

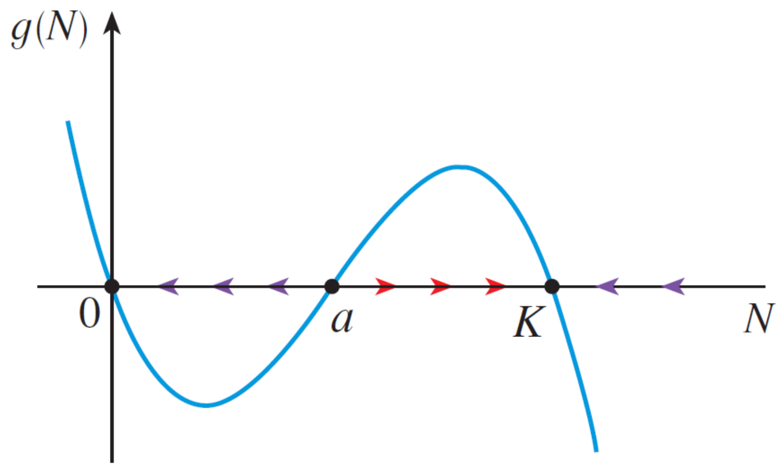
An equilibrium that is not stable is called **unstable**.

# Equilibria in the Allee effect

We need to find the values of $\check{N}$ that satisfy the equation

$$r(\check{N} - a)(1 - \check{N}/K)\,\check{N} = 0$$

We can see that these are $\check{N} = 0$ , $\check{N} = a$ , and $\check{N} = K$. These are the points at which the phase plot crosses the horizontal axis. From the arrows on the figure we can also see that both $\check{N} = 0$ and $\check{N} = K$ are locally stable whereas $\check{N} = a$ is unstable: no matter how close to $a$ we start $N$ , it always moves away from $a$ as time passes



# Equilibria and Stability

**Local Stability Criterion** Suppose that $\hat{y}$ is an equilibrium of the differential equation

$$\frac{dy}{dt} = g(y)$$

Then $\hat{y}$ is *locally stable* if $g'(\hat{y}) < 0$ , and $\hat{y}$ is *unstable* if $g'(\hat{y}) > 0$. If $g'(\hat{y}) = 0$, then the analysis is inconclusive.

