Stabilization via generalized homogeneous approximations

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***Abstract*—We introduce a notion of generalized homogeneous approximation at the origin and at infinity which extends the classical notions and captures a large class of nonlinear systems, including (lower and upper) triangular systems. Exploiting this extension and although this extension does not preserve the basic properties of the classical notion, we give basic results concerning stabilization and robustness of nonlinear systems, by designing a homogeneous (in the generalized sense) feedback controller which globally asymptotically stabilizes a chain of power integrators and makes it the dominant part at infinity and at the origin (in the generalized sense) of the dynamics. Stability against nonlinear perturbation follows from domination arguments.**

I. Introduction

The problem of designing stabilizing feedback control laws for nonlinear systems has been addressed by many authors with different approaches. Many of these use domination tools and robustness. In a domination approach the stability of a system $\dot{x} = f(x) + g(x)u + \phi(x)$ is ensured by designing a stabilizing feedback controller for $\dot{x} = f(x) + g(x)u$ provided that the stability property of the closed-loop system is robust with respect to the perturbation $\phi(x)$. This domination idea has been largely exploited by employing homogeneous feedback controllers with homogeneous systems $\dot{x} = f(x) + g(x)u$ ([14], [4], [7], [8], [9], [12], [13]). The idea of extending this approach to systems, which are not homogeneous but admit a homogeneous approximation as the state approaches the origin, is pursued in [15], [3], [14] and [4]. Firstly, conceptually in [15] and, more recently, in [2] this domination technique has been extended to systems which are not homogenous but become homogeneous as the state approaches either the origin or infinity but with different weights and degrees (homogeneous approximation in the bi-limit). However, the class of homogenous (or with a homogeneous approximation) vector fields is limited even for simple structures like triangular ones: for example $f(x) = x_2 \frac{\partial}{\partial x_1} + (c_0 x_2^q + c_\infty x_2^p)\frac{\partial}{\partial x_2}$, $0 < q < p$, admits a homogeneous approximation at the origin and at infinity only when $p < 2$. Homogeneity (in the bi-limit) in the classical sense of a given vector field $f(x)$ is characterized by some degree $\mathfrak{d}$ and weights $\mathfrak{r}$. In this paper, we introduce a generalized notion of homogeneity in the bi-limit, which significantly enlarges the class of homogeneous vector field (in the bi-limit in the classical sense). The difference is that a generalized homogeneous vector field $f(x)$ is characterized by some *vector* degree $\mathfrak{d}$ and weights $\mathfrak{r}$. Each degree $\mathfrak{d}_i$ of $f(x)$ is the homogeneity degree of the component $f_i(x)$ and each degree may be different from the other. For example, this generalization allows to establish that $f(x) = x_2 \frac{\partial}{\partial x_1} + (c_0 x_2^q + c_\infty x_2^p)\frac{\partial}{\partial x_2}$, is homogeneous in the bi-limit in the generalized sense whatever p and q are such that $0 < q < p$. A notion of generalized homogeneity with monotone degrees (i.e. the degrees are different but form a non-decreasing non-negative sequence) was previously introduced in [10] (see also [12] and [13]), while in this paper we consider the more general framework of homogeneous approximations with non-monotone degrees. Homogeneity in a generalized sense allows to establish useful (global and local) stability and robustness properties as homogeneity in the classical sense does ([14], [4], [7], [2]), although in this generalized framework we lose some nice properties of the trajectories associated with a homogeneous (in the classical sense) vector field X such as scalability or the existence of a homogeneous Lyapunov function for a globally asymptotically stable homogeneous vector field ([14]). However, a key observation shows how homogeneity in the generalized sense is still helpful in enlarging the class of vector fields which can be stabilized by (homogeneous) feedback: for a given vector field f, which is homogeneous in the generalized sense in the bi-limit, if we find a Lyapunov function V with homogeneous in the bi-limit partial derivatives (therefore the vector of partial derivatives $(\frac{\partial V}{\partial x})^T$ is homogeneous in the bi-limit in our generalized sense), then $\frac{\partial V}{\partial x} f$ is homogeneous in the bi-limit in the classical sense and retains the classical nice properties such as scalability. As a consequence, we will use our homogeneity in the generalized sense to design a homogeneous (in the bi-limit) stabilizing state-feedback, together with a Lyapunov function with a homogeneous (in the generalized sense) in the bi-limit derivative, for a chain of power integrators and then consider this dynamics as the dominant dynamics near the origin and infinity of a chain of power integrators with nonlinear perturbations. By the domination approach we establish the global asymptotic stability of the chain of power integrators with nonlinear perturbations. In [12] and [9] local homogeneous (with monotone degrees) stabilizers are designed for a chain of power integrators (with non-decreasing powers) perturbed by a class of homogeneous upper-triangular vector fields and then nested saturations are used to obtain a global stabilizers (but the resulting stabilizer is no more homogeneous). In [7] and [8] homogeneous (with monotone degrees) stabilizers are designed for a chain of power integrators (with non-decreasing powers) perturbed by a class of homogeneous lower-triangular vector fields.

The paper is organized as follows. In section III we define generalized homogeneity in the bi-limit with some examples. In section IV a stabilizing homogeneous in the bi-limit feedback $\alpha(x)$ is designed for a chain of power integrators together with a positive definite and radially unbounded function V with homogeneous in the bi-limit (in the generalized sense) derivatives. In section 5.1 we give stabilization results of a chain of power integrators perturbed by homogeneous in

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the bi-limit (in the generalized sense) vector fields using robustness and domination strategies.

II. Notation

- $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$) is the set of n-dimensional real column vectors (resp. $n\times n$ matrices). $\mathbb{R}_{\geqslant}$ denotes the set of real non-negative numbers with real non-negative entries) and $\mathbb{R}_{>}$ (resp. $\mathbb{R}^n_{>}$) denotes the set of real positive numbers (resp. vectors in $\mathbb{R}^n$ with real positive entries).
- For any $G\in\mathbb{R}^{p\times n}$ we denote by G_{ij} the (i,j)–th entry of G and for any $G\in\mathbb{R}^p$ by G_i the i–th element of G. We retain a similar notation for functions.
- Let $\delta_{\mathfrak{d},\mathfrak{h}}$ denote the Kronecker delta
$$(\mathfrak{d},\mathfrak{h})\in\mathbb{R}\times\mathbb{R}\mapsto\delta_{\mathfrak{d},\mathfrak{h}}=\begin{cases}1 \text{ if } \mathfrak{d}=\mathfrak{h}\\ 0 \text{ otherwise}\end{cases}\tag{1}$$
Moreover, by $\mathbf{1}_n$ we denote the vector $(1,\ldots,1)^T\in\mathbb{R}^n$.
- For any real $\mathfrak{r}>0$ we define $w\mapsto w^{\mathfrak{r}}$ as $w^{\mathfrak{r}}=\operatorname{sgn}\{w\}|w|^{\mathfrak{r}}$. Notice that
$$\begin{aligned}&w_1>w_2\;\Rightarrow w_1^{\mathfrak{r}}>w_2^{\mathfrak{r}},\\ &\frac{dw^{\mathfrak{r}}}{dw}=\mathfrak{r}|w|^{\mathfrak{r}-1}\;(\mathfrak{r}\geqslant 1),\;\frac{d|w|^{\mathfrak{r}}}{dw}=\mathfrak{r}w^{\mathfrak{r}-1}\;(\mathfrak{r}>1).\end{aligned}$$
- The *dilation* of a vector $x\in\mathbb{R}^n$ with weights $\mathfrak{r}\in\mathbb{R}^n$ and parameter $\varepsilon>0$ is defined as
$$\varepsilon^{\mathfrak{r}}\diamond x:=(\varepsilon^{\mathfrak{r}_1}x_1,\cdots,\varepsilon^{\mathfrak{r}_n}x_n)^T$$
We also use the notation $x^q:=(x_1^{q_1},\cdots,x_n^{q_n})^T$ for $x,q\in\mathbb{R}^n$.

