Design of observers with gain adaptation for systems with disturbances: an orbital symmetry-based approach

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Abstract

We propose a framework for designing global observers for nonlinear systems with disturbances under geometric conditions based on orbital symmetries. Under some additional restrictions these orbital symmetry-based conditions boil down to geometric homogeneity (at infinity) conditions. Our observers are the result of the combination of a first filter, a state norm estimator, with a second filter adaptively tuned by the first and when compared with the existing literature have a completely novel structure, inherited by the orbital symmetry-based conditions. The observers exploits the geometric properties of orbital symmetries which are one-parameter groups of transformations capable of mapping the system state into neighbourhoods with parametrized width.

Key words: Noisy systems, nonlinear dynamics, observers, gain adaptation.

1 Introduction

Among the various approaches to state estimation for nonlinear systems observer with adaptive or time-varying gains are by far the most popular [14]. Adaptive or time-varying gains allow to overcome classical limitations of high/low-gain observers (HGO), disturbance observers (DBO) and extended state observers (ESO) such as compact domains of error convergence or bounded (with their derivatives) nonlinearities. An adaptive high/low-gain observer is based on the idea of selecting or dynamically updating its gains via parallel filters in such a way as to dominate the nonlinear contribution to the dynamics of the estimation error. The combination of the tuning capabilities of the extended Kalman filter approach with the high-gain global stability properties is considered in [11]. Adaptive high-gain observers are investigated to achieve a tradeoff between transient response in a noise-free setting and sensitivity to disturbances in the presence of noise [3], [6], [15], [17], [22], [29]. In particular, [3] is a first attempt of mixing adaptive-based techniques with homogeneity conditions. Homogeneity is a particular type of symmetry, which is widely used in control theory for system analysis, regulation and observer design (see, for instance, [18], [25] for a geometric definition of homogeneity and [4] for observer design techniques based on weighted homogeneity). Such an interest to the homogeneity concept is based on various useful features of homogeneous systems. In particular, local stability properties are also global; the rate of convergence of homogeneous systems can be assessed by its homogeneity degree and homogeneous systems are robust with respect to external perturbations and time delays. All the previous observer design results with weighted homogeneity require specific system structures, such as chains of integrators, and in many cases restrictive conditions on the increments of the nonlinearities ([4], [8] and references therein). A symmetry-based approach has been initiated in [12] and subsequently in [20] and [23] with local asymptotic convergence results. A sliding-mode approach is also well-studied and recently combined with homogeneity in order to obtain (homogeneous) sliding-mode observers, with convergence in finite time and robustly with respect to bounded disturbances. Application however is still limited to chains of integrators with matched nonlinearities and disturbances and known, constant or time-varying, bound on the state trajectories ([13], [21]), globally Lipschitz nonlinearities ([31]) or incrementally dissipative nonlinearities ([1], [2]).

In this paper we follow and significantly improve the conference paper [5] (which is presented without proofs) by pointing out general observer design techniques resulting from mixing adaptive with symmetry-based methodologies. We stress the fact that our analysis and design is aimed at global state observers, i.e. the estimation error converges for any state and observer initial conditions, which are opposed to semi-global state observers, i.e. the estimation error converges for any state and observer initial conditions in a given bounded set $\mathcal{D} \subset \mathbb{R}^n$. With these premises, the main points in favour of the proposed results are:

i) Novel symmetry-based conditions for global observer design. Under some additional restrictive assumptions on the type of orbital symmetry involved, these conditions boils down to well-known homogeneity conditions ([3], [4], [26]). Indeed, homogeneity is a particular type of orbital symmetry. It follows that our results are applicable to homogeneous systems but most importantly may be used for systems which are not homogenous in the classical sense but still possess an orbital symmetry. Moreover, since symmetries are an intrinsic geometric notion it is possible to consider more general systems on homogeneous spaces: exploiting the symmetry structure has led to high performance observers and filters ([20], [23]).

ii) Unlike previous cited works on homogeneous observers and sliding-modes with homogeneity, the presented results do not require special forms or system's decompositions (integrator chains, relative-degree forms, etc) or restrictive/specific conditions on the observer initialization (semi-global observers) and/or on the state nonlinearities (globally Lipschitz nonlinearities, bounded state trajectories or incrementally dissipative nonlinearities: [1], [2], [13], [21], [31]).

iii) Unlike other existing symmetry-based approaches, we design observers with global convergence properties. Moreover, we consider the most general type of orbital symmetry (i.e. a nonlinear one) in comparison with more restricted classes of orbital symmetries (i.e. linear ones: [25]).

iv) Our results take into account the presence of nonvanishing disturbances and we give bounds for the estimation error, which can be potentially adjusted or optimized by adjusting the observer parameters. In this paper, the only information we use on the disturbance is a known bound d_∞ but our approach is amenable to various generalizations to unbounded disturbances (with known time-varying bound $d_\infty(t)$). To this aim, we introduce novel notions of symmetries for systems with disturbances.

The paper is organized as follows: in Section 3 we present the class of systems and we explain shortly the observer's structure (Figure 1), in Section 4 we discuss the existence of state-norm estimators with some constructive tips for special classes of systems (Examples 4.1 and 4.2), deferring to Section A of the Appendix the discussion of more abstract conditions based on backward/strong observability issues. In Section 5.1 we introduce actions, push-forwards and symmetries, specifically for readers less familiar with these issues, and a new notion of incremental symmetry in the ∞-limit, instrumental for our observer design, by describing in detail how to construct such types of symmetries for lower triangular vector fields (Section 5.3). We list the main assumptions A2-A6 in Section 6.1 and state the main result Theorem 6.1 in Section 6.2, together with some important corollaries which can be compared with existing results using high-gain or sliding-modes observers. A simulation for an academic unstable system is given in Section 6.4, by illustrating how the observer is capable to track diverging states, and the main observer design steps are sketched for the cart pendulum system.

2 Notation

(I)*(vector spaces).* $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times m}$) is the set of n-dimensional real column vectors (resp. $n \times m$ matrices). $\mathbb{R}_\geqslant$ (resp. $\mathbb{R}_>$, resp. $\mathbb{R}^n_>$) denotes the set of non-negative real numbers (resp. positive real numbers, resp. column vectors of n positive real numbers). $\mathbb{GL}(n)$ (general linear group) is the set of nonsingular matrices $A \in \mathbb{R}^{n\times n}$ and I_n is the identity element. For any $A \in \mathbb{R}^{n\times m}$ we denote by $A^\#$ we denote the Penrose pseudoinverse and $A^\# = (A^\top A)^{-1}A^\top$ when the rank of A is m. For any vector $v \in \mathbb{R}^n$ we denote by v_i or $[v]_i$ the i-th element of v and diag $\{v_1, \ldots, v_n\}$ denotes the $n \times n$ diagonal matrix with i-th diagonal element v_i. $\mathbb{C}$ (resp. $\mathbb{C}^-$) is the set of complex numbers (resp. with negative real part), $\mathrm{Re}\{\lambda\}$ denotes the real part of $\lambda \in \mathbb{C}$ and $\mathfrak{S}(S) \subset \mathbb{C}$ denotes the spectrum of $S \in \mathbb{R}^{n\times n}$. $\mathbb{R}^{n\times n}_>$ (resp. $\mathbb{R}^{n\times n}_\geqslant$) is the set of symmetric positive definite (resp. semi-definite) matrices $S \in \mathbb{R}^{n\times n}$ with $\lambda^S_{min} := \min\{\lambda : \lambda \in \mathfrak{S}(S)\}$ and $\lambda^S_{max} := \max\{\lambda : \lambda \in \mathfrak{S}(S)\}$.

(II)*(norms).* $|v|$ denotes the absolute value of $v \in \mathbb{R}$, $\|v\| := \sqrt{v^\top v}$ denotes the euclidean norm of $v \in \mathbb{R}^n$ and the induced norm of $S \in \mathbb{R}^{m\times n}$ is $\|S\| := \sup_{x\in\mathbb{R}^n}(\|Sx\|/\|x\|)$.

(III)*(monotone functions).* Let $\mathcal{K}_>$ (resp. $\mathcal{K}$, resp. $\mathcal{K}_\infty$) be the set of continuous non-decreasing (resp. strictly increasing) functions $f : \mathbb{R}_\geqslant \to \mathbb{R}_\geqslant$ such that $f(0) > 0$ (resp. such that $f(0) = 0$, resp. such that $f(0) = 0$ and $\lim_{s\to+\infty} f(s) = +\infty$). Let $\mathcal{L}$ be the set of continuous strictly decreasing functions $f : \mathbb{R}_\geqslant \to \mathbb{R}_>$ such that $\lim_{s\to+\infty} f(s) = 0$. Finally, let $\mathcal{KL}$ (resp. $\mathcal{KL}_>$) be the set of continuous functions $f : \mathbb{R}_\geqslant \times \mathbb{R}_\geqslant \to \mathbb{R}_\geqslant$ such that $f(\cdot, s) \in \mathcal{K}$ (resp. $f(\cdot, s) \in \mathcal{K}_>$) and $f(r, \cdot) \in \mathcal{L}$ for each $r, s \in \mathbb{R}_\geqslant$ and $\mathcal{KK}$ (resp. $\mathcal{KK}_>$) be the set of continuous functions $f : \mathbb{R}_\geqslant \times \mathbb{R}_\geqslant \to \mathbb{R}_\geqslant$ such that $f(\cdot, s) \in \mathcal{K}$ and $f(r, \cdot) \in \mathcal{K}$ (resp. $f(\cdot, s) \in \mathcal{K}_>$ and $f(r, \cdot) \in \mathcal{K}_>$) for each $r, s \in \mathbb{R}_\geqslant$.

Continuous, continuously differentiable and locally Lipschitz functions on a domain $\mathcal{X}$ are denoted by $C^0(\mathcal{X})$, $C^1(\mathcal{X})$ and $C^{0,1}_{\mathrm{loc}}(\mathcal{X})$, respectively, and we will omit the domain $\mathcal{X}$ when clear from the context. Let $\mathcal{K}^1_\infty$ be the set of functions $f \in \mathcal{K}_\infty \cap C^1(\mathbb{R}_>)$ such that $0 < \inf_{s>0} \dfrac{s}{f(s)}\dfrac{df}{ds}(s) \leqslant \sup\limits_{s>0} \dfrac{s}{f(s)}\dfrac{df}{ds}(s) < +\infty$. The set $\mathcal{K}^1_\infty$ is not standard but it is dense in $\mathcal{K}_\infty$. Monotonically increasing powers, roots, polynomials and rational functions are all in $\mathcal{K}^1_\infty$.

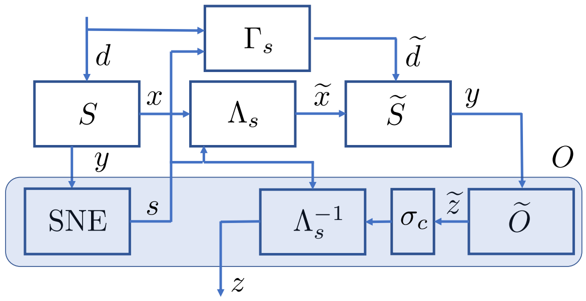


Figure 1. The structure of the observer O (shaded area) and the estimate z of x. S is the system, SNE the state norm estimator, Λ_s the symmetry for S and Γ_s the scaling factor for the disturbance d.

(IV) *(saturation functions)*. A $C^{0,1}_{\rm loc}$-function $\sigma : \mathbb{R}^n \times \mathbb{R}^n_> \to \mathbb{R}^n$, $(c, x) \mapsto \sigma_c(x) := (\sigma_1(x_1, c_1), \cdots, \sigma_n(x_n, c_n))^\top$, is a $C^{0,1}_{\rm loc}$-*saturation function with saturation levels* $c \in \mathbb{R}^n_>$ if $\sigma_i(x_i, c_i) = x_i$ for $x_i \in [-c_i, c_i]$, $|\sigma_i(x_i, c_i)| \leqslant c_i$ and $|\sigma_i(x_i, c_i) - \sigma_i(z_i, c_i)| \leqslant |x_i - z_i|$ for all $x_i, z_i \in \mathbb{R}$.

3 Class of systems

Consider a nonlinear system of the general form

$$\dot{x} = F(x, d),\ \ y = H(x, d), \tag{1}$$

with state $x \in \mathbb{R}^n$, measurements $y \in \mathbb{R}^p$ and disturbance vector d. We assume that the disturbances d are continuous and bounded functions $d : \mathbb{R} \to D \subset \mathbb{R}^m$ of a space denoted by $\mathcal{D}$ and endowed with the sup norm $\|d\|_\infty := \sup_{t\in\mathbb{R}} \|d(t)\|$ uniformly bounded by a known $d_\infty > 0$. $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are $C^{0,1}_{\rm loc}$-mappings with $F(0, 0) = 0$ and $H(0, 0) = 0$.

For stressing dependence on time, we will sometimes denote $x(t), y(t)$ (with initial state $x(0) = x_0$ at $t = 0$) and $d(t)$ by x_t, y_t and, respectively, d_t. In particular, $x_t(x, s; d)$ (resp. $y_t(x, s; d)$) will denote the value at time t of the unique solution (resp. output) of system (1) with input d and initialized at point x at time s, i.e. $x_s(x, s; d) = x$. Throughout the paper, we assume forward completeness of (1).

Assumption 1 *(Forward completeness). The solutions x_t of (1) are defined for all $(x_0, d, t) \in \mathbb{R}^n \times \mathcal{D} \times \mathbb{R}_\geqslant$.*

In order to formulate our assumptions on (1) which make possible our observer design, we discuss two key issues which are fundamental in the observer structure: a) the design of a state-norm estimator for (1) (which is less demanding than a full-state or reduced observer) and b) the construction of symmetries for (1), which basically are one-parameter groups of transformations. The structure of the observer is explained in Figure 1. The system (1), represented as S, is mapped into the system $\tilde{S}$ by a symmetry $\tilde{x} := \Lambda_s(x)$ (parametrized by s) together with a disturbance rescaling $\tilde{d} = \Gamma_s(x)d$. The parameter s of the symmetry is adapted on-line by a state norm estimator, labeled as (SNE), which keeps the norm of $\tilde{x} = \Lambda_s(x)$ (as well as the values of $\tilde{d}$) within prescribed small values c . Taking advantage of the fact that $\|\tilde{x}\|$ and $\|\tilde{d}\|$ are small, an estimate $\tilde{z}$ of $\tilde{x}$ is computed by a local observer $\tilde{O}$ and a global observer O is readily obtained by mapping $\tilde{z}$ back into z through Λ_s^{-1} after a pre-saturation σ_c (shaded area in Figure 1).