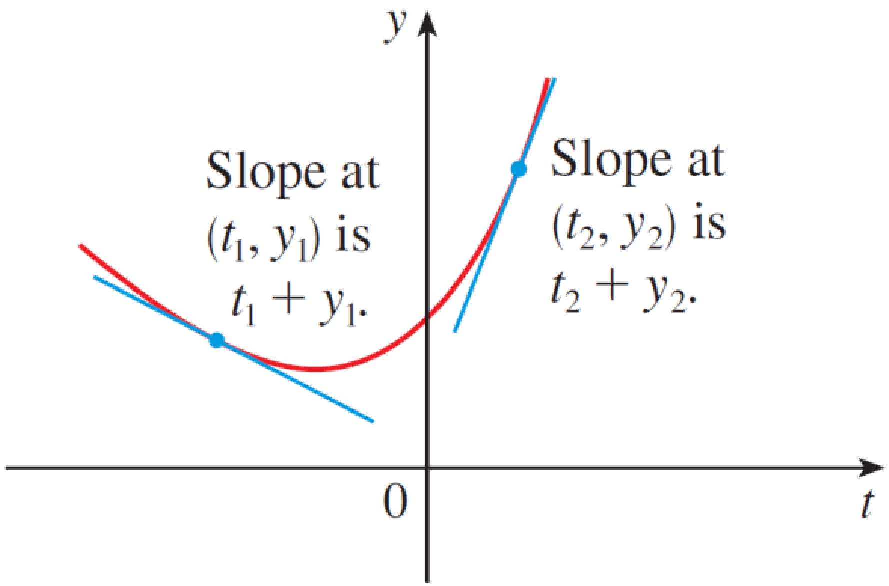
# Direction Fields

Suppose we want to sketch the graph of the solution of the initial-value problem

$$y' = t + y \qquad y(0) = 1$$

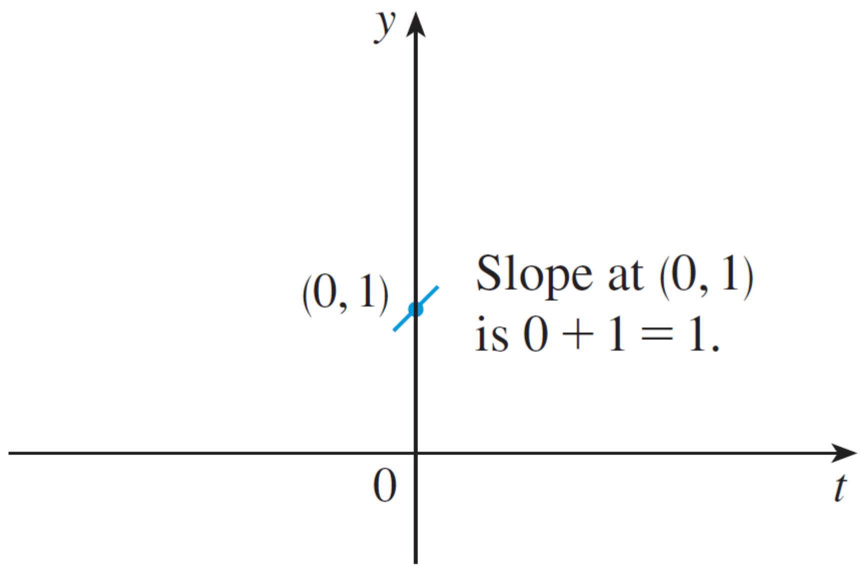
We don't know a formula for the solution, so how can we possibly sketch its graph?

The equation $y' = t + y$ tells us that the slope at any point $(t, y)$ on the graph of $y$ is the sum of the $t$- and $y$-coordinates of the point.



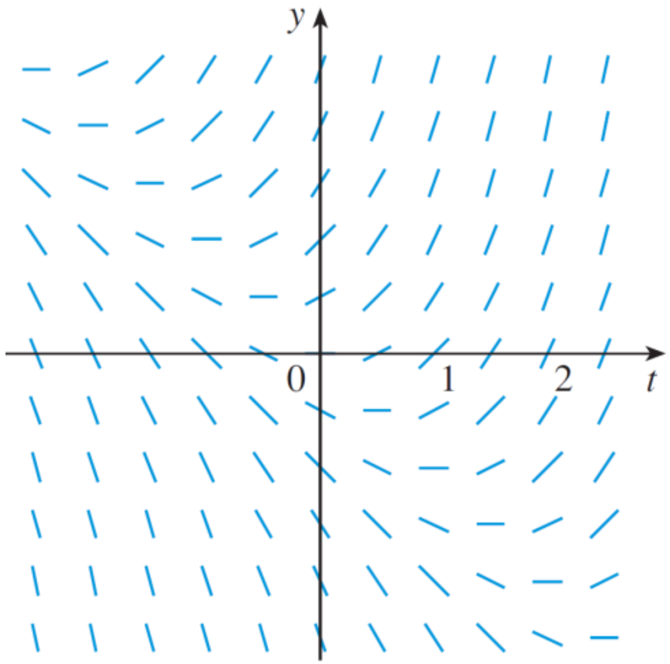
# Direction Fields

In particular, because the curve passes through the point (0, 1), its slope there must be 0 + 1 = 1. So a small portion of the solution curve near the point (0, 1) looks like a short line segment through (0, 1) with slope 1.



# Direction Fields

To sketch the rest of the curve, we draw short line segments at a number of points ($t$, $y$) with slope $t + y$. The result is called a *direction field.*




For instance, the line at the point (1, 2) has slope 1 + 2 = 3
This visualizes the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

# Direction Fields

In the general case, given the first-order differential equation $y' = F(t,y)$, where $F(t, y)$ is some function in $t$ and $y$.

The differential equation says that the slope of a solution curve at a point $(t, y)$ on the curve is $F(t, y)$.
If we draw short line segments with slope $F(t, y)$ at several points $(t, y)$, the result is called a **direction field** (or **slope field**).
These line segments indicate the direction in which a solution curve is heading.

# Example: use of Direction Field

(a) Sketch the direction field for the differential equation

$$\frac{dy}{dx} = x^2 + y^2 - 1.$$

(b) Use part (a) to sketch the solution curve that passes through the origin.