III. Homogeneity and homogeneous approximations in the generalized sense

A. Definitions

In this section we introduce the notion of homogeneity (in a generalized sense) in the bi-limit . The following definitions generalize the notion of homogeneous approximation of a function or vector field at infinity and around zero ([14], [2]) and go beyond the definition introduced in [10], in the sense that a vector field may be not homogeneous but it is so either near or far away from the origin.

Definition *3.1:* (Homogeneity (in the generalized sense) in the ∞-limit). *A continuous function $\phi:\mathbb{R}^n\to\mathbb{R}$ is said to be homogeneous in the ∞-limit with triple $(\mathfrak{r}_\infty,\mathfrak{d}_\infty,\phi_\infty)$ (weights $\mathfrak{r}_\infty\in\mathbb{R}^n_{>}$, degree $\mathfrak{d}_\infty\in\mathbb{R}$ and approximating function $\phi_\infty:\mathbb{R}^n\to\mathbb{R}$) if ϕ_∞ is continuous and for each $\lambda_\infty>0$ and compact set $\mathfrak{C}\subset\mathbb{R}^n\backslash\{0\}$ there exists $\varepsilon_\infty>0$ for which*
$$\max_{x\in\mathfrak{C}}\left\|\frac{\phi(\varepsilon^{\mathfrak{r}_\infty}\diamond x)}{\varepsilon^{\mathfrak{d}_\infty}}-\phi_\infty(x)\right\|\leqslant\lambda_\infty,\;\forall\varepsilon\geqslant\varepsilon_\infty.\tag{2}$$
A vector field $\phi=\sum_{i=1}^n\phi_i\frac{\partial}{\partial x_i}$ is said to be homogeneous (in the generalized sense) in the ∞-limit with triple $(\mathfrak{r}_\infty,\mathfrak{d}_\infty,\phi_\infty)$ (weights $\mathfrak{r}_\infty\in\mathbb{R}^n_{>}$, degrees $\mathfrak{d}_\infty\in\mathbb{R}^n$ and approximating vector field $\phi_\infty:=\sum_{i=1}^n\phi_{\infty,i}\frac{\partial}{\partial x_i}$) if ϕ_∞ is continuous and for each $i\in[1,n]$ ϕ_i is homogeneous in the ∞-limit with triple $(\mathfrak{r}_\infty,\mathfrak{r}_{\infty,i}+\mathfrak{d}_{\infty,i},\phi_{\infty,i})$.

Definition *3.2:* (Homogeneity (in the generalized sense) in the 0-limit). *A continuous function $\phi:\mathbb{R}^n\to\mathbb{R}$ is said to be homogeneous in the 0-limit with triple $(\mathfrak{r}_0,\mathfrak{d}_0,\phi_0)$ (weights $\mathfrak{r}_0\in\mathbb{R}^n_{>}$, degree $\mathfrak{d}_0\in\mathbb{R}$ and approximating function $\phi_0:\mathbb{R}^n\to\mathbb{R}$) if ϕ_0 is continuous and for each $\lambda_0>0$ and compact set $\mathfrak{C}\subset\mathbb{R}^n\backslash\{0\}$ there exists $\varepsilon_0>0$ for which*
$$\max_{x\in\mathfrak{C}}\left\|\frac{\phi(\varepsilon^{\mathfrak{r}_0}\diamond x)}{\varepsilon^{\mathfrak{d}_0}}-\phi_0(x)\right\|\leqslant\lambda_0,\;\forall\varepsilon\in(0,\varepsilon_0].\tag{3}$$
A vector field $\phi=\sum_{i=1}^n\phi_i\frac{\partial}{\partial x_i}$ is said to be homogeneous (in the generalized sense) in the 0-limit with triple $(\mathfrak{r}_0,\mathfrak{d}_0,\phi_0)$ (weights $\mathfrak{r}_0\in\mathbb{R}^n_{>}$, degrees $\mathfrak{d}_0\in\mathbb{R}^n$ and approximating vector field $\phi_0:=\sum_{i=1}^n\phi_{0,i}\frac{\partial}{\partial x_i}$) if ϕ_0 is continuous and for each $i\in[1,n]$ ϕ_i is homogeneous in the 0-limit with triple $(\mathfrak{r}_0,\mathfrak{r}_{0,i}+\mathfrak{d}_{0,i},\phi_{0,i})$.

A function (or a vector field) ϕ is said to be homogeneous (in the generalized sense) in the bi-limit with triples $(\mathfrak{r}_p,\mathfrak{d}_p,\phi_p)$, $p\in\{0,\infty\}$, if it is homogeneous (in the generalized sense) in the ∞-limit with triple $(\mathfrak{r}_\infty,\mathfrak{d}_\infty,\phi_\infty)$ and homogeneous (in the generalized sense) in the 0-limit with triple $(\mathfrak{r}_0,\mathfrak{d}_0,\phi_0)$.