4 Design of state norm estimators

In this section we discuss the design of a state norm estimator (*SNE*). To this aim, we propose the following set of conditions:

(SNE). There exist a C^1-function $v : \mathbb{R}^n \times \mathbb{R}_\geqslant \to \mathbb{R}$, $\lambda > 0$, $\alpha, \beta, \zeta \in \mathcal{K}_\infty$, $\delta, \xi \in \mathcal{K}_>$ and $\bar{t} \geqslant 0$ such that:

- *for all $(x, d, t) \in \mathbb{R}^n \times \mathcal{D} \times \mathbb{R}_\geqslant$*

$$\frac{\partial v}{\partial x}(x, t)F(x, d_t) + \frac{\partial v}{\partial t}(x, t) \leqslant -\lambda v(x, t) + \alpha(\|H(x, d_t)\|) + \delta(d_\infty), \tag{PDI}$$

- *for all $(x, d, t) \in \mathbb{R}^n \times \mathcal{D} \times [\bar{t}, +\infty)$*

$$v(x, t) \geqslant \beta(\|x\|), \tag{ULB}$$
$$\alpha(\|H(x, d_t)\|) \leqslant \zeta(v(x, t)) + \xi(d_\infty). \tag{UUB}$$

We also say that the tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$ *satisfies a* (SNE) condition for (1) and we use separate or combined terminologies as well: $(v, \lambda, \alpha, \delta)$ *satisfies a* (PDI) (*partial differential inequality*) condition, $(v, \beta, \bar{t})$ *satisfies a* (ULB) (*uniform lower bound*) condition, $(v, \lambda, \alpha, \delta, \beta, \bar{t})$ *satisfies a* (PDI) +(ULB) condition, etc. etc.

Notice that $v(x, t)$ is not required to be non-negative for all $t \geqslant 0$, still by (ULB) it must be positive only for $t \geqslant \bar{t}$. Our interest in the (PDI) + (ULB) condition is motivated by the following result (a similar result was originally proposed by [19] in slightly different terms and we omit the proof).

Proposition 4.1 *Assume $(v, \lambda, \alpha, \delta, \beta, \bar{t})$ satisfy a* (PDI) + (ULB) *condition for (1). For each $(x_0, d) \in \mathbb{R}^n \times \mathcal{D}$ there exists $T_{x_0,d} \geqslant \bar{t}$ such that for $t \geqslant T_{x_0,d}$:*

$$\|x_t\| \leqslant \beta^{-1}(v(x_t, t)) \leqslant \beta^{-1}(\hat{v}_t + 1), \tag{2}$$

where $\hat{v}_t$ is the output of the filter

$$\dot{\hat{v}}_t = -\lambda\hat{v}_t + \alpha(\|y_t\|) + \delta(d_\infty),\ \ \hat{v}_0 \geqslant 0. \tag{3}$$

Proposition 4.1 proves that a (PDI)+(ULB) condition for (1) is sufficient for the existence of a filter of the form (3) capable of estimating the state norm of (1) and clarifies in which sense, specified by (2), a state norm estimate for (1) has to be meant.

Remark 4.1 *For implementing the state norm estimator (2)-(3) we need the functions α, δ and β. A general approach (inspired by linear systems) is to stabilize (1) by output injection $\phi(y)$ and find a corresponding Lyapunov function $v(x)$. From v and ϕ we get the functions α, δ and β. For instance, for a detectable linear system $\dot{x} = Ax$, $y = Cx$, if K is such that $A - KC$ is Hurwitz and $P \in \mathbb{R}^{n\times n}_{>}$ such that $P(A-KC)+(A-KC)^\top P < 0$, then with $v(x) = x^\top Px$ and $\phi(y) = Ky$ we get $\dot{v} \leqslant -\lambda v + \alpha\|\phi(y)\|^2$ for some $\lambda, \alpha > 0$. Moreover, $v(x) \geqslant \lambda^P_{min}\|x\| := \beta(\|x\|)$. Below (Examples 4.1 and 4.2) we give a sketch of the constructive procedure for some important classes of nonlinear systems.*

Another approach (discussed in section A) is to pick α, define a time-varying v (using α and the backward solutions of (1)) and from α and v find δ and (under certain observability conditions) β.

The functions $\zeta \in \mathcal{K}$ and $\xi \in \mathcal{K}_>$ satisfying the (UUB) *(uniform upper bound) condition can be obtained from the functions β and α as follows: define $\tau(s,r) := \sup_{\|x\|\leqslant\beta^{-1}(s),\|d\|\leqslant r} \alpha(\|H(x,d)\|)$ and pick $\zeta \in \mathcal{K}$ and $\xi \in \mathcal{K}_>$ such that $\zeta(s) \geqslant \tau(s,s)$ and $\xi(s) \geqslant \tau(s,s)$.* $\lhd$

Example 4.1 *(Systems linear with respect to unmeasured state variables). Consider systems (1) of the form*

$$F(x,d) := \begin{pmatrix} A^{(1)}(x_1)x_2 + B^{(1)}(x_1)d_1 \\ A^{(2)}(x_1)x_2 + B^{(2)}(x_1)d_1 \end{pmatrix}, \tag{4}$$

$H(x,d) := x_1 + d_2$, with $x_1, x_2, d_1, d_2 \in \mathbb{R}$ and $A^{(1)}(x_1) > 0$ for all x_1. By direct computations, we can construct a tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, 0)$ satisfying a (SNE) *condition, with $v(x) := c_2(x_2 + \phi(x_1))^2 + c_1(\phi^2(x_1) + x_1^2) + c_0$ where $\phi(x_1) := -\int_0^{x_1}(A^{(2)}(s)+1){A^{(1)}}^{-1}(s)ds$ and for suitable $c_0, c_1, c_2 > 0$. By iterating this constructive paradigm, we come to a tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, 0)$ satisfying a* (SNE) *condition for a n-dimensional lower triangular system*

$$F(x,d) := \begin{pmatrix} A_2^{(1)}(x_1)x_2 + B^{(1)}(x_1)d_1 \\ \sum_{j=2}^{3} A_j^{(2)}(x_1)x_j + B^{(2)}(x_1)d_1 \\ \vdots \\ \sum_{j=2}^{n} A_j^{(n)}(x_1)x_j + B^{(n)}(x_1)d_1 \end{pmatrix}, \tag{5}$$

$H(x,d) := x_1 + d_2$, with $x_1, \cdots, x_n, d_1, d_2 \in \mathbb{R}$ and $A_{i+1}^{(i)}(x_1) > 0$, $i = 1, \ldots, n-1$, for all x_1. $\lhd$

Example 4.2 *(Homogeneous systems). Consider (1) with*

$$F(x,d) := Ax + \Phi(x) + d_1,\ H(x,d) := Cx + d_2 \tag{6}$$

with (C, A) in observer canonical form and $|\Phi_i(x)| \leqslant a\sum_{j=1}^{i} |x_j|^{r_j/r_i}$, $i = 1, \ldots, n$, with $r_1 := 1$, $r_i := \gamma_r + r_{i-1}$, $i = 2, \ldots, n$, $a, \gamma_r \geqslant 0$ and for all $x \in \mathbb{R}^n$. By borrowing some basic results on homogeneity from [4] we can construct a tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, 0)$ satisfying a (SNE) *condition with*

$$v(x) = c_0 + \sum_{j=1}^{n-1} \int_{x_{i+1}^{\frac{r_i}{r_{i+1}}}}^{c_i x_i} (h^{\frac{\gamma_v - r_i}{r_i}} - x_{i+1}^{\frac{\gamma_v - r_i}{r_{i+1}}})dh + c_n|x_n|^{\frac{\gamma_v}{r_n}}$$

for sufficiently large $c_0, \cdots, c_n, \gamma_v > 0$ (by r^s we mean $|r|^s\mathrm{sgn}(r)$). Notice that the case $\gamma_r = 0$ amounts to Φ being a globally Lipschitz lower triangular vector field.

Remark 4.2 *For symplifying the design of the functions v, β and α, it is possible to weaken the* (PDI) *condition as follows:*

$$\begin{aligned} \frac{\partial v}{\partial x}(x,t)F(x,d_t) + \frac{\partial v}{\partial t}(x,t) \\ \leqslant -\lambda(v(x,t)) + \alpha(\|H(x,d_t)\|) + \delta(d_\infty) \end{aligned} \tag{7}$$

where λ is a $\mathcal{K}$-class function for which there exist $c_1, c_2 \geqslant 0$ and $c_3 > 0$ such $\lambda((1+c_1)r + c_2) \geqslant (1+c_1)\lambda(r) + c_3$ for all $r \geqslant 0$. For instance, the $\mathcal{K}$-class function $\lambda(r) = \frac{r}{r+1}$ satisfies this additional condition. It is easy to see that a simpler function $v(x,t) = \ln(1 + x^T Px)$, with $P \in \mathbb{R}^{n\times n}_{>}$ and $\beta(r) = \ln(1+r)$, satisfies (7) *for the system* (5)*. Under condition* (7) *the filter* (3) *is modified as $\dot{\widehat{v}}_t = -\lambda(\widehat{v}_t) + \alpha(\|y_t\|) + \delta(d_\infty)$ and for $t \geqslant T_{x_0,d}$ we have $\|x_t\| \leqslant \beta^{-1}((1+c_1)\widehat{v}_t + c_2)$.* $\lhd$

Examples 4.1 and 4.2 provide functions $v(x,t)$ which are not time-varying. Notice that the function $v(x,t)$ must satisfy (PDI), (ULB) and (UUB) *for all* $d \in \mathcal{D}$ (i.e. $v(x,t)$ is the same whatever $d \in \mathcal{D}$ is). However, $v(x,t)$ may depend on $d \in \mathcal{D}$ as well, in the sense that *for each* $d \in \mathcal{D}$ we find $v(x,t)$ satisfying (PDI), (ULB) and (UUB). Indeed, these functions $v(x,t)$ are obtained along the backward solutions of (1). This different definition of $v(x,t)$ will not affect the result of Proposition 4.1. For not distracting the reader from the main flow of the presentation, we discuss in section A of the Appendix, supported by simple examples, how and under which conditions (related to backward/strong observability issues) it is possible in principle to construct time-varying $v(x,t)$ and satisfying (PDI)+(ULB). We only list below some useful and important properties, which we use for our main result:

(P1). *Let $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$ satisfy a* (SNE) *condition for (1). The tuples $(cv, \lambda, c\alpha, c\beta, c\zeta(\frac{s}{c}), c\xi,), \bar{t})$ or $(v +$*

$b, \lambda, \alpha, \delta + b\lambda, \beta, \zeta, \bar{t})$, *with any* $b, c > 0$, *still satisfy a* (SNE) *condition for (1)* (this trivially follows from the definition). In other words, we can change v by adding or multiplying by positive numbers while preserving the (SNE) condition.

(P2). *Let* $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$ *satisfy a* (PDI) + (UUB) *condition and* $v(x, t) \geqslant \beta(\|x\|) - b$ *for some* $b > 0$ *and for all* $(x, t) \in \mathbb{R}^n \times [\bar{t}, +\infty)$. *The tuple* $(v + b, \lambda, \alpha, \delta + \lambda b, \beta, \zeta, \bar{t})$ *satisfy a* (SNE) *condition for (1)* (this follows from (P1)).

(P3). *Given* $\widetilde{a}, \widetilde{b} > 0$, *it is possible to re-design a tuple* $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$, *with* $\beta, \zeta \in \mathcal{K}^1_\infty$ *and satisfying a* (SNE) *condition for (1), into a new tuple* $(\widetilde{v}, \widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\beta}, \widetilde{\zeta}, \widetilde{\xi}, \bar{t})$, *with* $\widetilde{\beta}(s) := \widetilde{a}s^{\widetilde{b}} \in \mathcal{K}^1_\infty$ *and* $\widetilde{\zeta}(s) = \widetilde{\ell}\zeta(s)$ *for all* $s \geqslant 1$ *and for some* $\widetilde{\ell} > 0$ *(depending on* λ, β, ζ, $\widetilde{a}$ *and* $\widetilde{b}$*), still satisfying a* (SNE) *condition for (1)* (the proof is available in Section B.1 of the Appendix). In other words, we can change $\beta \in \mathcal{K}^1_\infty$ into any other $\widetilde{\beta} \in \mathcal{K}^1_\infty$ while preserving the (SNE) condition.

5 Incremental orbital symmetries

A second key ingredient in our setup is to construct some types of symmetries for (1). Symmetries have a long history: see the recent book of Olver [27] for an introduction to symmetries in a differential geometric framework and with regard to their potential applications to nonlinear control systems. A breakthrough under this regard dates back to [18], where symmetries are linked to homogeneity, and more recently some developments for linear symmetries are contained in [25]. For defining a symmetry or any of its variant, we need some basic notions which we recall here for readers less familiar with these issues.

5.1 An excursus on actions and relevant properties

Let $\Lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(s, x) \mapsto \Lambda(s, x) := \Lambda_s(x)$, be a one-parameter group of C^1-transformations (i.e. a C^1 action $\Lambda : \mathcal{G} \times \mathbb{R}^n \to \mathbb{R}^n$ with $\mathcal{G} = (\mathbb{R}, +)$, the additive group of real numbers) with complete $C^{0,1}_{\text{loc}}$ infinitesimal generator W (i.e. the action is transitive). In equivalent terms, Λ_s has the group property $\Lambda_{s_1}(\Lambda_{s_2}(x)) = \Lambda_{s_1+s_2}(x)$ for all $(s_1, s_2, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ and satisfies the differential equation

$$\frac{\partial}{\partial s}\Lambda_s(x) = W(\Lambda_s(x)),\ \Lambda_0(x) = x \tag{8}$$

for all $(s, x) \in \mathbb{R} \times \mathbb{R}^n$. As a consequence of the group property, $\Lambda_s : \mathbb{R}^n \to \mathbb{R}^n$ is for each $s \in \mathbb{R}$ a C^1- transformation with inverse C^1-transformation $\Lambda_s^{-1} = \Lambda_{-s}$. In what follows, we use the more generic term "action" in place of "one-parameter group of transformations".