Solution:

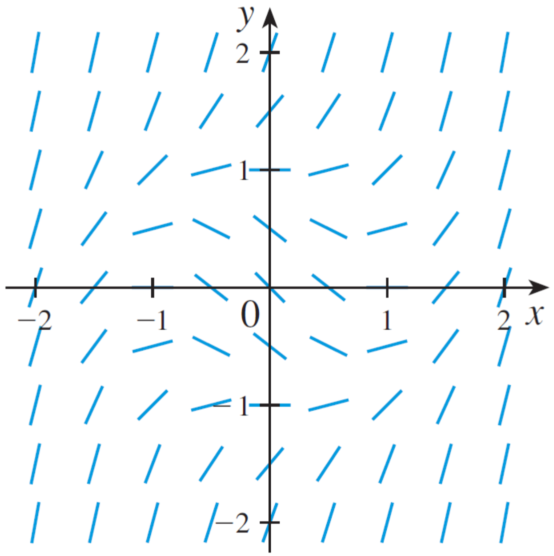
(a) We start by computing the slope at several points, as given in the following table:

$$y' = x^2 + y^2 - 1$$

| $y$ \ $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| 2 | ··· | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ··· |
| 1 | ··· | 4 | 1 | 0 | 1 | 4 | ··· |
| 0 | ··· | 3 | 0 | −1 | 0 | 3 | ··· |
| −1 | ··· | 4 | 1 | 0 | 1 | 4 | ··· |
| −2 | ··· | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ··· |

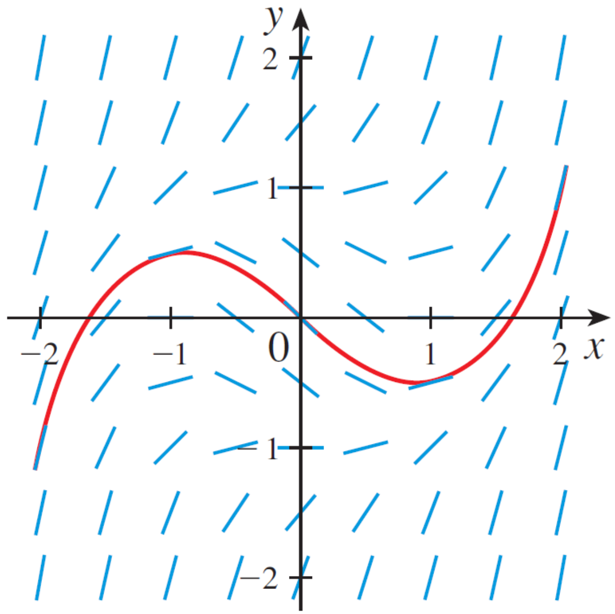
# Example: use of Direction Field

Now we draw short line segments with these slopes at the indicated points.



# Example: use of Direction Field

(b) We start at the origin and move to the right in the direction of the line segment (which has slope –1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. Then, we draw the solution curve to the left as well.



# Euler Method

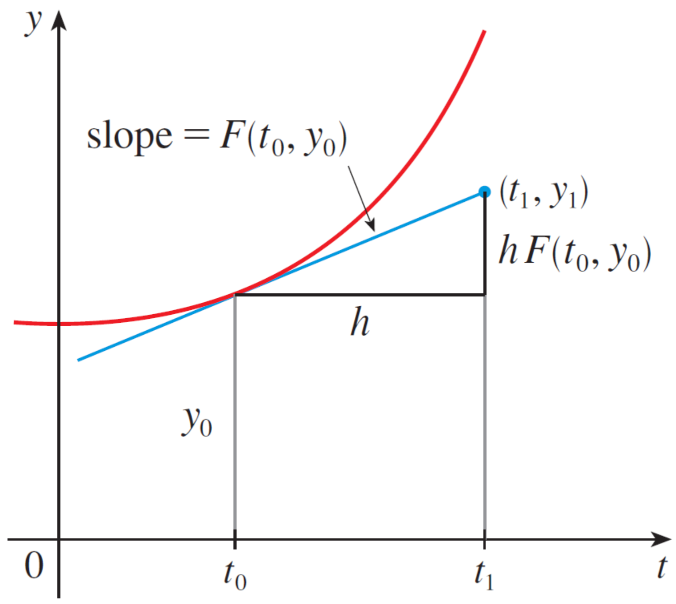
For the general first-order initial-value problem $y' = F(t, y)$, $y(t_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers $t_0$, $t_1 = t_0 + h$, $t_2 = t_1 + h$,… where $h$ is a step size (the smaller, the better approximation we have). The red curve is the real unknown $y(t)$.