Remark *3.1:* The definition of homogeneity (in the generalized sense) in the bi-limit generalizes the definition of homogeneity (in the classical sense: [15] and, more recently, [2]) in the bi-limit as follows: ϕ is homogeneous (in the generalized sense) in the bi-limit if each ϕ_i is homogeneous (in the classical sense) in the bi-limit and the homogeneity degrees of each ϕ_i are possibly different one from each other. In particular, for functions our generalized homogeneity in the bi-limit coincides with the classical one. Note also that it may happen that the homogeneity degrees at zero are monotonically decreasing (resp. increasing) while the homogeneity degrees at infinity are monotonically increasing (resp. decreasing), therefore in our generalized notion the degrees cannot be considered (after a renaming of the state variables) monotonically decreasing or increasing and a generalized notion with monotone degrees turns out to be restrictive. □

Remark *3.2:* A vector field ϕ homogeneous (in the generalized sense) in the bi-limit with triple $(\mathfrak{r}_p,\mathfrak{d}_p,\phi_p)$ is also homogeneous (in the generalized sense) in the bi-limit with triple $(k\mathfrak{r}_p,k\mathfrak{d}_p,\phi_p)$ for any $k>0$. □

Example *3.1:* The vector field
$$\phi(x):=x_2\frac{\partial}{\partial x_1}+(\sum_{p\in\{0,\infty\}}\sum_{j=1}^2 c_{p,j}x_j^{h_{p,j}})\frac{\partial}{\partial x_2}\tag{4}$$
$c_{p,1},c_{p,2}\in\mathbb{R}$, $p\in\{0,\infty\}$, $0<h_{0,2}<h_{\infty,2}$ and $0<h_{0,1}<h_{\infty,1}$, is homogeneous in the bi-limit in the classical sense only if $h_{p,2}<2$ and $h_{p,1}=\frac{h_{p,2}}{2-h_{p,2}}$, $p\in\{0,\infty\}$. Notice also that ϕ is not homogeneous with monotone degrees in the sense of [10] for the presence of non-homogeneous terms $c_{p,1}x_1^{h_{p,1}}+c_{p,2}x_2^{h_{p,2}}$, $p\in\{0,\infty\}$. However, ϕ is homogeneous (in the generalized sense) in the bi-limit with triples
$$\begin{aligned}(\mathfrak{r}_p,\mathfrak{d}_p,\phi_p):=\Big(&(1,\frac{h_{p,1}}{h_{p,2}})^T,(\frac{h_{p,1}}{h_{p,2}}-1,h_{p,1}-\frac{h_{p,1}}{h_{p,2}})^T,\\ &x_2\frac{\partial}{\partial x_1}+(\sum_{j=1}^2 c_{p,j}x_j^{h_{p,j}})\frac{\partial}{\partial x_2}\Big),\end{aligned}\tag{5}$$
$p\in\{0,\infty\}$, whatever $0<h_{0,2}<h_{\infty,2}$ and $0<h_{0,1}<h_{\infty,1}$ are. □

Remark *3.3:* By introducing homogeneity (in the generalized sense) in the bi-limit we significantly enlarge the class of homogeneous (in the classical sense) in the bi-limit vector fields. For example, consider the lower triangular vector field (for $n \geqslant 1$)

$$g(x) \coloneqq \sum_{j=1}^{n} g_j(x_1, x_2, \ldots, x_j)\frac{\partial}{\partial x_j} \qquad (6)$$

where each g_j is a multi-power function, i.e. a sum of terms of the form $\prod_{i\in\{1,\ldots,j\}} x_i^{h_{j,i}}$ for some $h_{j,i} \in \mathbb{R}_{\geqslant}$. It is possible to show that (the proof is omitted for lack of space) there exists a non-decreasing sequence $\{\mathfrak{d}_{\infty,j}\}_{j\in[1,n]}$ such that (6) is homogeneous (in the generalized sense) in the ∞-limit with triple $(\mathfrak{r}_\infty, \mathfrak{d}_\infty, g_\infty)$, weigths $\mathfrak{r}_{\infty,1} > 0$, $\mathfrak{r}_{\infty,j} := \frac{\mathfrak{r}_{\infty,j-1}+\mathfrak{d}_{\infty,j-1}}{q_j}$, $j \in [2, n]$. Therefore, $g(x)$ can be dominated at infinity by a given chain of n integrators $\sum_{j=1}^{n-1} x_{j+1}\frac{\partial}{\partial x_j} + u\frac{\partial}{\partial x_n}$ (with weights $\mathfrak{r}$ for x and weight $\mathfrak{s} = \mathfrak{r}_{\infty,n} + \mathfrak{d}_{\infty,n}$ for u), u being the control input. On the contrary, we note that for the vector field $g(x) \coloneqq x_1^{\frac{1}{4}}\frac{\partial}{\partial x_1} + (x_1^{\frac{3}{2}} + x_2^3)\frac{\partial}{\partial x_2}$ there is no non-increasing sequence $\{\mathfrak{d}_{0,j}\}_{j\in[1,n]}$ such that (6) is homogeneous (in the generalized sense) in the 0-limit. Under this regard, it is possible to show that, if we add some restriction on (6) around the origin (and we do not discuss this here for lack of space), there exists a non-increasing sequence $\{\mathfrak{d}_{0,j}\}_{j\in[1,n]}$ such that (6) is homogeneous (in the generalized sense) in the 0-limit with triple $(\mathfrak{r}_0, \mathfrak{d}_0, g_0)$, weigths $\mathfrak{r}_{0,1} > 0$, $\mathfrak{r}_{0,j} := \frac{\mathfrak{r}_{0,j-1}+\mathfrak{d}_{0,j-1}}{q_j}$, $j \in [2, n]$. Non-decreasing degrees at infinity and non-increasing degrees at 0 give a favorable situation for constructing stabilizing feedback laws (see section IV).

To further motivate our generalized notion, consider the upper triangular vector field (for $n \geqslant 3$)

$$g(x) \coloneqq \sum_{j=1}^{n-2} g_j(x_{j+2}, \ldots, x_n)\frac{\partial}{\partial x_j} \qquad (7)$$

where each g_j is a multi-power function. It is possible to show that (the proof is omitted for lack of space) there exists a non-decreasing sequence $\{\mathfrak{d}_{\infty,j}\}_{j\in[1,n]}$ such that (7) is homogeneous (in the generalized sense) in the ∞-limit with triple $(\mathfrak{r}_\infty, \mathfrak{d}_\infty, g_\infty)$, weigths $\mathfrak{r}_{\infty,j} := q_{j+1}\mathfrak{r}_{\infty,j+1} - \mathfrak{d}_{\infty,j}$, $j \in [1, n-1]$. Therefore, $g(x)$ can be dominated at infinity by a given chain of n integrators $\sum_{j=1}^{n-1} x_{j+1}\frac{\partial}{\partial x_j} + u\frac{\partial}{\partial x_n}$ (with weights $\mathfrak{r}$ for x and weight $\mathfrak{s} = \mathfrak{r}_{\infty,n} + \mathfrak{d}_{\infty,n}$ for u), u being the control input. On the contrary, we note that for the vector field $g(x) \coloneqq x_3^{\frac{1}{4}}\frac{\partial}{\partial x_1}$ there is no non-increasing sequence $\{\mathfrak{d}_{0,j}\}_{j\in[1,n]}$ such that (7) is homogeneous (in the generalized sense) in the 0-limit. Under this regard, it is possible to show that, if we add some restriction on (7) around the origin (and we do not discuss this here for lack of space), there exists a non-increasing sequence $\{\mathfrak{d}_{0,j}\}_{j\in[1,n]}$ such that (7) is homogeneous (in the generalized sense) in the 0-limit with triple $(\mathfrak{r}_0, \mathfrak{d}_0, g_0)$, weigths $\mathfrak{r}_{0,j} := q_{j+1}\mathfrak{r}_{0,j+1} - \mathfrak{d}_{0,j}$, $j \in [1, n-1]$.