If $W(x) = Wx$, with $W \in \mathbb{R}^{n\times n}$, then $\Lambda_s(x) = e^{sW}x$ and we say that Λ_s *is a linear action*. If, in addition, $W \in \mathbb{R}^{n\times n}$ is diagonal we say that Λ_s *is a decoupled linear action*. For our purposes we consider actions Λ_s with the following specific properties:

(Stability margin (SM)). There exists $\lambda \in \mathcal{KL}$ *such that for all* $(s, x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$

$$\|\Lambda_s(x)\| \leqslant \lambda(\|x\|, s). \tag{9}$$

The (SM) property asks for the origin of (8) being *globally asymptotically stable*: we say that *the action* Λ_s *is GAS*. If (9) holds with $\lambda(r, s) = ke^{-hs}r$ for some $k, h > 0$, we say that *the action* Λ_s *is GES* (i.e. the origin of (8) is *globally exponentially stable*). A GAS linear action $\Lambda_s(x) = e^{sW}x$ is such that W is Hurwitz, hence it is also GES.

(Contraction (C)). For each $\tau \in \mathcal{L}$ *there exist* $\widehat{s} \in \mathcal{K}_>$ *such that for all* $r \in \mathbb{R}_\geqslant$

$$\lambda(r + 1, \widehat{s}(r)) \leqslant \tau(r) \tag{10}$$

For each $\tau \in \mathcal{L}$ the function $\widehat{s} \in \mathcal{K}_>$ can be always obtained from the stability margin $\lambda \in \mathcal{KL}$: indeed, there always exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $a > 0$ such that $\lambda(r, s) \leqslant \alpha_1(\alpha_2(r)e^{-as})$ for all $r, s \in \mathbb{R}_\geqslant$ (see for instance [16], Lemma 7) and $\widehat{s}$ can be readily obtained as $\widehat{s}(r) = (1/a)\ln(\alpha_2(r + 1)/\alpha_1^{-1}(\tau(r)))$. For example, for a GES action we get $\widehat{s}(r) = \frac{1}{h}\ln\left(\frac{k(r+1)}{\tau(r)}\right)$ for all $r \geqslant 0$.

(Incremental Rate (IR)). There exists $\rho \in \mathcal{KK}_>$ *such that for all* $(s, x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$

$$\left\|\frac{\partial\Lambda_s^{-1}}{\partial x}(x)\right\| \leqslant \rho(\|x\|, s). \tag{11}$$

The incremental rate property is a direct consequence of Λ_s^{-1} being C^1. For a linear action we get for instance $\rho(r, s) = e^{\|W\|s}$.

(Normalized upper bounds (NUB)). There exist a C^0 *function* $\psi : \mathbb{R}_\geqslant \to \mathbb{R}^n_>$, $\psi(s) = (\psi_1(s), \cdots, \psi_n(s))^\top$, *such that for all* $(s, x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$

$$\frac{|[\Lambda_s(x)]_i|}{\lambda(\|x\|, s)} \leqslant \psi_i(s) \leqslant 1,\ i = 1, \ldots, n. \tag{12}$$

The (NUB) property is a direct consequence of the (SM) property (since $\frac{\|\Lambda_s(x)\|}{\lambda(\|x\|,s)} \leqslant 1$ for all $(s, x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$). Each ψ_i can be either a $\mathcal{L}$- or a $\mathcal{K}_>$-class function (bounded by 1) and represents an upper bound, normalized by the stability margin, for each component of the vector $\Lambda_s(x)$. A trivial choice of ψ_i is clearly 1.

Throughout this paper, for a GAS action with generator W as in (8) and (λ, ρ, ψ) satisfying (9), (11) and (12) we say that *the action Λ_s is GAS with associated quadruple* (W, λ, ρ, ψ).

We mention an important generalization of an action Λ_s, suitable for allowing non-uniform generators $W(s,x)$ which we may come across in general nonlinear contexts. More in detail, $\Lambda : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(s,x) \mapsto \Lambda(s,t,x) := \Lambda_{s,t}(x)$, be a C^1-map such that $\Lambda_{s_1,t}(\Lambda_{t,s_2}(x)) = \Lambda_{s_1,s_2}(x)$ for all $(s_1,t,s_2,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ and satisfying the differential $\frac{\partial}{\partial s}\Lambda_{s,t}(x) = W(s, \Lambda_{s,t}(x))$, $\Lambda_{t,t}(x) = x$ for all $(s,t,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. In this case, we lose the one-parameter structure and group property, still $\Lambda_{s,t} : \mathbb{R}^n \to \mathbb{R}^n$ is for each pair $(s,t) \in \mathbb{R} \times \mathbb{R}$ a C^1-transformation with C^1-inverse $\Lambda_{s,t}^{-1} = \Lambda_{t,s}$. The (non-uniform) generator of $\Lambda_{s,t}$ is $W(s,x)$ and we still say (with some abuse of terminology) that $\Lambda_{s,t}$ is a C^1-action (with associated quadruple (W, λ, ρ, ψ)). For our purposes, the second parameter t is not important and we may set $t = 0$ and consider $\overline{\Lambda}_s(x) = \Lambda(s, 0, x)$.

5.2 Incremental orbital symmetries and some examples

An action Λ_s transforms vector fields and maps as follows. For a given $C^{0,1}_{\mathrm{loc}}$ vector field $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the *push-forward of F by Λ_s* is defined as $\Lambda_{s*}F(x,d) := \frac{\partial \Lambda_s}{\partial z}(z)F(z,d)|_{z=\Lambda_s^{-1}(x)}$, while, for a given $C^{0,1}_{\mathrm{loc}}$ mapping $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, the *push-forward of H by Λ_s* is $\Lambda_{s*}H(x,d) := H(z,d)|_{z=\Lambda_s^{-1}(x)}$.

According to previous notions of symmetry (see for instance [18]), an orbital C^1-symmetry for a $C^{0,1}_{\mathrm{loc}}$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is a GAS C^1-action $\Lambda_s(x)$ such that $\Lambda_{s*}F(x) = e^{\overline{\gamma}_F s}F(x)$ for all $(s,x) \in \mathbb{R}_{\geqslant} \times \mathbb{R}^n$ and for some $\overline{\gamma}_F \in \mathbb{R}$. The above definition has the following drawbacks: a) it requires the exact equality $\Lambda_{s*}F(x) = e^{\overline{\gamma}_F s}F(x)$ which may be somewhat restrictive for our purposes, b) it is not suitable for vector fields with exogenous inputs and c) no insight is given on the incremental behaviour of $\Lambda_{s*}F(x)$ at different points, which is a key information for observer and contraction-based design. Under this regard, we propose the following definition of *orbital incremental symmetry*.

Definition 5.1 *A GAS C^1-action $\Lambda_s(x)$ with associated quadruple (W, λ, ρ, ψ) is an incremental orbital symmetry in the ∞-limit of a $C^{0,1}_{\mathrm{loc}}$ vector field $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with scaling factors (γ_F, Γ_d) and C^1 limit vector field $F_\infty : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F_\infty(0,0) = 0$, if there exist C^0 functions $\gamma_F : \mathbb{R} \to \mathbb{R}_>$, $\Gamma_d : \mathbb{R} \times \mathbb{R}^n \to \mathbb{GL}(m)$ and $\lambda_F \in \mathcal{KL}_>$ such that for all $s \in \mathbb{R}_{\geqslant}$, $x, z \in \mathbb{R}^n : |x_i|, |z_i| \leqslant \psi_i(s)$, $i = 1, \ldots, n$, and $d \in \mathbb{R}^m$*

$$\|\Phi(s,x,d) - \Phi(s,z,0)\| \leqslant \lambda_F(\|d\|, s)(\|x - z\| + \|d\|) \quad (13)$$

where

$$\Phi(s,x,d) := \frac{1}{\gamma_F(s)}\Lambda_{s*}F\left(x, \Gamma_d^{-1}(s,x)d\right) - F_\infty(x,d).$$

When $\Phi(s,x,d) \equiv 0$ for all $(s,x,d) \in \mathbb{R}_{\geqslant} \times \mathbb{R}^n \times \mathbb{R}^m$ we say that $\Lambda_s(x)$ *is an orbital symmetry of F with scaling factors* (γ_F, Γ_d). In this case, (13) is trivially satisfied and, since $\Lambda_0(x) \equiv x$ and $\frac{\partial \Lambda_s}{\partial x}|_{s=0} = \frac{\partial \Lambda_0}{\partial x}$, it follows that

$$F_\infty(x,d) \equiv \frac{1}{\gamma_F(0)}F(x, \Gamma_d^{-1}(0,x)d).$$

Hence, F_∞ is a rescaled version of F. In view of this particular case, it is clear from (13) that an incremental orbital symmetry in the ∞-limit can be seen as an orbital symmetry both in an “approximating” sense (quantified by $\lambda_F \in \mathcal{KL}_>$) and “incremental” sense (quantified by the increments of Φ).

A necessary and sufficient condition for a C^1-action $\Lambda_s(x)$ with C^1-generator $W(x)$ to be an orbital symmetry of a disturbance-free vector field $F(x)$ is given by the following result which we state without proof ($[F,W](x)$ denotes the Lie bracket between the vector fields $F(x)$ and $W(x)$).

Proposition 5.1 *A C^1-action $\Lambda_s(x)$ with generator $W(x)$ is an orbital symmetry of a C^1-vector field $F(x)$ with exponential scaling factor $\gamma_F(s) \equiv e^{\overline{\gamma}_F s}$ if and only if $[F,W](x) = \overline{\gamma}_F F(x)$ for all $x \in \mathbb{R}^n$.*

Proposition 5.1 provides the geometric definition of homogeneity with degree $\overline{\gamma}_F$ given in [18] for a disturbance-free vector field $F(x)$. Hence, definition 5.1 extends in many directions the geometric definition of homogeneity of a vector field. In classical homogeneity frameworks ([3], [4], [26], [28]) only decoupled linear actions are considered with exponential scaling factors while in more general contexts ([25]) also non-decoupled linear actions are taken into account.

Below we give directly some examples of incremental symmetries, which can be also computed with the more general methodology given later in Section 5.3.

Example 5.1 *(Decoupled linear actions: homogeneity revisited). Consider the vector field $F(x,d) = (x_2, -x_2x_1^2 + d)^\top$. The decoupled linear action $\Lambda_s(x) = e^{Ws}x$ with $W = \mathrm{diag}(-1,-3)$ is an orbital symmetry of F with scaling factors $(\gamma_F, \Gamma_d) = (e^{2s}, e^{-5s})$. As associated quadruple of Λ_s we have $(W, \lambda, \rho, \psi) = (Wx, e^{-s}r, e^{3s}, (1, e^{-s})^\top)$. The vector field $F(x,0)$ is homogeneous with weigths $(1,3)$ and degree 2.* ◁

Example 5.2 *(Nonlinear actions). Consider the vector field* $F(x,d) = \left(x_2\sqrt{1+x_1^2}, \frac{x_1x_2^2+x_1^2+d}{\sqrt{1+x_1^2}}\right)^\top$.

There is no linear action Λ_s which is a symmetry of F: in particular, $F(x,0)$ is not even homogeneous. On the other hand, the GES nonlinear action $\Lambda_s(x) = (e^{-s}x_1, \sqrt{\frac{1+x_1^2}{1+e^{-2s}x_1^2}}e^{-2s}x_2)^\top$ is an incremental orbital symmetry in the ∞-limit of F with scaling factors $(\gamma_F, \Gamma_d) = (e^s, e^{-3s})$ and limit $F_\infty(x,d) = \left(x_2\sqrt{1+x_1^2}, \frac{x_1x_2^2+d}{\sqrt{1+x_1^2}}\right)^\top$. The generator of $\Lambda_s(x)$ is $W(x) = \left(-x_1, -\frac{2+x_1^2}{1+x_1^2}x_2\right)^\top$. As associated quadruple of Λ_s we have $(W, \lambda, \rho, \psi) = (W(x), e^{-s}r, e^{3s}, (1, e^{-s})^\top)$.

For a $C^{0,1}_{\rm loc}$ mapping $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ a C^1-action Λ_s defines a symmetry of H exactly as in Definition 5.1 on account of the different meaning of the push-forward of H.

5.3 Design of incremental symmetries for lower triangular vector fields

It turns out that wide classes of nonlinear vector fields and maps admit an incremental symmetry. In this section we will construct an incremental symmetry in the ∞-limit for lower triangular vector fields

$$F(x,d) = \begin{pmatrix} F_1(x_1) + G_1(x_1)x_2 \\ \vdots \\ F_{n-1}(x^{(n-1)}) + G_{n-1}(x_1)x_n \\ F_n(x^{(n)}) + G_n(x^{(n)})d \end{pmatrix}, \tag{14}$$

with $x \in \mathbb{R}^n$, $d \in \mathbb{R}$, $x^{(i)} := (x_1, \cdots, x_i)^\top$, $i = 1, \ldots, n$,

(I) *$C^{0,1}_{\rm loc}$ functions F_i, $i = 1, \ldots, n$, and G_n such that $F_i(0) = 0$ and $0 < G_n(x)$ for all $x \in \mathbb{R}^n$ and C^1-functions G_i, $i = 1, \ldots, n-1$, such that:*

$$\frac{1}{\theta^{(i)}(\|x_1\|)} \leqslant G_i(x_1) \leqslant \nu^{(i)}, \ \left\|\frac{\partial G_i}{\partial x_1}(x_1)\right\| \leqslant \overline{\mu}^{(i)}(\|x_1\|),$$

for all $x_1 \in \mathbb{R}$ and for some $\theta^{(i)}, \overline{\mu}^{(i)} \in \mathcal{K}_>$ and $\nu^{(i)} > 0$.