The from the equation, the slope at $(t_0, y_0)$ is $y' = F(t_0, y_0)$

At $t_1$ the $y_1$ is about $y_0 + hF(t_0, y_0)$

At $t_2$ the $y_2$ is about $y_1 + hF(t_1, y_1)$

And so on: at $t_{n+1}$ the $y_{n+1}$ is about $y_n + hF(t_n, y_n)$



# Example of the Euler Method

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$\frac{dy}{dx} = x + y \qquad y(0) = 1$$

Solution:

We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$

So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

……

# Example of the Euler Method

Proceeding with the calculations, we get the values in the table, which are approximate values for the $y(t)$

| $n$ | $x_n$ | $y_n$ | $n$ | $x_n$ | $y_n$ |
|---|---|---|---|---|---|
| 1 | 0.1 | 1.100000 | 6 | 0.6 | 1.943122 |
| 2 | 0.2 | 1.220000 | 7 | 0.7 | 2.197434 |
| 3 | 0.3 | 1.362000 | 8 | 0.8 | 2.487178 |
| 4 | 0.4 | 1.528200 | 9 | 0.9 | 2.815895 |
| 5 | 0.5 | 1.721020 | 10 | 1.0 | 3.187485 |

# Separable Equations

A **separable equation** is a first-order differential equation in which the expression for *dy*/*dt* can be factored as a function of *t* times a function of *y* , so the equation can be written in the form:

$$\frac{dy}{dt} = f(t)\, g(y)$$

Equivalently, if $g(y) \neq 0$, we could write:

**(1)**

$$\frac{dy}{dt} = \frac{f(t)}{h(y)}$$

where $h(y) = 1/g(y)$. To solve this equation we rewrite it in the differential form: all *y* on one side and all *t* on the other side

$$h(y)\, dy = f(t)\, dt$$

# Separable Equations

$$h(y)\,dy = f(t)\,dt$$

Then we integrate both sides of the equation:

**(2)** $$\int h(y)\,dy = \int f(t)\,dt$$

Equation 2 defines *y* implicitly as a function of *t*. In some cases we may be able to solve for *y* in terms of *t*

# Example

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$

Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2 dy = x^2 dx \qquad \int y^2 dy = \int x^2 dx \qquad \tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

Where $C$ is an arbitrary constant. We could have used two constants $C_1$ and $C_2$ for the 2 integrals, but then we could combine these constants with $C = C_2 - C_1$

# Example

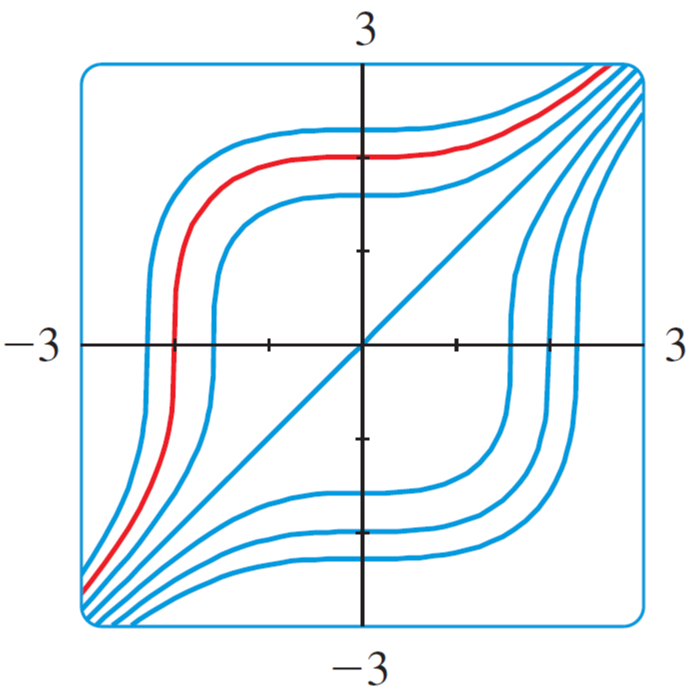
Solving for *y*, we get $y = \sqrt[3]{x^3 + 3C}$

Since C is an arbitrary constant, we define another arbitrary constant K = 3C

$$y = \sqrt[3]{x^3 + K}$$

We can plots this family of solutions.

The one with initial value y(0)=2 is in red



# Example

From $y = \sqrt[3]{x^3 + K}$

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$.

To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$

Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$