Although in our generalized framework we lose nice properties of the trajectories associated with a homogeneous in the classical sense vector field X (such as scalability), however we are still able to convey the properties of generalized homogeneity to achieve global stabilization for large classes of vector fields. The aim of this paper is to prove that homogeneity in the bi-limit (in the generalized sense), rather than being a "generalization" of the classical corresponding notion by retaining most of its properties (we use the terminology 'in the generalized sense" only for distinguishing our notion from homogeneity in the classical sense), it is a useful stabilization tool and significantly enlarges the class of vector fields which can be globally stabilized by feedback using classical homogeneity tools.

IV. Homogeneous state feedback design for a chain of power integrators

The notion of homogeneity (in the generalized sense) in the bi-limit is instrumental to adopt the following stabilization method. First, fix some stabilizable dynamics such as a chain of power integrators:

$$\dot{x} = Ax^q + Bu \qquad (8)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, (A, B) is in Brunowski form and $q \in \mathbb{R}^n_{>,\text{odd}}$. Find a feedback control law $u = \alpha(x)$ which globally stabilizes (8) and such that the resulting closed-loop vector field $\sum_{i=1}^{n-1} x_{i+1}^{q_{i+1}}\frac{\partial}{\partial x_i} + \alpha(x)\frac{\partial}{\partial x_n}$ associated to (8) is homogeneous (in the generalized sense) in the bi-limit with some weights and degrees. With this at hand, the stability of the closed-loop vector field is preserved under any nonlinear perturbation such that its degrees are dominated at the origin and at infinity by the degrees of the chain of power integrators. In order that $\sum_{i=1}^{n-1} x_{i+1}^{q_{i+1}}\frac{\partial}{\partial x_i} + \alpha(x)\frac{\partial}{\partial x_n}$ be homogeneous (in the generalized sense) in the bi-limit with degrees $\mathfrak{d}_p$ and weights $\mathfrak{r}_p$, $p \in \{0, \infty\}$, no matter what α is, the vectors $\mathfrak{d}_p$ and $\mathfrak{r}_p$ must satisfy

$$\mathfrak{r}_{p,j} = \frac{\mathfrak{r}_{p,j-1} + \mathfrak{d}_{p,j-1}}{q_j},\ j \in [2, n]. \qquad (9)$$

Moreover, we assume that $\{\mathfrak{d}_{\infty,j}\}_{j\in[1,n]}$ is a non-decreasing sequence (resp. $\{\mathfrak{d}_{0,j}\}_{j\in[1,n]}$ is non-increasing) and

$$\mathfrak{r}_{p,i} + \mathfrak{d}_{p,i} > 0,\ \forall i \in [1, n],\ p \in \{0, \infty\} \qquad (10)$$

(for each $i \in [1, n-1]$ this is a necessary condition for being $\mathfrak{r}_{p,i+1} > 0$ on account of (9)).

It is known that a chain of integrators can be rendered homogeneous (in the classical sense) in the bi-limit by means of a stabilizing homogeneous in the bi-limit state feedback ([2]). In this section we show that this property can be extended to the case of homogeneity (in the generalized sense) in the bi-limit and for the more general class of chains of power integrators (8). More precisely, in subsection IV-A we show that there exists a homogeneous (in the generalized sense) in the bi-limit function $\alpha(x)$ such that the vector field $f(x) := Ax^q + B\alpha(x)$ (associated to the closed-loop system (8) with $u = \alpha(x)$) is homogeneous (in the generalized sense) in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{d}_p, f_p)$, $p \in \{0, \infty\}$, and $\dot{x} := f(x)$ is globally asymptotically stable. We will prove this stability property in subsection IV-B by defining a positive definite and radially unbounded function V for the closed-loop system $\dot{x} := f(x)$ such that $\left(\frac{\partial V}{\partial x}\right)^T$, $j \in [1, n]$, is homogeneous in the bi-limit (in the generalized sense) with triple $(\mathfrak{r}_p, \mathfrak{w}_p - \mathfrak{r}_{p,j} - \mathfrak{d}_{p,j}, \left(\frac{\partial V}{\partial x}\right)_p^T)$, $p \in \{0, \infty\}$, for a suitably large $\mathfrak{w}_p > 0$. In this way, the product $(\frac{\partial V}{\partial x} f)(x)$ is homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{w}_p, \sum_{j=1}^n \left(\frac{\partial V}{\partial x_j}\right)_p f_{p,j})$, $p \in \{0, \infty\}$. This guarantees robustness properties against nonlinear perturbations dominated at infinity and at the origin by f (see section 5.1). Our constructive procedure provides new types of controllers even in the case of homogeneity in the bi-limit in the classical sense.

A. Definition of the feedback law

Let $\mathfrak{X}_j := (x_1, \ldots, x_j)$, $j \in [1, n]$, and $q_1 := 1$. Define the following function

$$\begin{aligned} &\alpha(x) := \alpha_n(\mathfrak{X}_n), \\ &\alpha_j(\mathfrak{X}_j) := \mathfrak{J}^{\frac{1}{\mathfrak{a}_{j+1}}}(\|\zeta_j(\mathfrak{X}_j)\|^{\mathfrak{m}}_{\mathfrak{R}_{0,j}}, \alpha_j^{(\infty)}(\mathfrak{X}_j), \alpha_j^{(0)}(\mathfrak{X}_j)), \\ &j \in [1, n],\ \alpha_0 := 0, \end{aligned} \tag{11}$$

with $\mathfrak{J}$ defined in (38), $\|\cdot\|_{\mathfrak{R}_{0,j}}$ the weighted homogeneous norm (with weights $\mathfrak{R}_{0,j}$),

$$\begin{aligned} &\alpha_j^{(p)}(\mathfrak{X}_j) := -\gamma_j^{\mathfrak{a}_{j+1}}\left(x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}(\mathfrak{X}_{j-1})\right)^{\frac{\mathfrak{a}_{j+1}(\mathfrak{r}_{p,j}+\mathfrak{d}_{p,j})}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}, \\ &\zeta_j(\mathfrak{X}_j) := \sum_{i=1}^j |x_i^{\mathfrak{a}_i q_i} - \alpha_{i-1}^{\mathfrak{a}_i}(\mathfrak{X}_{i-1})|^{\frac{1}{\mathfrak{a}_i q_i}} \frac{\partial}{\partial x_i}, \end{aligned} \tag{12}$$