Before stating the constructive result, for a given $\sigma \in \mathcal{L}$ find $\overline{\gamma} \in \mathcal{K}_>$ such that for all $s \geqslant 0$ and $i = 1, \ldots, n$

$$\|\Psi^{(i)}(s, x^{(i)}) - \Psi^{(i)}(s, z^{(i)})\| \leqslant \overline{\gamma}(s)\sigma(s)\|x^{(i)} - z^{(i)}\| \tag{15}$$

for all $x^{(i)}, z^{(i)} \in \mathbb{R}^i : \|x_j^{(i)}\|, \|z_j^{(i)}\| \leqslant 1$, $j = 1, \ldots, i$, with

$$\Psi^{(i)}(s, x^{(i)}) := e^{-s}\overline{\Gamma}_i(s, x_1)F_i\left(\frac{e^s x_1}{\overline{\Gamma}_1(s, x_1)}, \ldots, \frac{e^s x_i}{\overline{\Gamma}_i(s, x_1)}\right)$$
$$+\frac{x_i}{\overline{\Gamma}_i(s, x_1)}\frac{\partial \overline{\Gamma}_i}{\partial x_1}(s, x_1)\left(e^{-s}F_1(e^s x_1) + G_1(x_1)x_2\right) \tag{16}$$

and

$$\overline{\Gamma}_1(s, x_1) := 1, \overline{\Gamma}_i(s, x_1) := \prod_{j=1}^{i-1}\frac{G_j(e^s x_1)}{G_j(x_1)}, i = 2, \ldots, n.$$

For instance, we can pick any $\overline{\gamma}$ such that for all $s \geqslant 0$ and $i = 1, \ldots, n$:

$$\overline{\gamma}(s) \geqslant \frac{1}{\sigma(s)} \sup_{\substack{\|x_j^{(i)}\|, \|z_j^{(i)}\| \leqslant 1 \\ j=1,\ldots,i}} \frac{\|\Psi^{(i)}(s, x^{(i)}) - \Psi^{(i)}(s, z^{(i)})\|}{\|x^{(i)} - z^{(i)}\|}.$$

Also, if $\sigma(0)$ is taken properly, we can assume $\overline{\gamma}(0) = 1$.

Also, from (I) we obtain the following upper bound: for all $(s, x_1) \in \mathbb{R}_\geqslant \times \mathbb{R}$ and $i = 1, \ldots, n-1$

$$\left\|\frac{\partial}{\partial x_1}\left(\frac{G_i(e^s x_1)}{G_i(x_1)}\right)\right\| \leqslant \theta^{(i)}(\|x_1\|)(e^s\overline{\mu}^{(i)}(e^s\|x_1\|)$$
$$+\nu^{(i)}\theta^{(i)}(\|x_1\|)\overline{\mu}^{(i)}(\|x_1\|)) := \mu^{(i)}(\|x_1\|, s) \in \mathcal{KK}_>. \tag{17}$$

Proposition 5.2 *Under assumption (I), any $\gamma_F \in \mathcal{K}_> \cap C^1$, with $\gamma_F(0) = 1$ and $\gamma_F(s) \geqslant \overline{\gamma}(s)$ for all $s \geqslant 0$, is such that*

$$\Lambda_s(x) := (\Gamma_{d_0}(s, e^{-s}x_1)x_1, \cdots, \Gamma_{d_{n-1}}(s, e^{-s}x_1)x_n)^\top,$$
$$\Gamma_{d_j}(s, x_1) := e^{-s}\gamma_F^{-j}(s)\overline{\Gamma}_{j+1}(s, x_1), \tag{18}$$

is a GAS C^1-action with non-uniform generator

$$W(s, x) := (\overline{W}^{(1)}(s, x_1)x_1, \cdots, \overline{W}^{(n)}(s, x_1)x_n)^\top,$$

where

$$\overline{W}^{(i)}(s, x_1) = \frac{\partial \ln \Gamma_{d_{i-1}}}{\partial s}(s, x_1) - x_1\frac{\partial \ln \Gamma_{d_{i-1}}}{\partial x_1}(s, x_1) \tag{19}$$

and an incremental orbital symmetry in the ∞-limit of (14) with limit

$$F_\infty(x, d) = (G_1(x_1)x_2, \cdots, G_{n-1}(x_1)x_n, G_n(x_1)d)^\top \tag{20}$$

and scaling factors (γ_F, Γ_d), where

$$\Gamma_d(s, x) := e^{-s}\gamma_F^{-n}(s)\overline{\Gamma}_n(s, x_1)\frac{G_n(\Lambda_s^{-1}(x))}{G_n(x)}. \tag{21}$$

Proof. The constructive proof is by steps.

Step 1. First, we construct an incremental symmetry $\Lambda_s^{(1)}(x_1)$ for $F^{(1)}(x_1, d_1) = F_1(x_1) + G_1(x_1)d_1$ with disturbance $d_1 := x_2$. To this aim, consider the GAS action $\Lambda_s^{(1)}(x_1) = e^{-s}x_1$ with associated quadruple $(W^{(1)}, \lambda^{(1)}, \rho^{(1)}, \psi^{(1)})$, where $W^{(1)}(x_1) := -x_1, \lambda^{(1)}(r,s) := e^{-s}r, \rho^{(1)}(r,s) := e^s$ and $\psi^{(1)}(s) := 1$. Let Γ_{d_1} be defined as in (18), $\Psi^{(1)}$ as in (16) and $(\overline{\gamma}, \sigma)$ as in (15). With $\gamma_F \in \mathcal{K}_> \cap C^1$ such that $\gamma_F(0) = 1$ and $\gamma_F(s) \geqslant \overline{\gamma}(s)$ for all $s \geqslant 0$, notice that

$$\frac{1}{\gamma_F(s)} \Lambda^{(1)}_{s*} F^{(1)} \left(x_1, \Gamma^{-1}_{d_1}(s, x_1) d_1\right) = \frac{\Psi^{(1)}(s, x_1)}{\gamma_F(s)} + G_1(x_1) d_1. \tag{22}$$

But $\gamma_F(s) \geqslant \overline{\gamma}(s)$, hence by (15)

$$\frac{1}{\gamma_F(s)} \|\Psi^{(1)}(s, x_1) - \Psi^{(1)}(s, z_1)\| \leqslant \sigma(s) \|x_1 - z_1\| \tag{23}$$

for all $(s, x_1, z_1) \in \mathbb{R}_\geqslant \times \mathbb{R} \times \mathbb{R}$ such that $\|x_1\|, \|z_1\| \leqslant \psi^{(1)}(s) = 1$. It follows from (22) and (23) that $\Lambda^{(1)}_s$ is an incremental symmetry in the ∞-limit of $F^{(1)}(x_1, d_1)$ with limit $F^{(1)}_\infty(x_1, d_1) = G_1(x_1) d_1$ and scaling factors (γ_F, Γ_{d_1}). Moreover, using assumption (I) we have the following inequalities (to be used in the subsequent steps) for all $(s, x_1) \in \mathbb{R}_\geqslant \times \mathbb{R}$

$$\begin{aligned}
&\frac{\|\Gamma^{-1}_{d_1}(s, x_1)\|}{\gamma_F(s)} \leqslant e^s \nu^{(1)} \theta^{(1)}(e^s \|x_1\|) := \overline{\delta}^{(1,1)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>, \\
&\|\Gamma^{-1}_{d_1}(s, x_1) \frac{\partial \Gamma_{d_1}}{\partial x_1}(s, x_1)\| \leqslant \nu^{(1)} \mu^{(1)}(\|x_1\|, s) \theta^{(1)}(e^s \|x_1\|) \\
&\qquad := \overline{\lambda}^{(1)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>, \\
&\frac{1}{\gamma_F(s)} \|\frac{\partial \Gamma^{-1}_{d_1}}{\partial x_1}(s, x_1)\| \leqslant \overline{\lambda}^{(1)}(\|x_1\|, s) \overline{\delta}^{(1,1)}(\|x_1\|, s) \\
&\qquad := \overline{\delta}^{(1,2)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>.
\end{aligned}$$

Step $i \geqslant 2$. We construct an incremental symmetry $\Lambda^{(i)}_s(x^{(i)})$ for the vector field

$$F^{(i)}(x^{(i)}, d_i) := (F^{(i-1)^\top}(x^{(i-1)}, x_i), F_i(x^{(i)}) + G_i(x_1) d_i)^\top$$

with disturbance $d_i := x_{i+1}$. To this aim, consider the GAS C^1-action

$$\Lambda^{(i)}_s(x^{(i)}) := (\Lambda^{(i-1)}_s(x^{(i-1)}), \Gamma_{d_{i-1}}(s, e^{-s} x_1) x_i)^\top.$$

with associated quadruple $(W^{(i)}, \lambda^{(i)}, \rho^{(i)}, \psi^{(i)})$, where

$$\begin{aligned}
&W^{(i)}(s, x^{(i)}) = (W^{(i-1)}(s, x^{(i-1)}), \overline{W}^{(i)}(s, x_1) x_{i+1})^\top, \\
&\lambda^{(i)}(r, s) := \lambda^{(i-1)}(r, s) + \left(\prod_{j=1}^{i-1} \nu^{(j)} \theta^{(j)}(r)\right) e^{-s} r, \\
&\rho^{(i)}(r, s) := \rho^{(i-1)}(r, s) \\
&\qquad + \gamma^{i-1}_F(s) \overline{\delta}^{(i-1,1)}(r, s)(1 + \overline{\lambda}^{(i-1)}(r, s) r), \\
&\psi^{(i)}(s) := (\psi^{(i-1)^\top}(s), \gamma^{i-1}_F(s))^\top.
\end{aligned} \tag{24}$$

Let Γ_{d_i} be as in (18), $\Psi^{(i)}$ as in (16) and $(\overline{\gamma}, \sigma)$ as in (15). Notice that

$$\frac{1}{\gamma_F(s)} \Lambda^{(i)}_{s*} F^{(i)} \left(x^{(i)}, \Gamma^{-1}_{d_i}(s, x_1) d_i\right) = \begin{pmatrix} \frac{1}{\gamma_F(s)} \Lambda^{(i-1)}_{s*} F^{(i-1)} \left(x^{(i-1)}, \Gamma^{-1}_{d_{i-1}}(s, x_1) x_i\right) \\ \frac{1}{\gamma^i_F(s)} \Psi^{(i)}(s, \Gamma^{(i)}_F(s) x^{(i)}) + G_i(x_1) d_i \end{pmatrix} \tag{25}$$

where $\Gamma^{(i)}_F(s) := \operatorname{diag}\{1, \gamma_F(s), \cdots, \gamma^{i-1}_F(s)\}$. Since $\gamma_F(s) \geqslant \overline{\gamma}(s)$ by (15) we have

$$\begin{aligned}
&\frac{1}{\gamma^i_F(s)} \|\Psi^{(i)}(s, \Gamma^{(i)}_F(s) x^{(i)}) - \Psi^{(i)}(s, \Gamma^{(i)}_F(s) z^{(i)})\| \\
&\leqslant \sigma(s) \|x^{(i)} - z^{(i)}\|
\end{aligned} \tag{26}$$

for all $(s, x^{(i)}, z^{(i)}) \in [0, +\infty) \times \mathbb{R}^i \times \mathbb{R}^i$ such that $\|x^{(i)}_j\|, \|z^{(i)}_j\| \leqslant \psi^{(i)}_j(s) := \gamma^{-j+1}_F(s)$, $j = 1, \ldots, i$. By (25) and (26) and since $\Lambda^{(i-1)}_s$ is an incremental symmetry in the ∞-limit of $F^{(i-1)}$ with limit $F^{(i-1)}_\infty$, $\Lambda^{(i)}_s$ is an incremental symmetry in the ∞-limit of the vector field $F^{(i)}(x^{(i)}, d_i)$ with limit

$$F^{(i)}_\infty(x^{(i)}, d_i) = (F^{(i-1)^\top}_\infty(x^{(i-1)}, x_i), G_i(x_1) d_i)^\top$$

and scaling factors (γ_F, Γ_{d_i}). Moreover, using assumption (I) we have the following inequalities (to be used in the subsequent steps) for all $(s, x_1) \in \mathbb{R}_\geqslant \times \mathbb{R}$

$$\begin{aligned}
&\frac{\|\Gamma^{-1}_{d_i}(s, x_1)\|}{\gamma^i_F(s)} \leqslant \nu^{(i)} \theta^{(i)}(e^s \|x_1\|) \overline{\delta}^{(i-1,1)}(\|x_1\|, s) \\
&\qquad := \overline{\delta}^{(i,1)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>, \\
&\|\Gamma^{-1}_{d_i}(s, x_1) \frac{\partial \Gamma_{d_i}}{\partial x_1}(s, x_1)\| \leqslant \nu^{(i)} \mu^{(i)}(\|x_1\|, s) \theta^{(i)}(e^s \|x_1\|) \\
&\qquad + \overline{\lambda}^{(i-1)}(\|x_1\|, s) := \overline{\lambda}^{(i)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>, \\
&\frac{1}{\gamma^i_F(s)} \|\frac{\partial \Gamma^{-1}_{d_i}}{\partial x_1}(s, x_1)\| \leqslant \overline{\lambda}^{(i)}(\|x_1\|, s) \overline{\delta}^{(i,1)}(\|x_1\|, s) \\
&\qquad := \overline{\delta}^{(i,2)}(\|x_1\|, s) \in \mathcal{K}\mathcal{K}_>.
\end{aligned}$$

Finally, $i \to i + 1$ and jump to step i. The claim of the Proposition is proved when $i > n$. □

Remark 5.1 *If the functions $\theta^{(i)}$, $i = 1, \ldots, n-1$, of Assumption (I) are all bounded then $\lambda^{(i)}(r, s) = k^{(i)} e^{-s} r$ for some $k^{(i)} > 0$ and for all $i = 1, \ldots, n$, hence Λ_s is a GES action.*

If, in addition to Assumption (I), G_n satisfies $\frac{1}{\theta^{(n)}(\|x_1\|)} \leqslant G_n(x) \leqslant \nu^{(n)}(\|x_1\|)$ for all $x \in \mathbb{R}^n$ and for some $\nu^{(n)}, \theta^{(n)} \in \mathcal{K}_>$ and we define Γ_d slightly differently from

(21) *as follows:*

$$\Gamma_d(s,x_1) := e^{-s}\gamma_F^{-n+1-k}(s)\overline{\Gamma}_n(s,x_1)\frac{G_n(\Lambda_s^{-1}(x))}{G_n(x)} \quad (27)$$

for any $k \in (0,1)$, the scaling factor $\gamma_F(s)(\geqslant \overline{\gamma}(s))$ can be additionally selected so that the limit vector field is

$$F_\infty(x,d) = (G_1(x_1)x_2, \cdots, G_{n-1}(x_1)x_n, 0)^\top.$$

This means that $\frac{\partial F_\infty}{\partial d}|_{(0,0)} = 0$, which is one of the conditions ensuring practical error convergence of the observer in the main result of Section 6.2. If $G_n(0) = 0$ it would be still possible to have $\frac{\partial F_\infty}{\partial d}|_{(0,0)} = 0$ with Γ_d as in (21) *and F_∞ in* (20)*. Also, notice that $\frac{\partial F_\infty}{\partial x}|_{(0,0)} = \mathrm{diag}\{G_1(0), \ldots, G_n(0)\}J$, where J is a Jordan matrix and $G_i(0) > 0$ for all $i = 1, \ldots, n$.*