and for some $\mathfrak{a}_1, \ldots, \mathfrak{a}_{n+1}, \mathfrak{m} \geqslant 1$ such that

$$\mathfrak{a}_j \geqslant \max_{k \in [1,j]} \frac{\mathfrak{r}_{0,k}}{q_j \mathfrak{r}_{0,j}},\ j \in [1, n] \tag{13}$$

$$\mathfrak{a}_{j+1} \geqslant \frac{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}{\mathfrak{r}_{p,j} + \mathfrak{d}_{p,j}},\ j \in [1, n], \tag{14}$$

$$\mathfrak{m} \geqslant 1 + \mathfrak{a}_j q_j \mathfrak{r}_{p,j},\ j \in [1, n], \tag{15}$$

and $\gamma_1, \ldots, \gamma_n \geqslant 1$ positive reals to be specified later in the design. Conditions (13)-(15) are needed for guaranteeing suitable smoothness properties on each α_j. The function α_j is designed, according to remark 7.1, by combining $\alpha_j^{(0)}$ and $\alpha_j^{(\infty)}$ which, as we will see, are homogeneous in the 0-limit and, respectively, in the ∞-limit. This combination procedure is different from how homogeneous in the 0-limit and ∞-limit functions are combined in [2]. The exponential decay of $\mathfrak{J}$ with respect to the first argument is specifically designed to guarantee the homogeneity in the bi-limit of the derivatives of $\mathfrak{J}$. Our first result is to prove that α is homogeneous in the bi-limit.

Proposition *4.1: The vector field $f(x) := \sum_{i=1}^{n-1} x_{i+1}^{q_{i+1}} \frac{\partial}{\partial x_i} + \alpha(x) \frac{\partial}{\partial x_n}$ is continuous over $\mathbb{R}^n$. Moreover, it is homogeneous (in the generalized sense) in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{d}_p, f_p)$, $p \in \{0, \infty\}$, where f_p and α_p are defined as follows*

$$f_p(x) := \sum_{i=1}^{n-1} x_{i+1}^{q_{i+1}} \frac{\partial}{\partial x_i} + \alpha_p(x) \frac{\partial}{\partial x_n},\ \alpha_p(x) := \alpha_{p,n}(\mathfrak{X}_n)$$

$$\begin{aligned} &\alpha_{p,0} := 0,\ \alpha_{p,j}(\mathfrak{X}_j) := -\gamma_j\left(x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}(\mathfrak{X}_j)\right)^{\frac{\mathfrak{r}_{p,j}+\mathfrak{d}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}, \\ &j \in [1, n]. \end{aligned} \tag{16}$$

Proof *4.1:* On account of remark 3.2, we can assume without loss of generality that for each $p \in \{0, \infty\}$

$$1 > \mathfrak{a}_i q_i \mathfrak{r}_{p,i},\ i \in [1, n]. \tag{17}$$

together with (13)-(15). From (10) it follows that α is continuous over $\mathbb{R}^n$. Therefore, f is continuous over $\mathbb{R}^n$, which proves the first part of the proposition. Each f_i, $i = 1, \ldots, n-1$, is homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{r}_{p,i} + \mathfrak{d}_{p,i}, f_i)$, $p \in \{0, \infty\}$, as a consequence of (9). Application of properties P1-P3 and lemma 2.10 of [2] together with proposition 7.1 (remark 7.1) gives that α homogeneous (in the generalized sense) in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{r}_{p,n} + \mathfrak{d}_{p,n}, f_p)$, $p \in \{0, \infty\}$. This proves the second part of the proposition. □

Remark *4.1:* If $\frac{\mathfrak{a}_{j+1}(\mathfrak{r}_{\infty,j}+\mathfrak{d}_{\infty,j})}{\mathfrak{a}_j q_j \mathfrak{r}_{\infty,j}} > \frac{\mathfrak{a}_{j+1}(\mathfrak{r}_{0,j}+\mathfrak{d}_{0,j})}{\mathfrak{a}_j q_j \mathfrak{r}_{0,j}}$ for some $j \in [1, n]$, then the design of α_j in (11)) can be simplified as follows: $\alpha_j := \alpha_j^{(0)} + \alpha_j^{(\infty)}$. □

Example *4.1:* Let us consider the situation in which

$$\mathfrak{r}_\infty = (\frac{1}{4}, \frac{1}{4q_2})^T,\ \mathfrak{d}_\infty = (0, \frac{1}{4}(h_{\infty,1} + \frac{h_{\infty,2}}{q_2}))^T, \tag{18}$$

$$\mathfrak{r}_0 = (\frac{1}{2}, \frac{1}{2q_2})^T,\ \mathfrak{d}_0 = (0, \frac{1}{4}\min\left\{\min\{h_{0,1}, \frac{h_{0,2}}{q_2}\} - \frac{1}{2q_2}, 0\right\})^T$$

with $0 < h_{0,p} < h_{\infty,p}$, $p \in \{0, \infty\}$. Clearly, (9)-(10) are satisfied and $\{\mathfrak{d}_{\infty,j}\}_{j \in [1,2]}$ is non-decreasing and $\{\mathfrak{d}_{0,j}\}_{j \in [1,2]}$ is non-increasing. In this situation, the chain of power integrators $x_2^{q_2} \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial x_2}$ (with $u = \alpha$ given in (11)) dominates at the origin and at infinity any vector field $g(x) := (\sum_{p \in \{0,\infty\}} \sum_{j=1}^2 c_{p,j} x_j^{h_{p,j}}) \frac{\partial}{\partial x_2}$. The function α is given in (11) with $\mathfrak{a}_2 := \frac{q_2+1}{q_2}$, $\mathfrak{a}_3 := (q_2 + 1)\max\{1, \frac{2}{\min\{q_2 h_{0,1}, h_{0,2}\}}\}$ and $\mathfrak{m} := 3$ (chosen in such a way to satisfy (13)-(15) and (17)).□

B. Construction of the Lyapunov function

Define the following numbers

$$\mathfrak{f}_{p,j} := \mathfrak{w}_p - \mathfrak{d}_{p,j} - \mathfrak{r}_{p,j} - \mathfrak{a}_j q_j \mathfrak{r}_{p,j},\ j \in [1, n],\ p \in \{0, \infty\}, \tag{19}$$

with $\mathfrak{w}_p > 0$ such that

$$\mathfrak{f}_{p,j} > \mathfrak{r}_{p,j},\ \mathfrak{f}_{p,j} \geqslant 1,\ j \in [1, n] \tag{20}$$

$$\mathfrak{f}_{\infty,j} + \mathfrak{a} q_j \mathfrak{r}_{\infty,j} > \mathfrak{f}_{0,j} + \mathfrak{a} q_j \mathfrak{r}_{0,j} > 0,\ j \in [1, n]. \tag{21}$$

Define the following function

$$\begin{aligned} &V(x) := V_n(\mathfrak{X}_n),\ V_0 := 0, \\ &V_j(\mathfrak{X}_j) := V_{j-1}(\mathfrak{X}_{j-1}) + \sum_{p \in \{0,\infty\}} V_j^{(p)}(\mathfrak{X}_j),\ j \in [1, n], \end{aligned} \tag{22}$$

where

$$\begin{aligned} V_j^{(p)}(\mathfrak{X}_j) &:= |x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{0,j}}} \cdot \\ &\cdot \int_{\alpha_{j-1}^{\frac{1}{q_j}}(\mathfrak{X}_{j-1})}^{x_j} [s^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}(\mathfrak{X}_{j-1})] ds,\ p \in \{0, \infty\}. \end{aligned} \tag{23}$$

Conditions (20)-(21) are needed for guaranteeing suitable smoothness properties on each V_j. The function V_j is designed by summing $V_j^{(0)}$ and $V_j^{(\infty)}$, which, as we will see, are homogeneous in the 0-limit and, respectively, in the ∞-limit, and using dominance at the origin and, respectively, at infinity as in [2]. On the other hand, the construction of (23) is somewhat different from the one in [2]: indeed, together with the integral term in (23), we use $|x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{1}{\mathfrak{a}_j q_j \mathfrak{r}_{0,j}}}$ as a rescaling homogeneous in the bi-limit factor. We have the following result on V and $\left(\frac{\partial V}{\partial x}\right)^T$ (the proof is given in the appendix).

Proposition *4.2: The function V is continuous, positive definite and radially unbounded over $\mathbb{R}^n$ and $\left(\frac{\partial V}{\partial x}\right)^T$ is continuous over $\mathbb{R}^n$ and homogeneous (in the generalized sense) in the bi-limit with triple $(\mathfrak{r}_p, \mathbf{1}_n \cdot \mathfrak{w}_p - \mathfrak{r}_p - \mathfrak{d}_p, \left(\frac{\partial V}{\partial x}\right)_p^T)$, $p \in \{0, \infty\}$, where each coordinate function $(\frac{\partial V}{\partial x})_{p,j}$ is defined as follows*

$$\begin{aligned}
(\frac{\partial V}{\partial x})_{p,k} \;&:=\; v_{p,k,n}, k \in [1,n],\\
v_{p,k,j} \;&:=\; \delta_{\mathfrak{d}_{p,k},\mathfrak{d}_{p,j}}\Big\{\Big(\frac{\partial}{\partial x_k}|x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}\Big)\cdot\\
&\cdot\; \int_{\alpha_{p,j-1}^{\frac{1}{q_j}}}^{x_j}\Big(s^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}\Big)ds\\
&+\; |x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}\Big[\delta_{j,k}\Big(x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}\Big)\\
&+\; (1-\delta_{j,k})\Big(x_j - \alpha_{p,j-1}^{\frac{1}{q_j}}\Big)w_{p,k,j-1}\Big]\Big\}\\
&+\; v_{p,k,j-1}\;, j \in [1,n],\; k \in [1,j],\\
v_{p,k,k-1} \;&:=\; 0,\; k \in [1,j],
\end{aligned} \quad (24)$$

and

$$\begin{aligned}
&w_{p,k,j} := -\gamma_j^{\mathfrak{a}_{j+1}}\frac{\mathfrak{a}_{j+1}(\mathfrak{r}_{p,j} + \mathfrak{d}_{p,j})}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}\cdot\\
&\cdot\Big|x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}\Big|^{\frac{\mathfrak{a}_{j+1}(\mathfrak{r}_{p,j}+\mathfrak{d}_{p,j})}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}-1}\Big[\delta_{j,k}\mathfrak{a}_j q_j |x_j|^{\mathfrak{a}_j q_j - 1}\\
&+(1-\delta_{j,k})w_{p,k,j-1}\Big],\; w_{p,k,k-1} := 0.
\end{aligned} \quad (25)$$

Example *4.2:* Let us go back to example (18). The function V is given, according to proposition 4.2, by $V := V_1^{(0)} + V_1^{(\infty)} + V_2^{(0)} + V_2^{(\infty)}$ where

$$\begin{aligned}
&V_2^{(p)} := \Big|x_2^{\mathfrak{a}_2 q_2} - \alpha_1^{\mathfrak{a}_2}\Big|^{\frac{\mathfrak{f}_{p,2}}{\mathfrak{a}_2 q_2 \mathfrak{r}_{p,2}}},\\
&\int_{\alpha_1^{\frac{1}{q_2}}}^{x_2}[s^{\mathfrak{a}_2 q_2} - \alpha_1^{\mathfrak{a}_2}]ds,\; V_1^{(p)} := |x_1|^{\frac{\mathfrak{f}_{p,1}}{\mathfrak{r}_{p,1}}+2},
\end{aligned} \quad (26)$$

and $\mathfrak{f}_{p,1} := \mathfrak{w}_p - \mathfrak{d}_{p,1} - 2\mathfrak{r}_{p,1}$, $\mathfrak{f}_{p,2} := \mathfrak{w}_p - \mathfrak{d}_{p,2} - \mathfrak{r}_{p,2}(1+\mathfrak{a}_2 q_2)$, with $\mathfrak{w}_0 := 1 + \frac{1}{q_2} + \min\{h_{0,1}, \frac{h_{0,2}}{q_2}\}(q_2+1)$ and $\mathfrak{w}_\infty := \mathfrak{w}_0 + 1 + \frac{1}{q_2} + (h_{\infty,1} + \frac{h_{\infty,2}}{q_2})(q_2+1)$ (chosen in such a way to satisfy (19)).□

As a second, step, we establish that the derivative of V along the trajectories of the closed-loop system (8) with $u = \alpha(x)$, where α is defined in (11), is negative definite for a suitable choice of $\gamma_1, \ldots, \gamma_n \geqslant 1$ (the proof follows by repeated application of propositions 4.1, 4.2 and properties P1-P3 of [2]).

Proposition *4.3: The function $\frac{\partial V}{\partial x}f$ is homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{w}_p, (\frac{\partial V}{\partial x}f)_p)$, $p \in \{0,\infty\}$, where $(\frac{\partial V}{\partial x}f)_p := \sum_{j=1}^n (\frac{\partial V}{\partial x_j})_p f_{p,j}$. Moreover, there exist $\gamma_1^*, \ldots, \gamma_n^* > 0$ such that*

$$(\frac{\partial V}{\partial x}f)(x) < 0\;\forall x \neq 0,\; (\frac{\partial V}{\partial x}f)_p(x) < 0,\;\; p \in \{0,\infty\}, \quad (27)$$

for all $x \neq 0$, for all $\gamma_j \geqslant \gamma_j^*$, $j \in [1,n]$.