Moreover, with Γ_d as in (27) *and $\rho \in \mathcal{KK}_>$ being the incremental rate of $\Lambda_s^{-1}(x)$, for all $(s,x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$ we have*

$$\|\Gamma_d(s,x_1)\| \leqslant \frac{e^{-s}\nu^{(n)}(e^s\|x_1\|)}{\gamma_F^{-n+1-k}(s)}\prod_{j=1}^{n-1}\nu^{(j)}\prod_{j=1}^{n}\theta^{(j)}(\|x_1\|),$$

$$\|\Gamma_d(s,x_1)\|\rho(\|x\|,s) \leqslant \frac{\nu^{(n)}(e^s\|x_1\|)}{\gamma_F^k(s)}\prod_{j=1}^{n-1}\nu^{(j)}\prod_{j=1}^{n}\theta^{(j)}(\|x_1\|)$$

$$\times\Big(1 + e^{-s}\sum_{j=2}^{n}\overline{\delta}^{(j-1,1)}(\|x\|,s)\Big(1+\overline{\lambda}^{(j-1)}(\|x\|,s)\|x\|\Big)\Big)$$

and the scaling factor $\gamma_F(s)(\geqslant \overline{\gamma}(s))$ can be further selected so that for all $(s,x_1) \in \mathbb{R}_\geqslant \times \mathbb{R}$ and for some $\alpha_d \in \mathcal{K}_>$ and $\gamma_d < 0$:

$$\|\Gamma_d(s,x_1)\| \leqslant \alpha_d(\|x_1\|)e^{\gamma_d s} \quad (28)$$

and the following "small gain" condition is satisfied for all $(s,x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n$ and for some $\omega \in \mathcal{K}_>$:

$$\|\Gamma_d(s,x_1)\|\rho(\|x\|,s) \leqslant \omega(\|x\|). \quad (29)$$

Conditions like (28) *and* (29) *will be required in the main result of Section 6.2.*

Finally, we remark that if F_i, $i = 1, \ldots, n$, are polynomial and all the functions $\theta^{(i)}$, $\nu^{(i)}$ and $\overline{\mu}^{(i)}$, $i = 1, \ldots, n$, of Assumption (I) are bounded, γ_F is exponential. □

6 Main assumptions and results

In this section we list the main assumptions on (1) and state the main result together with the observer equations and an upper bound for the asymptotic estimation error.

6.1 Main assumptions

The first assumption is concerned with the existence of orbital incremental symmetries Λ_s in the ∞-limit for F and H: this assumption takes care of the global behaviour of the nonlinear observer and prevents large or even unbounded errors.

Assumption 2 *The GES action Λ_s, with associated quadruple (W, λ, ρ, ψ), is an orbital incremental symmetry for F (resp. for H) in the ∞-limit with scaling factors (γ_F, Γ_d) (resp. (γ_H, Γ_d)) and limit F_∞ (resp. H_∞). Moreover, $\gamma_F(s) = e^{\overline{\gamma}_F s}$ (resp. $\gamma_H(s) = e^{\overline{\gamma}_H s}$) for some $\overline{\gamma}_F > 0$ (resp. $\overline{\gamma}_H > 0$) and*

$$\|\Gamma_d(s,x)\| \leqslant \gamma_d(\|x\|,s) := \alpha_d(\|x\|)e^{\overline{\gamma}_d s}, \quad (30)$$

for all $(s,x) \in \mathbb{R}^n \times \mathbb{R}_\geqslant$ and for some $\alpha_d \in \mathcal{K}_>$ and $\overline{\gamma}_d < 0$.

Our next assumption is on the observer's capability of tolerating the disturbances d in terms of the interaction between the incremental rate $\rho \in \mathcal{KK}_>$ of Λ_s^{-1} and the scaling factor Γ_d of the disturbance d.

Assumption 3 *There exists $\omega \in \mathcal{K}_>$ such that $\gamma_d(r,s)\rho(r,s) \leqslant \omega(r)$ for all $r, s \in \mathbb{R}_\geqslant$.*

Practical relevance of Assumptions 2 and 3 is substantiated by the result of Section 5.3 on lower triangular systems (see in particular Remark 5.1).

Assumption 4 *There exists $\beta \in \mathcal{K}$ such that for all $(x,d) \in \mathbb{R}^n \times \mathbb{R}^m$*

$$\|F_\infty(x,d) - F_\infty(x,0) - \frac{\partial F_\infty}{\partial d}\Big|_{(0,0)}d\|$$
$$+\|H_\infty(x,d) - H_\infty(x,0) - \frac{\partial H_\infty}{\partial d}\Big|_{(0,0)}d\| \leqslant \beta(\|x\|)\|d\|.$$

Assumption 5 *There exist $\Pi \in \mathbb{R}_>^{n\times n}$ and $\mu > 0$ such that*

$$\Pi\frac{\partial F_\infty}{\partial x}\Big|_{(0,0)} + \frac{\partial F_\infty}{\partial x}\Big|_{(0,0)}^\top\Pi - \frac{\partial H_\infty}{\partial x}^\top\Big|_{(0,0)}\frac{\partial H_\infty}{\partial x}\Big|_{(0,0)} \leqslant -\mu\Pi.$$

Assumption 4 is satisfied for instance when F_∞ (resp. H_∞) is linear with respect to d. Assumption 5 amounts to the detectability of the pair $(\frac{\partial H_\infty}{\partial x}|_{(0,0)}, \frac{\partial F_\infty}{\partial x}|_{(0,0)})$. This assumption takes care of the local behaviour of the nonlinear observer and guarantees local convergence.

The next assumption is on the existence of a state norm estimator, which is needed to provide an on-line correction s_t of the parameter s of the symmetry Λ_s and keep $\|\Lambda_{s_t}(x_t)\|$ small.

Assumption 6 *The tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$ satisfies a* (SNE) *condition for (1) with $\beta, \zeta \in \mathcal{K}^1_\infty$.*

Assumption 6 is a (SNE) condition (section 4) reinforced with the mild restriction $\beta, \zeta \in \mathcal{K}^1_\infty$, so that we can reshape $\beta \in \mathcal{K}^1_\infty$ into any desired $\widetilde{\beta} \in \mathcal{K}^1_\infty$, while preserving the (SNE) condition (Property (P3) at the end of section 4).

6.2 Main result

We are now ready to state (and prove) the main result of this paper. For a $C^{0,1}_{\rm loc}$ saturation function $\sigma_{\widehat{c}}$ (see Notation section), we consider the composition $\sigma_{\widehat{c}(s)}$ of $\sigma_{\widehat{c}}$ with the function $\widehat{c}(s) := c\psi(s)$ where $c > 0$ is a design parameter and $\psi = (\psi_1, \dots, \psi_n)^\top$ comes from the quadruple associated with the incremental symmetry Λ_s.

Theorem 6.1 *Under Assumptions 1-6, for each $\varepsilon > 0$ there exist $c > 0$, $\widehat{s} \in \mathcal{K}_> \cap C^1$, $\widetilde{\alpha} \in \mathcal{K}_\infty$ and $\widetilde{\delta} \in \mathcal{K}_>$ such that along the solutions of (1) and*

$$\begin{aligned}\dot{\widetilde{z}}_t &= \Lambda_{s_t *} F(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t), 0) + W(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t)) \frac{\partial \widehat{s}}{\partial r}\Big|_{\widehat{v}_t} \dot{\widehat{v}}_t \\ &+ \gamma_F(s_t) \left(\frac{\partial F_\infty}{\partial x}|_{(0,0)} - K \frac{\partial H_\infty}{\partial x}|_{(0,0)} \right) (\widetilde{z}_t - \sigma_{\widehat{c}(s_t)}(\widetilde{z}_t)) \\ &+ \frac{\gamma_F(s_t)}{\gamma_H(s_t)} K \Big(y_t - \Lambda_{s_t *} H(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t), 0) \Big), \end{aligned} \tag{31}$$

where $K := \Pi^{-1} \frac{\partial H_\infty}{\partial x}^\top|_{(0,0)}$ and $s_t = \widehat{s}(\widehat{v}_t)$ with

$$\dot{\widehat{v}}_t = -\lambda \widehat{v}_t + \widetilde{\alpha}(\|y_t\|) + \widetilde{\delta}(d_\infty), \ \widehat{v}_0 \geqslant 0, \tag{32}$$

the estimation error $e_t := x_t - \Lambda^{-1}_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t))$ is bounded for all times and

$$\limsup_{t \to +\infty} \|e_t\| \leqslant \varepsilon + \frac{6\omega(nc)\|D\|\sqrt{\lambda^\Pi_{max}} d_\infty}{\mu\sqrt{\lambda^\Pi_{min}}}, \tag{33}$$

where $D := \frac{\partial F_\infty}{\partial d}|_{(0,0)} - K \frac{\partial H_\infty}{\partial d}|_{(0,0)}$.

The parameter c and the functions $\widehat{s}$, $\widetilde{\alpha}$ and $\widetilde{\delta}$ are all determined in the proof of Theorem 6.1 (subsections (B.4.1)-(B.4.3) of the Appendix). The bound (33) depends on D, ω, d_∞ and the (arbitrary) small number ε and can be reduced by reducing the bound d_∞ on the disturbance d and/or by taking smaller values of the saturation level c (since $\omega \in \mathcal{K}_>$). Remarkably, notice that if in addition $\frac{\partial F_\infty}{\partial d}|_{(0,0)} = 0$ and $\frac{\partial H_\infty}{\partial d}|_{(0,0)} = 0$ then $D = 0$ and practical error convergence is achieved (i.e. arbitrarily small asymptotic error). This happens, for instance, in lower triangular systems with disturbance affecting only the last state equation (see Section 5.3 and Remark 5.1). If we put this together with Examples 4.1 and 4.2 on state norm estimators, by Theorem 6.1 we conclude that practical error convergence can be achieved for lower triangular systems with $C^{0,1}_{\rm loc}$ nonlinearities, homogeneous or linear with respect to unmeasured state variables, and disturbance affecting only the last state equation. Similar results with exact convergence in finite time have been obtained in [8] for lower triangular systems with Hölder nonlinearities satifying suitable homogeneity conditions and in [31] for chain of integrators with globally Lipschitz nonlinearities in the last state equation.

6.3 State solutions with known bound

The observer (31) simplifies remarkably and gets closer to a more familiar high-gain observer whenever a known bound $L > 0$ for the solutions of (1) is at hand, i.e. $\|x_t\| \leqslant L$ for all $t \geqslant 0$. In this case, the "small gain" condition of Assumption 3 and a state-norm estimator are no more needed. Let $\widehat{c}(s)$ be as in Theorem 6.1.

Theorem 6.2 *Under Assumptions 2, 4 and 5 and for all state solutions of (1) such that $\|x_t\| \leqslant L < +\infty$ for all $t \geqslant 0$, for each $\varepsilon > 0$ there exist $c > 0$ and $s_L > 0$ such that along the solutions of (1) and*

$$\begin{aligned}\dot{\widetilde{z}}_t &= \Lambda_{s_L *} F(\sigma_{\widehat{c}(s_L)}(\widetilde{z}_t), 0) \\ &+ \gamma_F(s_L) \left(\frac{\partial F_\infty}{\partial x}|_{(0,0)} - K \frac{\partial H_\infty}{\partial x}|_{(0,0)} \right) (\widetilde{z}_t - \sigma_{\widehat{c}(s_L)}(\widetilde{z}_t)) \\ &+ \frac{\gamma_F(s_L)}{\gamma_H(s_L)} K \Big(y_t - \Lambda_{s_L *} H(\sigma_{\widehat{c}(s_L)}(\widetilde{z}_t), 0) \Big), \end{aligned} \tag{34}$$

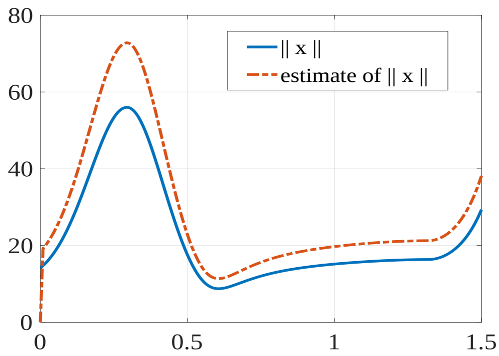
where $K := \Pi^{-1} \frac{\partial H_\infty}{\partial x}^\top|_{(0,0)}$, the estimation error $e_t := x_t - \Lambda^{-1}_{s_L}(\sigma_{\widehat{c}(s_L)}(\Lambda_{s_L}(z_t)))$ is bounded for all times and

$$\limsup_{t \to +\infty} \|e_t\| \leqslant \varepsilon + \frac{6\omega(nc, s_L)\|D\|\sqrt{\lambda^\Pi_{max}} d_\infty}{\mu\sqrt{\lambda^\Pi_{min}}} \tag{35}$$

where $\omega(r, s) := \gamma_d(r, s)\rho(r, s)$ and $D := \frac{\partial F_\infty}{\partial d}|_{(0,0)} - K \frac{\partial H_\infty}{\partial d}|_{(0,0)}$.

If in addition $\frac{\partial F_\infty}{\partial d}|_{(0,0)} = 0$ and $\frac{\partial H_\infty}{\partial d}|_{(0,0)} = 0$ then $D = 0$ and practical convergence is achieved. With Section 5.3 and Remark 5.1 in mind, this proves that, as long as a known bound for the state solutions is available, practical estimation error convergence can be achieved for lower triangular systems with $C^{0,1}_{\rm loc}$ nonlinearities and disturbance affecting only the last equation. Similar results with convergence in finite time have been obtained in [8] with C^0 nonlinearities and in [21], [24] and [13] for a chain of integrators $(x_1, \dots, x_n)$ with $|\dot{x}_n(t)| \leqslant L$ for all $t \geqslant 0$ and for some known $L > 0$.

The known bound L on the state solutions of (1) may be more generally assumed *time-varying*, i.e. $\|x_t\| \leqslant L(t)$



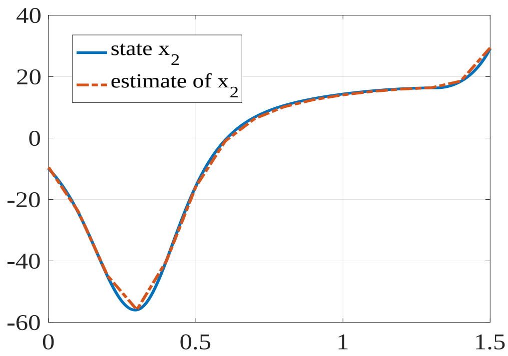


Figure 2. With $x(0) = (10, -10)^\top$: (left figure) state norm $\|x\|$ of (36) and its estimate, (right figure) State x_2 of (36) and its estimate.

for all $t \geqslant 0$, with $\sup_{t\geqslant 0} \|\dot{L}(t)/L(t)\| < +\infty$. However, if $L(t)$ is unbounded, yet for error convergence we need to satisfy the "small gain" condition of Assumption 3 and the observer has the form (31) with $\hat{v}_t := L(t)$, although we may experience serious implementation problems due to the unboundedly growing observer gains. If in addition $\frac{\partial F_\infty}{\partial d}|_{(0,0)} = 0$ and $\frac{\partial H_\infty}{\partial d}|_{(0,0)} = 0$ practical convergence is achieved. State estimation in finite time have been obtained in [21] and [24] using sliding mode observers for a disturbance-free chain of integrators $(x_1, \ldots, x_n)$ with $|\dot{x}_n(t)| \leqslant L(t)$ and $\dot{L}(t) \geqslant 0$ for all $t \geqslant 0$ and for some known $L(t)$: however, when $L(t)$ is unbounded the effect of bounded disturbances on $\dot{x}_n(t)$ results in unbounded estimation errors.

6.4 Illustrative applications

6.4.1 An academic example

In this section we want to illustrate how to design (using the results of Theorem 6.1) a state observer for the system:

$$\dot{x} = F(x, d) = (x_2, F_2(x) + G_2(x)d)^\top, \; y = H(x, d) = x_1 \tag{36}$$

with $F_2(x) = x_1 x_2$, $G_2(x) = \sqrt{1 + x_1^2}$ and $d(t) = 10\sin(t)$. The solution $x(t)$ is unbounded if, for instance, $x(0) = (10, -10)^\top$. For this type of systems and as long as we want to design *global* state observer, we cannot use the sliding-mode observers of [1], [2] or [31] because F_2 is not incrementally dissipative (nor Lipschitz) and G_2 is not bounded and we do not consider homogeneus observers ([4], [3]), since there is no (homogeneous) function $\overline{F}_2$ such that $|F_2(x+e) - F_2(x)| \leqslant \overline{F}_2(e)$ for all x, e, although $F_2(x)$ is itself homogeneous. Here, "global observer" means that the estimation error converges for any state and observer initial conditions. On the other hand, if also semi-global state-observers (i.e. the estimation error converges for any state and observer initial conditions in a given bounded set $\mathcal{D} \subset \mathbb{R}^n$) come into play, sliding-mode observers (in the simple version of [21] and [24]) as well as homogeneous observers can be used for (35) obtaining disturbance suppression with finite-time error convergence. Our (global) observer will suppress the disturbance d up to any given degree or tolerance with asymptotic error convergence.

The vector field $F(x, d)$ has the form (14) with $F_1(x_1) \equiv 0$ and $G_1(x_1) \equiv 1$. According to Proposition 5.2 (with Remark 5.1) we can find $\gamma_F \in \mathcal{K} \cap C^1$ such that the decoupled linear action $\Lambda_s(x) := (e^{-s}x_1, e^{-s}\gamma_F^{-1}(s)x_2)^\top$ is GES with generator $W(x) = \mathrm{diag}\{-x_1, -(1 + \frac{d}{ds}\ln\gamma_F(s))x_2\}$ and an incremental orbital symmetry in the ∞-limit of F with associated quadruple $(W, e^{-s}, e^s\gamma_F, (1, \gamma_F^{-1})^\top)$, limit $F_\infty = (x_2, 0)^\top$ and scaling factors (γ_F, Γ_d), where $\Gamma_d(s, x) := \gamma_F^{-3/2}(s)$. Moreover, Λ_s and Γ_d should satisfy also (30) and Assumption 3. Hence, we get $\gamma_F(s) = e^{2s}$. In addition, Λ_s is also an incremental orbital symmetry in the ∞-limit of H with limit $H_\infty = x_1$ and scaling factors (γ_H, Γ_d), with $\gamma_H = e^s$. The linearization of F_∞ and H_∞ at $x = 0$ is observable and satisfies Assumptions 4 and 5. A state norm estimator for (36) (Assumption 6) is designed as the output of $\dot{\hat{v}} = -\hat{v} + 20(y^2 + 1)$, $\hat{v}(0) = 0$. A simulation with $x(0) = (10, -10)^\top$ using an observer of the form (31)-(32)

$$\dot{\tilde{z}} = \tilde{s}^2 \begin{pmatrix} \tilde{z}_2 + \frac{4}{\tilde{s}}(y - \tilde{s}\tilde{z}_1) - \frac{\dot{\hat{v}}}{\tilde{s}^3}\sigma_1(\tilde{z}_1) \\ \frac{1}{\tilde{s}}\sigma_{\tilde{s}^{-2}}(\tilde{z}_2)\sigma_1(\tilde{z}_1) + \frac{2}{\tilde{s}}(y - \tilde{s}\tilde{z}_1) - \frac{3\dot{\hat{v}}}{\tilde{s}^3}\sigma_{\tilde{s}^{-2}}(\tilde{z}_2) \end{pmatrix}$$

where $\tilde{s} = 50 + \hat{v}$, has been worked out and an estimate $z = (\tilde{s}\sigma_1(\hat{z}_1), \tilde{s}^3\sigma_{\tilde{s}^{-2}}(\hat{z}_1))^\top$ of x has been obtained with an error tolerance $\varepsilon = 0.5$ (in this case the upper bound on the estimation error norm is exactly ε since $D = 0$: see (33)). The results for the $\|x\|$-estimate and the x_2-estimate are illustrated in Figure 2, in which we see how the state norm estimator provides (after a very short transient) an upper bound for the state norm $\|x\|$, while the observer keeps track of x_2 within the given error tolerance using the upper bound on the state norm.

6.4.2 Cart pendulum system

Consider $H(x, d) = y$ and

$$F(x, d) = ((G_1(y)z)^\top, (F_2(x) + G_2(y)d)^\top)^\top \tag{37}$$

where $x = (y^\top, z^\top)^\top$, $y, z \in \mathbb{R}^2$, $d \in \mathbb{R}$ with

$$F_2(x) = \Psi^\top(y)\nabla\phi(y),\ \phi(y) = a\cos(y_1),$$
$$\Psi(y) = \begin{pmatrix} \frac{\sqrt{m}}{\sqrt{m - b^2\cos^2(y_1)}} & 0 \\ \frac{-b\cos(y_1)}{\sqrt{m}\sqrt{m-b^2\cos^2(y_1)}} & \frac{1}{\sqrt{m}} \end{pmatrix},\ G_2(y) = \Psi^\top(y)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $a > 0$ and $0 < b < \sqrt{m}$. As far as we see, there is no global observer design with disturbance suppression for (37) available in the literature. Let us see how to design our observer for (37) according to the lines of Theorem 6.1. By following the constructive procedure of section 5.3 (with vector-valued states x_1, x_2) we obtain that

$$\Lambda_s(x) = (e^{-s}y^\top, e^{-2s}(\Psi^{-1}(e^{-s}y)\Psi(y)z)^\top)^\top \tag{38}$$

where $\Gamma_d(s, y) = e^{-2s}\Psi^{-1}(y)\Psi(e^s y)$, is an incremental symmetry in the ∞-limit of F with scaling factors $(\gamma_F, \Gamma_d) = (e^s, e^{-2s}G_2^{\#}(y)\ \Psi^{-1}(y)\Psi(e^s y)G_2(e^s y))$ and limit $F_\infty = ((G_1(y)z)^\top, 0^\top)^\top$ and, at the same time, an incremental symmetry in the ∞-limit of H with scaling factor $\gamma_H = e^s$ and limit $H_\infty = y$. A state norm estimator can be designed following Example 4.1. $\lhd$

7 Conclusions

New classes of global observers with on-line adapted gains have been presented based on state-norm estimators and incremental orbital symmetries. The geometric aspects and peculiarities of these symmetries reflect the observer structure. Future works will be devoted to the global output feedback stabilization problem using orbital symmetry-based observers and/or controllers.

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A Conditions for satisfying (PDI) and (ULB)

A.1 Closed-form solutions of (PDI)

In this section we study some general conditions under which it is possible to construct time-varying solutions $v(x, t)$ of (PDI). To this aim, we discard finite exit times from $\mathbb{R}^n$ for the backward solutions of (1).

(BWC). *(Backward completeness) The solutions $x_s(x, t; d)$ of (1) are defined for all $(x, t, s) \in \mathbb{R}^n \times \mathcal{D} \times \mathbb{R}_\geqslant \times [0, t]$.*

Let $Y_s(x, t; d) := H(x_s(x, t; d), d_s) - H(0, d_s)$.

Proposition A.1 *Under Assumption* (BWC) *and for any given $\lambda > 0$ and $\alpha \in \mathcal{K}_\infty$, $(v, \lambda, \overline{\alpha}, \overline{\delta})$ satisfies a* (PDI) *condition for (1) with*

$$v(x, t) := \int_0^t e^{-\lambda(t-s)}\alpha(\|Y_s(x, t; d)\|)ds \tag{A.1}$$

and $\overline{\alpha}(s) := \alpha(2s)$, $\overline{\delta}(s) := \sup_{\|d\|\leqslant s}\alpha(2\|H(0, d)\|)$.

The proof is deferred in section B.2 of the Appendix. The paradigm we follow for the construction of the function $v(x, t)$ in (A.1) may be compared for similarities with the one for constructing the map $T(x, t)$ in [9], although the purpose and the use of $v(x, t)$ is far different from that of $T(x, t)$. The function $v(x, t)$ is defined using the backward solutions of (1) and, implicitly, the function d_t.

A.2 Sufficient conditions for (ULB)

In order to have the function $v(x, t)$, defined in (A.1), uniformly lower bounded as required in (ULB) , we invoke a sort of uniform (backward) state reconstructibility property from the outputs.

(BWR). *(Uniform backward reconstructibility). There exist $\bar{t}, b > 0$, $\alpha, \beta \in \mathcal{K}_\infty$ such that for all $(x, d) \in \mathbb{R}^n \times \mathcal{D}$*

$$\int_0^{\bar{t}} \alpha(\|Y_s(x, \bar{t}; d)\|)ds \geqslant \beta(\|x\|) - b. \tag{A.2}$$

A similar reconstructibility property was used in [30] in a noise-free context for state-dependent solutions of differential Riccati equations. Also, comparisons can be made with [9], where a uniform backward reconstructibility condition is used for the purpose of injectively reconstructing the state x from the map $T(x,t)$. Here, we introduce the uniform backward reconstructibility condition for estimating the state norm $\|x\|$ from the function $v(x,t)$. A proof of the following result is given in section B.3 of the Appendix.

Proposition A.2 *Under Assumptions* (BWC)+(BWR) *and for any given $\lambda > 0$ and $\alpha \in \mathcal{K}_\infty$ there exist $\overline{t}, \overline{b} > 0$ and $\overline{\beta} \in \mathcal{K}_\infty$ such that $(v + \overline{b}, \lambda, \overline{\alpha}, \overline{\delta} + \lambda\overline{b}, \overline{\beta}, \overline{t})$ satisfies a (PDI)+(ULB) condition for (1) with $v : \mathbb{R}^n \times \mathbb{R}_\geqslant$ defined in (A.1) and $\overline{\alpha} \in \mathcal{K}_\infty$ and $\overline{\delta} \in \mathcal{K}_>$ as in Proposition A.1.*

Using Proposition A.2, we can construct for instance $v(x,t)$ as in (A.1) satisfying a (PDI)+(ULB) condition for the system $\dot{x}_1 = x_2 + x_2^3$, $\dot{x}_2 = 0$, $y = x_1 + d$ (feedforward systems are in general amenable to this kind of construction).

The (BWC)+(BWR) condition can be replaced by a condition based on the Lie derivatives of the output map H along the vector field F, which is very closely related to strong differential observability (see for instance [9]).

B Proofs of main and auxiliary results

B.1 Proof of Property (P3)

Let $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \overline{t})$ satisfies a (SNE) condition. Given $\widetilde{a}, \widetilde{b} > 0$ we will show how to construct a new tuple $(\widetilde{v}, \widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\beta}, \widetilde{\zeta}, \widetilde{\xi}, \overline{t})$, with $\widetilde{\beta}(s) \equiv \widetilde{a}s^{\widetilde{b}}$ and still satisfying a (SNE) condition. First, we will construct a tuple $(v^\circ, \lambda^\circ, \alpha^\circ, \delta^\circ, \beta^\circ, \zeta^\circ, \xi^\circ, \overline{t})$ with $\beta^\circ(s) \equiv s^{\widetilde{b}}$ and satisfying a (SNE) condition.