Remark *4.2:* Consider, in place of (8), the system

$$\dot{\tilde{x}} = L^{\hat{\mathfrak{d}}} \diamond [A\tilde{x}^q + Bu] \quad (28)$$

with $L \geqslant 1$ (resp. $L \in (0,1]$) and $\hat{\mathfrak{d}} \in \mathbb{R}^n$ with non-decreasing sequence $\{\hat{\mathfrak{d}}_i\}_{i\in[1,n]}$ (resp. non-increasing sequence $\{\hat{\mathfrak{d}}_i\}_{i\in[1,n]}$), and consider, in place of (22),

$$\begin{aligned}
&\tilde{V}(\tilde{x}) := \tilde{V}_n(\tilde{\mathfrak{X}}^{(n)}),\\
&\tilde{V}_j(\tilde{\mathfrak{X}}_j) := \tilde{V}_{j-1}(\tilde{\mathfrak{X}}_{j-1}) + L^{-\hat{\mathfrak{d}}_j}[V_j^{(0)}(\tilde{\mathfrak{X}}_j) + V_j^{(\infty)}(\tilde{\mathfrak{X}}_j)],\\
&j \in [1,n],\; V_0 := 0,
\end{aligned} \quad (29)$$

where $\tilde{\mathfrak{X}}_j := (\tilde{x}_1, \ldots, \tilde{x}_j)$, $j \in [1,n]$, and $V_j^{(p)}$, $p \in \{0,\infty\}$, is defined in (23). Notice that $\tilde{V}$ (as V) is positive definite and radially unbounded for each $L > 0$. Moreover, let α be defined in (11). Since $\{\hat{\mathfrak{d}}_i\}_{i\in[1,n]}$ is a non-decreasing (resp. $\{\hat{\mathfrak{d}}_i\}_{i\in[1,n]}$ is a non-increasing) sequence and $V_j^{(p)}$, $p \in \{0,\infty\}$, does not depend on $\tilde{x}_{j+1}, \ldots, \tilde{x}_n$, we have

$$\begin{aligned}
&\frac{\partial \tilde{V}}{\partial \tilde{x}}(\tilde{x})\{L^{\hat{\mathfrak{d}}} \diamond [A\tilde{x}^q + B\alpha(\tilde{x})]\} \leqslant \sum_{j=1}^{n-1}\frac{\partial V_j}{\partial \tilde{x}_j}(\tilde{\mathfrak{X}}_j)\tilde{x}_{j+1}^{q_{j+1}}\\
&+\frac{\partial V_n}{\partial \tilde{x}_n}(\tilde{\mathfrak{X}}_n)\alpha(\tilde{x}) + \sum_{j=2}^{n}\sum_{i=1}^{j-1}\Big|\frac{\partial V_j}{\partial \tilde{x}_i}(\tilde{\mathfrak{X}}_j)\Big||\tilde{x}_{i+1}^{q_{i+1}}|
\end{aligned} \quad (30)$$

for all $\tilde{x} \in \mathbb{R}^n$, $L \geqslant 1$ (resp. $L \in (0,1]$), where V_j is defined in (22). By repeating the constructive procedure of proposition 4.3 we find out that

$$\begin{aligned}
W(\tilde{x}) :=& \sum_{j=1}^{n-1}\frac{\partial V_j}{\partial \tilde{x}_j}(\tilde{\mathfrak{X}}_j)\tilde{x}_{j+1}^{q_{j+1}} + \frac{\partial V_n}{\partial \tilde{x}_n}(\tilde{\mathfrak{X}}_n)\alpha(\tilde{x})\\
&+\sum_{j=2}^{n}\sum_{i=1}^{j-1}\Big|\frac{\partial V_j}{\partial \tilde{x}_i}(\tilde{\mathfrak{X}}_j)\Big||\tilde{x}_{i+1}^{q_{i+1}}|
\end{aligned}$$

is homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{w}_p, W_p)$, $p \in \{0,\infty\}$ and there exist $\gamma_1^*, \ldots, \gamma_n^* > 0$ (possibly different from those selected in proposition 4.3) such that

$$W(\tilde{x}) < 0,\; W_p(\tilde{x}) < 0$$

for all $\tilde{x} \neq 0$, $L \geqslant 1$ (resp. $L \in (0,1]$), $\gamma_j \geqslant \gamma_j^*$ and $j \in [1,n]$. On account of (30)

$$\begin{aligned}
&\dot{\tilde{V}}|_{(28)\text{ with } u=\alpha(\tilde{x})} = \frac{\partial \tilde{V}}{\partial \tilde{x}}(\tilde{x})\{L^{\hat{\mathfrak{d}}} \diamond [A\tilde{x}^q + B\alpha(\tilde{x})]\}\\
&\leqslant W(\tilde{x}) < 0
\end{aligned} \quad (31)$$

for all $\tilde{x} \neq 0$, $L \geqslant 1$ (resp. $L \in (0,1]$), $\gamma_j \geqslant \gamma_j^*$ and $j \in [1,n]$. This, together with proposition 4.3, means that there exist $\gamma_1^*, \ldots, \gamma_n^* > 0$ such that both (28) with $u = \alpha(\tilde{x})$ and (8) with $u = \alpha(x)$ are globally asymptotically stable, whatever $L \geqslant 1$ (resp. $L \in (0,1]$) and $\gamma_j \geqslant \gamma_j^*$, $j \in [1,n]$. This fact will be used in proposition 5.2 for establishing key robustness results. □

V. STATE FEEDBACK STABILIZATION AND ROBUSTNESS VIA HOMOGENEITY IN THE GENERALIZED SENSE

The following result is the key result for establishing the global asymptotic stability of a given system from the global asymptotic stability of its approximations at zero and at infinity. In particular, the global asymptotic stability of $\dot{x} = f(x) + \phi(x)$ from the global asymptotic stability of the dominating dynamics $\dot{x} = f(x)$.

Proposition *5.1: Assume*
(i) the existence of positive definite and radially unbounded $V : \mathbb{R}^n \to \mathbb{R}$ *such that* $\left(\frac{\partial V}{\partial x}\right)^T$ *is continuous over* $\mathbb{R}^n$ *and homogeneous (in the generalized sense) in the bi-limit with triple* $(\mathfrak{r}_p, \mathbf{1}_n \cdot \mathfrak{w}_p - \mathfrak{d}_p - \mathfrak{r}_p, (\frac{\partial V}{\partial x})_p^T)$, $p \in \{0, \infty\}$, *for some* $\mathfrak{w}_p \in \mathbb{R}_>$ *and* $\mathfrak{d}_p \in \mathbb{R}^n$,
(ii) f *and* g *are continuous vector fields and homogeneous (in the generalized sense) in the bi-limit with triple* $(\mathfrak{r}_p, \mathfrak{d}_p, f_p)$ *and, respectively,* $(\mathfrak{r}_p, \mathfrak{h}_p, g_p)$, $p \in \{0, \infty\}$, *with* $\mathfrak{d}_{\infty,j} \geqslant \mathfrak{h}_{\infty,j}$, $\mathfrak{d}_{0,j} \leqslant \mathfrak{h}_{0,j}$, $j \in [1, n]$,
(iii) $(\frac{\partial V}{\partial x} f)(x) < 0$, $(\frac{\partial V}{\partial x} f)_p(x) < 0$, $p \in \{0, \infty\}$, *for all* $x \neq 0$, *where* $(\frac{\partial V}{\partial x} f)_p := \sum_{i=1}^n (\frac{\partial V}{\partial x_i})_p f_{p,i}$, $p \in \{0, \infty\}$, *are the approximating functions associated with* $\frac{\partial V}{\partial x} f$ *at zero and, respectively, at infinity.*
There exists $c^* > 0$ *such that the origin of* $\dot{x} = f(x) + cg(x)$ *is globally asymptotically stable for all* $c \in [0, c^*]$.