Since $\beta, \zeta \in \mathcal{K}_\infty^1$ there exist $\beta_l, \zeta_u > 0$ for which $\beta_l \leqslant \frac{s}{\beta(s)}\frac{d\beta}{ds}$ and $\frac{s}{\zeta(s)}\frac{d\zeta}{ds} \leqslant \zeta_u$ for all $s \geqslant 0$: this implies $b_l s^{\beta_l} \leqslant \beta(s)$ and $\zeta(s) \leqslant z_u s^{\zeta_u}$ for all $s \geqslant 1$ and for some $b_l, z_u > 0$.Hence, up to modifications of β and ζ, we can assume $\beta(s) = b_l s^{\beta_l}$ and $\zeta(s) = z_u s^{\zeta_u}$ for all $s \geqslant 1$. Define the new tuple $(\overline{v}, \lambda, \alpha, \overline{\delta}, \overline{\beta}, \zeta, \overline{\xi}, \overline{t})$ with $\overline{v} \equiv v + 2b_l$, $\overline{\delta} \equiv \delta + 2b_l$, $\overline{\beta} \equiv b_l s^{\overline{\beta}_l}$ for $\overline{\beta}_l := \min\{\widetilde{b}, \beta_l\}$ and $\overline{\xi} \equiv \xi + z_u$. Indeed, since $b_l s^{\beta_l} = \beta(s)$ and $\zeta(s) = z_u s^{\zeta_u}$ for all $s \geqslant 1$, we have $b_l s^{\beta_l} - b_l \leqslant \beta(s)$ and $\zeta(s) \leqslant z_u s^{\zeta_u} + z_u$ for all $s \geqslant 0$. Hence, on account of the (ULB)+(UUB) condition on $(v, \beta, \zeta, \xi, \overline{t})$, for all $(x,d,t) \in \mathbb{R}^n \times \mathcal{D} \times [\overline{t}, +\infty)$ and any $\overline{\beta}_l \in (0, \min\{\widetilde{b}, \beta_l\}]$

$$
\begin{aligned}
&v(x,t) \geqslant \beta(\|x\|) \geqslant b_l\|x\|^{\beta_l} - b_l \geqslant \overline{\beta}(\|x\|) - 2b_l,\\
&\alpha(\|H(x,d_t)\|) \leqslant \zeta(v(x,t)) + \xi(d_\infty) \leqslant z_u s^{\zeta_u} + z_u + \xi(d_\infty).
\end{aligned}
$$

This, on account of Property (P2), implies that $(\overline{v}, \lambda, \alpha, \overline{\delta}, \overline{\beta}, \zeta, \overline{\xi}, \overline{t})$ satisfies a (SNE) condition. By considering the modified function $v^\circ(x,t) := (\overline{\beta}^{-1}(\overline{v}(x,t)))^{\widetilde{b}}$ and on account of the (PDI) condition on $(\overline{v}, \lambda, \alpha, \overline{\delta})$, after some lengthy passages we get for all $(x,d,t) \in \mathbb{R}^n \times \mathcal{D} \times \mathbb{R}_\geqslant$

$$
\begin{aligned}
&\frac{\partial v^\circ}{\partial x}(x,t)F(x,d_t) + \frac{\partial v^\circ}{\partial t}(x,t) \leqslant -\lambda v^\circ(x,t)\\
&+\frac{2^{\frac{\widetilde{b}}{\overline{\beta}_l}-1}}{\lambda^{\frac{\widetilde{b}}{\overline{\beta}_l}-1} b_l^{\frac{\widetilde{b}}{\overline{\beta}_l}}}\left[\alpha^{\frac{\widetilde{b}}{\overline{\beta}_l}}(\|H(x,d_t)\|) + \overline{\delta}^{\frac{\widetilde{b}}{\overline{\beta}_l}}(d_\infty)\right]\\
&:= -\lambda v^\circ(x,t) + \alpha^\circ(\|H(x,d_t)\|) + \delta^\circ(d_\infty),
\end{aligned}
\tag{B.1}
$$

where we used Young inequality and the inequality $|x + y|^p \leqslant 2^{p-1}||x|^p + |y|^p|$ for all $x, y \in \mathbb{R}$ and $p \geqslant 1$. Moreover, on account of the (ULB) condition on $(\overline{v}, \overline{\beta}, \overline{t})$, we obtain for all $(x,t) \in \mathbb{R}^n \times [\overline{t}, +\infty)$

$$
v^\circ(x,t) \geqslant (\overline{\beta}^{-1}(\overline{\beta}(\|x\|)))^{\widetilde{b}} = \|x\|^{\widetilde{b}} := \beta^\circ(\|x\|).
$$

Upon the (UUB) condition on $(v, \zeta, \xi, \overline{t})$, it follows for all $(x,d,t) \in \mathbb{R}^n \times \mathcal{D} \times [\overline{t}, +\infty)$

$$
\begin{aligned}
&\alpha^\circ(\|H(x,d_t)\|) \leqslant (4/\lambda)^{\frac{\widetilde{b}}{\overline{\beta}_l}-1} b_l^{-\frac{\widetilde{b}}{\overline{\beta}_l}}\left[\zeta^{\frac{\widetilde{b}}{\overline{\beta}_l}}(b_l \widehat{v}^{\frac{\overline{\beta}_l}{\widetilde{b}}}(x,t)) + \xi^{\frac{\widetilde{b}}{\overline{\beta}_l}}\right]\\
&:= \zeta^\circ(\widehat{v}(x,t)) + \xi^\circ(d_\infty).
\end{aligned}
$$

Moreover, since $\zeta(s) = z_u s^{\zeta_u}$ for $s \geqslant 1$, we have for $s \geqslant 1$

$$
\zeta^\circ(s) = (4/\lambda)^{\frac{\widetilde{b}}{\overline{\beta}_l}-1}(\zeta(b_l s^{\frac{\overline{\beta}_l}{\widetilde{b}}})/b_l)^{\frac{\widetilde{b}}{\overline{\beta}_l}} = (4/\lambda)^{\frac{\widetilde{b}}{\overline{\beta}_l}-1}\zeta(s).
$$

It follows that the tuple $(v^\circ, \lambda, \alpha^\circ, \delta^\circ, \beta^\circ, \zeta^\circ, \xi^\circ, \overline{t})$ with $\beta^\circ(s) \equiv s^\mu$ satisfy a (SNE) condition and, on account of Property (P1), the modified tuple $(\widetilde{v}, \lambda, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\beta}, \widetilde{\zeta}, \widetilde{\xi}, \overline{t}) = (\widehat{s}v^\circ, \lambda, \widehat{s}\alpha^\circ, \widehat{s}\delta^\circ, \widehat{s}\beta^\circ, \widehat{s}\zeta^\circ, \widehat{s}\xi^\circ, \overline{t})$ is the claimed one in Property (P3) satisfying a (SNE) condition.

B.2 Proof of Proposition A.1

Notice that $x_s(x_{t+\Delta t}(x,t;d), t + \Delta t; d) = x_s(x,t;d)$ for any $\Delta t \in \mathbb{R}$. We have

$$
\begin{aligned}
&\frac{v(x_{t+\Delta t}(x,t;d), t+\Delta t) - v(x,t)}{\Delta t} = \Big(\frac{e^{-\lambda\Delta t} - 1}{\Delta t}\Big)v(x,t)\\
&+\frac{e^{-\lambda\Delta t}}{\Delta t}\int_t^{t+\Delta t} e^{-\lambda(t-s)}\alpha(\|Y_s(x,t;d)\|)ds.
\end{aligned}
$$

Letting Δt tend to 0 we get

$$
\begin{aligned}
&\frac{\partial v}{\partial x}(x,t)F(x,d_t) + \frac{\partial v}{\partial t}(x,t)\\
&= -\lambda v(x,t) + \alpha(\|H(x,d_t) - H(0,d_t)\|)\\
&\leqslant -\lambda v(x,t) + \overline{\alpha}(\|H(x,d_t)\|) + \overline{\delta}(\|d_t\|),
\end{aligned}
$$

where we used $\alpha(s + r) \leqslant \alpha(2s) + \alpha(2r)$ for all $s, r \geqslant 0$.

B.3 Proof of Proposition A.2

Proof. Let $\bar{t}, b > 0$, $\alpha, \beta \in \mathcal{K}_\infty$ be as in (BWR) and $v : \mathbb{R}^n \times \mathbb{R}_\geqslant$ as in (A.1). For all $t \geqslant \bar{t}$ we have

$$v(x,t) \geqslant \int_{t-\bar{t}}^{t} e^{-\lambda(t-s)} \alpha(\|Y_s(x,t;d)\|) ds \\ \geqslant e^{-\lambda\bar{t}} \int_0^{\bar{t}} \alpha(\|Y_\psi(x,\bar{t};\bar{d})\|) d\psi \tag{B.2}$$

and $\bar{d}_\psi := d_{\psi+t-\bar{t}}$. But $\bar{d} \in \mathcal{D}$ and on account of (BWR), $v(x,t) \geqslant e^{-\lambda\bar{t}}(\beta(\|x\|) - b) := \bar{\beta}(\|x\|) - \bar{b}$ for all $(x,t) \in \mathbb{R}^n \times [\bar{t}, +\infty)$, which proves that $(v + \bar{b}, \bar{\beta}, \bar{t})$ satisfy a (ULB) condition. On the other hand, $(v + \bar{b}, \lambda, \bar{\alpha}, \bar{\delta} + \lambda\bar{b})$ satisfies a (PDI) condition by proposition A.1 and property $(P2)$.

B.4 Proof of Theorem 6.1

Let Λ_s be the GES action in Assumption 2 with associated quadruple (W, λ, ρ, ψ) and $\hat{c}(s) := c\psi(s)$, with $c \in (0,1]$, be the function introduced before Theorem 6.1 with $\hat{s} \in \mathcal{K}_>$ to be determined by the design.

We notice at once that, by the properties of $C^{0,1}_{\text{loc}}$-saturation functions (see also Notation section), for all $(x,z,s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_\geqslant$

$$\|\sigma_{\hat{c}(s)}(x)\| \leqslant \|\hat{c}(s)\|, \ \|\sigma_{\hat{c}(s)}(x) - \sigma_{\hat{c}(s)}(z)\| \leqslant \|x - z\| \tag{B.3}$$

with $\|\hat{c}(s)\| = c\|\psi(s)\| \leqslant nc$. By inspection of the state norm estimator's equation (32) and by forward completeness of (1) (Assumption 1), it follows that $\hat{v}_t$ has no finite escape time (i.e. it is defined for all $t \geqslant 0$). Also, by inspection of the observer's equation (31), since $\hat{v}_t$ has no finite escape time and $\|\sigma_{\hat{c}(\hat{v}_t)}(\Lambda_{\hat{s}(\hat{v}_t)}(z_t))\|$ is bounded by $2\|\hat{c}(\hat{v}_t)\| \leqslant 2n$, it is seen that $\Lambda_{\hat{s}(\hat{v}_t)}(z_t)$ (and therefore, by completeness of the action Λ_s, also the observer's solution z_t) has no finite escape time.

Define the following operators Δ, Δ_s acting on (respectively, C^1 and $C^{0,1}_{\text{loc}}$) vector fields $F_\infty, F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: for $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $s \in \mathbb{R}_\geqslant$

$$\Delta F_\infty(x,d) := F_\infty(x,d) - \frac{\partial F_\infty}{\partial x}\Big|_{(0,0)} x - \frac{\partial F_\infty}{\partial d}\Big|_{(0,0)} d \\ \Delta_s(F, F_\infty)(x,d) := \frac{1}{\gamma_F(s)} (\Lambda_s)_* F\left(x, \Gamma_d^{-1}(s,x)d\right) \\ - F_\infty(x,d) \tag{B.4}$$

Similar definitions are adopted for mappings H, H_∞. Using the mean value theorem, for all $x, z \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $s \in \mathbb{R}_\geqslant$

$$\|\Delta F_\infty(\sigma_{\hat{c}(s)}(x), 0) - \Delta F_\infty(\sigma_{\hat{c}(s)}(z), 0)\| \\ \leqslant \left\| \int_0^1 \frac{\partial \Delta F_\infty}{\partial x}\Big|_{(\theta\sigma_{\hat{c}(s)}(x) + (1-\theta)\sigma_{\hat{c}(s)}(z), 0)} d\theta \right\| \|x - z\|, \tag{B.5}$$

Since $\frac{\partial \Delta F_\infty}{\partial x}\Big|_{(0,0)} = 0$ and using (B.3), there exists some $\alpha_1 \in \mathcal{K}$ such that for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^n$ and $s \in \mathbb{R}_\geqslant$

$$\|\Delta F_\infty(\sigma_{\hat{c}(s)}(x), 0) - \Delta F_\infty(\sigma_{\hat{c}(s)}(z), 0)\| \\ \leqslant \alpha_1(\|\sigma_{\hat{c}(s)}(x)\| + \|\sigma_{\hat{c}(s)}(z)\|)\|x - z\| \leqslant \alpha_1(2nc)\|x - z\| \tag{B.6}$$

Since W is $C^{0,1}_{\text{loc}}$ and using (B.3), there also exists some $\alpha_2 \in \mathcal{K}_>$ such that for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^n$ and $s \in \mathbb{R}_\geqslant$

$$\|W(\sigma_{\hat{c}(s)}(x)) - W(\sigma_{\hat{c}(s)}(z))\| \\ \leqslant \alpha_2(\|\sigma_{\hat{c}(s)}(x)\| + \|\sigma_{\hat{c}(s)}(z)\|)\|x - z\| \leqslant \alpha_2(2nc)\|x - z\|. \tag{B.7}$$

Moreover, by Assumption 4 and (B.3), we have $\beta \in \mathcal{K}$ such that for all $(x,d) \in \mathbb{R}^n \times \mathbb{R}^m$ and $s \in \mathbb{R}_\geqslant$

$$\|\Delta F_\infty(\sigma_{\hat{c}(s)}(x), d) - \Delta F_\infty(\sigma_{\hat{c}(s)}(x), 0)\| \leqslant \beta(nc)\|d\|.$$

On account of (B.3) and since Λ_s is by Assumption 2 an incremental symmetry of F in the ∞-limit with limit F_∞ and scaling factors (γ_F, Γ_d), we have $\lambda_F \in \mathcal{KL}_>$ such that for all $(x,z,d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and $s \in \mathbb{R}_\geqslant$

$$\|\Delta_s(F,F_\infty)(\sigma_{\hat{c}(s)}(x), 0) - \Delta_s(F,F_\infty)(\sigma_{\hat{c}(s)}(z), 0)\| \\ \leqslant \lambda_F(0,s)\|\sigma_{\hat{c}(s)}(x) - \sigma_{\hat{c}(s)}(z)\| \leqslant \lambda_F(0,s)\|x - z\|, \\ \|\Delta_s(F,F_\infty)(\sigma_{\hat{c}(s)}(x), d) - \Delta_s(F,F_\infty)(\sigma_{\hat{c}(s)}(z), 0)\| \\ \leqslant \lambda_F(\|d\|, s)\|d\|. \tag{B.8}$$

The same inequalities can be worked out for H and H_∞.

B.4.1 Selection of parameter c

In this section we choose the parameter $c \in (0,1]$ of the function $\hat{c}$ in (31). Let μ, Π be as in Assumption 5, $\gamma_d \in \mathcal{KL}_>$ as in Assumption 2 and pick $\varepsilon > 0$ (the estimation error tolerance). Using inequalities (B.6)-(B.8) for F and F_∞, together with the corresponding ones for H and H_∞, find $s_0 \geqslant 1$ and $\delta_0, c \in (0,1]$ such that for all $x, z \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$ such that $\|d\| \leqslant \gamma_d(nc, 0)d_\infty$, $|\delta| \leqslant \delta_0$ and for all $s \geqslant s_0$:

$$\bar{A}_s(x,z;\delta) \leqslant \frac{\mu\lambda^\Pi_{min}}{4}, \tag{B.9}$$

$$\bar{P}_s(x;d) \leqslant \frac{\varepsilon\mu\lambda^\Pi_{min}}{32\omega(nc)\lambda^\Pi_{max}d_\infty} + \|D\|, \tag{B.10}$$

where $D := \frac{\partial F_\infty}{\partial d}|_{(0,0)} - K\frac{\partial H_\infty}{\partial d}|_{(0,0)}$ and

$$\begin{aligned}
\overline{A}_s(x,z;\delta) &:= \frac{\|2\Pi(A_s(\sigma_{\widehat{c}(s)}(x);\delta) - A_s(\sigma_{\widehat{c}(s)}(z);\delta)\|}{\|x-z\|} \\
&+ \frac{\|2\Pi K(C_s(\sigma_{\widehat{c}(s)}(x)) - C_s(\sigma_{\widehat{c}(s)}(z)))\|}{\|x-z\|} \\
A_s(x;\delta) &:= \Delta F_\infty(x,0) + \Delta_s(F,F_\infty)(x,0) + \delta W(x) \\
C_s(x) &:= \Delta H_\infty(x,0) + \Delta_s(H,H_\infty)(x,0), \\
\overline{P}_s(x;d) &:= \frac{\|P_s(\sigma_{\widehat{c}(s)}(x);d) - KQ_s(\sigma_{\widehat{c}(s)}(x);d)\|}{\|d\|}, \\
P_s(x;d) &:= \frac{\partial F_\infty}{\partial d}\Big|_{(0,0)} d + \Delta F_\infty(x,d) - \Delta F_\infty(x,0), \\
&\quad + \Delta_s(F,F_\infty)(x,d) - \Delta_s(F,F_\infty)(x,0), \\
Q_s(x;d) &:= \frac{\partial H_\infty}{\partial d}\Big|_{(0,0)} d + \Delta H_\infty(x,d) - \Delta H_\infty(x,0) \\
&\quad + \Delta_s(H,H_\infty)(x,d) - \Delta_s(H,H_\infty)(x,0). \qquad \text{(B.11)}
\end{aligned}$$

B.4.2 Selection of the functions $\widehat{s} \in \mathcal{K}_>$, $\widetilde{\alpha} \in \mathcal{K}_\infty$ and $\widetilde{\delta} \in \mathcal{K}_>$ in (32)

Since Λ_s is a GES action, by the (SM) property (9) of Λ_s there exist $k, h > 0$ such that for all $(s,x) \in \mathbb{R}^n \times \mathbb{R}_\geqslant$

$$\|\Lambda_s(x)\| \leqslant \lambda(\|x\|, s) := ke^{-hs}\|x\|. \qquad \text{(B.12)}$$

On the other hand, by the (C) property (10) of Λ_s, we obtain for each $\tau \in \mathcal{L}$ a function $\widehat{s} \in \mathcal{K}_>$, $\widehat{s}(r) := \frac{1}{h}\ln(\frac{k(r+1)}{\tau(r)})$, such that

$$\lambda(r+1, \widehat{s}(r)) \leqslant \tau(r), \ \forall r \geqslant 0. \qquad \text{(B.13)}$$

Hence, choosing in particular $\tau(r) = \frac{k(r+1)}{(q+r)^p}$, $q,p > 1$, we get the function $\widehat{s}(r) = \frac{p}{h}\ln(q+r)$ in (31). The parameters $q,p > 1$ are picked out later.

According to property *(P3)* (section 4), we can transform the tuple $(v, \lambda, \alpha, \delta, \beta, \zeta, \xi, \bar{t})$ of Assumption 6, with $\beta \in \mathcal{K}_\infty^1$, into a new tuple $(\widetilde{v}, \lambda, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\zeta}, \widetilde{\beta}, \widetilde{\xi}, \bar{t})$ satisfying a (SNE) condition with $\widetilde{\beta}(r) := r \in \mathcal{K}_\infty^1$ and state norm estimate

$$\|x_t\| \leqslant \widetilde{v}(x_t, t) \leqslant \widehat{v}_t + 1, \ t \geqslant \max\{\bar{t}; T_{x_0,d}\}, \qquad \text{(B.14)}$$

where $\widehat{v}_t$ is the output of the filter (32), and

$$\widetilde{\zeta}(r) = \widetilde{\ell}\zeta(r) \qquad \text{(B.15)}$$

for all $r \geqslant 1$ and for some $\widetilde{\ell} > 0$. This determines the functions $\widetilde{\alpha}$ and $\widetilde{\delta}$ in (32).

B.4.3 Selection of the parameters $q,p > 1$ of the function $\widehat{s}$

Let $\overline{\gamma}_F > 0$ and $\overline{\gamma}_d < 0$ be as in Assumption 2 and $\zeta \in \mathcal{K}_\infty^1$ as in Assumption 6. Pick $p > 1$ such that

$$\lim_{r \to +\infty} \frac{p}{h}\frac{r + \zeta(r)}{r^{1+\frac{p}{h}\overline{\gamma}_F}} = 0. \qquad \text{(B.16)}$$

By (B.16) it is possible to pick $q > \max\{e^{\frac{s_0 h}{p}}, (k/c)^{\frac{1}{p-1}}, 1\}$ such that for all $r \geqslant q$

$$\begin{aligned}
&\frac{p/h}{r^{1+\frac{p}{h}\overline{\gamma}_F}}\Big((\lambda + \widetilde{\ell})(r + \zeta(r)) + \widetilde{\xi}(d_\infty) + \widetilde{\delta}(d_\infty)\Big) \\
&\leqslant \min\left\{\delta_0, \frac{\mu}{2(1+4|\overline{\gamma}_d|)}\right\} \qquad \text{(B.17)}
\end{aligned}$$

Since $q > (k/c)^{\frac{1}{p-1}}$ and on account of (B.12) and (B.13)

$$\begin{aligned}
&\forall (r,x) \in \mathbb{R}_\geqslant \times \mathbb{R}^n : \|x\| \leqslant r+1 \\
&\Rightarrow \|\Lambda_{\widehat{s}(r)}(x)\| \leqslant \lambda(r+1, \widehat{s}(r)) \leqslant \tau(r) \leqslant c. \qquad \text{(B.18)}
\end{aligned}$$

From (B.14) and (B.18), since $c \leqslant 1$, $\lambda \in \mathcal{KL}$ and the (NUB) property (12) of Λ_s, it follows that for each $i = 1, \ldots, n$ and $t \geqslant \max\{\bar{t}; T_{x_0,d}\}$

$$|[\Lambda_{s_t}(x_t)]_i| \leqslant \frac{|[\Lambda_{s_t}(x_t)]_i|}{\lambda(\|x_t\|, s_t)}\lambda(\widehat{v}_t + 1, s_t) \leqslant c\psi_i(s_t) \leqslant \psi_i(s_t)$$

where $s_t := \widehat{s}(\widehat{v}_t)$. Hence, for all $t \geqslant \max\{\bar{t}; T_{x_0,d}\}$

$$\sigma_{\widehat{c}(s_t)}(\Lambda_{s_t}(x_t)) = \sigma_{c\psi(s_t)}(\Lambda_{s_t}(x_t)) = \Lambda_{s_t}(x_t). \qquad \text{(B.19)}$$

By the (SNE) condition on $(\widetilde{v}, \lambda, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\zeta}, \widetilde{\beta}, \widetilde{\xi}, \bar{t})$ and (B.14), we get $\widetilde{\alpha}(\|y_t\|) \leqslant \widetilde{\zeta}(\widehat{v}_t + 1) + \widetilde{\xi}(d_\infty)$ for $t \geqslant \max\{\bar{t}; T_{x_0,d}\}$. Hence, from (B.15) and (B.17), for $t \geqslant \max\{\bar{t}; T_{x_0,d}\}$ (and since $\widehat{v}_t \geqslant 0$ for all $t \geqslant 0$) we obtain after few passages

$$\frac{|\dot{s}_t|}{\gamma_F(s_t)} \leqslant \min\left\{\delta_0, \frac{\mu}{2(1+4|\overline{\gamma}_d|)}\right\}. \qquad \text{(B.20)}$$

B.4.4 Estimation error convergence

Since Λ_s has generator W, mapping the system's solutions x_t into $\widetilde{x}_t := \Lambda_{s_t}(x_t)$ and rescaling the disturbances d_t as $\widetilde{d}_t := \Gamma_d(s_t, \widetilde{x}_t)d_t$, we get for all $t \geqslant 0$

$$\dot{\widetilde{x}}_t = \Lambda_{s_t *}F(\widetilde{x}_t, \Gamma_d^{-1}(s_t, \widetilde{x}_t)\widetilde{d}_t) + W(\widetilde{x}_t)\dot{s}_t,$$

and finally, using (B.19) and definitions (B.11), for all $t \geqslant T_{x_0,d}$ and with $\delta_t := \frac{\dot{s}_t}{\gamma_F(s_t)}$

$$\dot{\widetilde{x}}_t = \gamma_F(s_t)\Big\{\frac{\partial F_\infty}{\partial x}\Big|_{(0,0)}\widetilde{x}_t + A_{s_t}(\widetilde{x}_t; \delta_t) + P_{s_t}(\widetilde{x}_t; \widetilde{d}_t)\Big\}.$$

Notice that on account of Assumption 2 and 3 and (B.19)

$$\|\widetilde{d}_t\| \leqslant \|\Gamma_d(s_t, \widetilde{x}_t)\| d_\infty \leqslant \gamma_d(\|\widetilde{x}_t\|, s_t) d_\infty \leqslant \gamma_d(nc, 0) d_\infty. \tag{B.21}$$

On the other hand, from the observer's equations (31) with definitions (B.11) we get

$$\begin{aligned}\dot{\widetilde{z}}_t = \gamma_F(s_t)\Big\{ & \frac{\partial F_\infty}{\partial x}\Big|_{(0,0)} \widetilde{z}_t + A_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t); \delta_t) \\ & + \\ & + K \frac{\partial H_\infty}{\partial x}\Big|_{(0,0)} (\widetilde{x}_t - \widetilde{z}_t) \\ & + K\Big(C_{s_t}(\widetilde{x}_t) - C_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t)) + Q_{s_t}(\widetilde{x}_t; \widetilde{d}_t)\Big)\Big\}.\end{aligned}$$

Let $\widetilde{e} := \widetilde{x} - \widetilde{z}$, the estimation error, and $V(\widetilde{e}) := \widetilde{e}^\top \Pi \widetilde{e}$. Using (B.20), (B.21), the fact that $\widehat{s}(0) \geqslant s_0$ (since $q > e^{\frac{s_0 h}{p}}$) and (B.9) and (B.10) with (B.11), we get in the end for all $t \geqslant \max\{\overline{t}; T_{x_0,d}\}$

$$\begin{aligned}\dot{V} \leqslant & \gamma_F(s_t)\Big\{ -\mu V + \overline{A}_{s_t}(\widetilde{x}_t, \widetilde{z}_t; \delta_t)\|\widetilde{e}\|^2 \\ & + \frac{4\lambda^\Pi_{max}}{\mu} \overline{P}^2_{s_t}(\widetilde{x}_t; \widetilde{d}_t)\|\widetilde{d}_t\|^2 \Big\} \leqslant \gamma_F(s_t)\Big\{ -\frac{\mu}{2} V \\ & + \Big(\frac{\varepsilon^2 \lambda^\Pi_{min} \mu}{4\omega^2(nc)} + \frac{8\|D\|^2 \lambda^\Pi_{max} d^2_\infty}{\mu} \Big) \gamma^2_d(nc, s_t) \Big\}\end{aligned}$$

With $\widetilde{V} := \frac{V}{\gamma^2_d(nc,s_t)}$ and once more on account of (B.20) and since $\gamma_d(r,s) = \alpha_d(r) e^{\overline{\gamma}_d s}$ (Assumption 2), we find

$$\dot{\widetilde{V}} \leqslant \gamma_F(s_t)\Big\{ -\frac{\mu}{4}\widetilde{V} + \Big(\frac{\varepsilon^2 \lambda^\Pi_{min} \mu}{4\omega^2(nc)} + \frac{8\|D\|^2 \lambda^\Pi_{max} d^2_\infty}{\mu} \Big) \Big\}$$

for all $t \geqslant \max\{\overline{t}; T_{x_0,d}\}$. Since $\lambda^\Pi_{min}\|\widetilde{e}\|^2 \leqslant V(\widetilde{e}) \leqslant \lambda^\Pi_{max}\|\widetilde{e}\|^2$ for all $\widetilde{e}$, it follows that for all $t \geqslant t^* := \max\{\overline{t}; T_{x_0,d}\}$

$$\begin{aligned}\|\widetilde{x}_t - \widetilde{z}_t\| \leqslant & \Big[\frac{\sqrt{\lambda^\Pi_{max}} e^{-\frac{\mu}{4}(t-t^*)}}{\sqrt{\lambda^\Pi_{min}} \gamma_d(nc, s_{t^*})} \|\widetilde{x}_{t^*} - \widetilde{z}_{t^*}\| \\ & + \frac{\varepsilon}{\omega(nc)} + \frac{6\|D\|\sqrt{\lambda^\Pi_{max}} d_\infty}{\mu\sqrt{\lambda^\Pi_{min}}} \Big] \gamma_d(nc, s_t).\end{aligned} \tag{B.22}$$

On the other hand, on account of (B.3) and (B.19), the (IR) property (11) of Λ_s and the mean value theorem, we get for $t \geqslant \max\{\overline{t}; T_{x_0,d}\}$

$$\begin{aligned}& \|x_t - \Lambda^{-1}_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t))\| = \|\Lambda^{-1}_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{x}_t)) - \Lambda^{-1}_{s_t}(\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t))\| \\ & \leqslant \int_0^1 \Big\| \frac{\partial \Lambda^{-1}_{s_t}}{\partial x}\Big|_{\theta\sigma_{\widehat{c}(s_t)}(\widetilde{x}_t) + (1-\theta)\sigma_{\widehat{c}(s_t)}(\widetilde{z}_t)} \Big\| d\theta \times \\ & \times \|\sigma_{\widehat{c}(s_t)}(\widetilde{x}_t) - \sigma_{\widehat{c}(s_t)}(\widetilde{z}_t)\| \leqslant \rho(nc, s_t)\|\widetilde{x}_t - \widetilde{z}_t\|.\end{aligned} \tag{B.23}$$

On account of (B.22) and $\rho(r,s)\gamma_d(r,s) \leqslant \omega(r)$ for all $s, r \geqslant 0$ (Assumption 3), by applying $\limsup_{t\to+\infty}$ to the left and right part of (B.23) we prove the claim of the theorem. ◁

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