Proof *5.1:* It follows proposition 2.13 of [2] since $\sigma := -\frac{\partial V}{\partial x} f$ and $\xi := \sum_{i=1}^n |\frac{\partial V}{\partial x_i}||g_i|$ are homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{w}_p, \sigma_p)$, $p \in \{0, \infty\}$, where $\sigma_p := -\sum_{i=1}^n (\frac{\partial V}{\partial x_i})_p f_{p,i}$, and, respectively, $(\mathfrak{r}_p, \mathfrak{w}_p, \xi_p)$, $p \in \{0, \infty\}$, where $\xi_p := \sum_{i=1}^n \delta_{\mathfrak{d}_{p,i}, \mathfrak{h}_{p,i}} |(\frac{\partial V}{\partial x_i})_p||g_{p,i}|$ (where δ is the Kronecker delta), with non-negative σ and $\{x \neq 0 : \sigma(x) = 0\} \subset \{x \neq 0 : \xi(x) < 0\}$ (by (iii)). □

In other words, if the degree of f dominates the degree of g both at the origin and at infinity and $\dot{x} = f(x)$ is globally asymptically stable, $\dot{x} = f(x) + cg(x)$ is globally asymptotically stable for c sufficiently small.

From proposition 5.1 with $f(x) := \sum_{i=1}^{n-1} x_{i+1}^{q_{i+1}} \frac{\partial}{\partial x_i} + \alpha(x) \frac{\partial}{\partial x_n}$ and α defined in (11), together with propositions 4.1 and 4.2, we readily obtain a stabilization result on a chain of power integrators with a nonlinear additive perturbation. Throughout this section we consider $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, (A, B) in Brunowski form and $q \in \mathbb{R}^n_{>,\mathrm{odd}}$.

Theorem *5.1: Let* $\{\mathfrak{d}_{\infty,j}\}_{j \in [1,n]}$ *(resp.* $\{\mathfrak{d}_{0,j}\}_{j \in [1,n]}$*) be non-decreasing (resp. non-increasing) sequence of reals and* $\mathfrak{r}_p$, $p \in \{0, \infty\}$, *be as in (9) and satisfy (10). Assume*
(i) g *is a continuous vector field and homogeneous (in the generalized sense) in the bi-limit with triple* $(\mathfrak{r}_p, \mathfrak{h}_p, g_p)$, $p \in \{0, \infty\}$, *with* $\mathfrak{d}_{\infty,j} \geqslant \mathfrak{h}_{\infty,j}$, $\mathfrak{d}_{0,j} \leqslant \mathfrak{h}_{0,j}$, $j \in [1, n]$.

There exist $c^*, \gamma_1^*, \ldots, \gamma_n^* > 0$ *such that the origin of* $\dot{x} = Ax^q + Bu + cg(x)$ *with* $u = \alpha(x)$ *and* α *defined in (11), is globally asymptotically stable for all* $c \in [0, c^*]$ *and* $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$.

Theorem 5.1 guarantees robustness with respect to homogeneous in the bi-limit perturbations cg with c sufficiently small and g with homogeneity degrees dominated (at the origin and at infinity) by the degrees of the chain of power integrators. The next result considers the entire class of perturbations cg with either large or small c.

Theorem *5.2: Let* $\{\mathfrak{d}_{\infty,j}\}_{j \in [1,n]}$ *(resp.* $\{\mathfrak{d}_{0,j}\}_{j \in [1,n]}$*) be non-decreasing (resp. non-increasing) sequence of reals and* $\mathfrak{r}_p$, $p \in \{0, \infty\}$, *be as in (9) and satisfy (10). Assume (i) of theorem 5.1 and, additionally,*
(ii) ϕ *is a continuous vector field for which there exist a non-decreasing sequence* $\{\hat{\mathfrak{d}}_i\}_{i \in [1,n]}$, $\hat{\mathfrak{z}} > 0$ *and* $L^* > 0$ *such that for all* $L \geqslant L^*$

$$|\phi_i(L^{\hat{\mathfrak{r}}} \diamond x)| \leqslant L^{\hat{\mathfrak{r}}_i + \hat{\mathfrak{d}}_i - \hat{\mathfrak{z}}} |g_i(x)|, \ i \in [1, n], \ \forall x \in \mathbb{R}^n, \quad (32)$$

with $\hat{\mathfrak{r}}_1 > 0$, $\hat{\mathfrak{r}}_i := \frac{\hat{\mathfrak{r}}_{i-1} + \hat{\mathfrak{d}}_{i-1}}{q_i}$, $i \in [2, n]$.

There exist $\hat{L}, \gamma_1^*, \ldots, \gamma_n^* > 0$ *such that the origin of*

$$\dot{x} = Ax^q + Bu + \phi(x) \quad (33)$$

with

$$u = \hat{\alpha}(x) := L^{\hat{\mathfrak{r}}_n + \hat{\mathfrak{d}}_n} \alpha(L^{-\hat{\mathfrak{r}}} \diamond x) \quad (34)$$

and α *defined in (11), is globally asymptotically stable for all* $L \geqslant \hat{L}$ *and* $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$.

Proof *5.2:* With $\tilde{x} := L^{-\hat{\mathfrak{r}}} \diamond x$ and u as in (34), from (33) we obtain

$$\dot{\tilde{x}} = L^{\hat{\mathfrak{d}}} \diamond [A\tilde{x}^q + B\alpha(\tilde{x}) + L^{-\hat{\mathfrak{r}} - \hat{\mathfrak{d}}} \diamond \phi(L^{\hat{\mathfrak{r}}} \diamond \tilde{x})]. \quad (35)$$

Let $\tilde{V}$ be as in (29) and V_j, $j \in [1, n]$, as in (22). According to remark 4.2,

$$W(\tilde{x}) := \sum_{j=1}^{n-1} \frac{\partial V_j}{\partial \tilde{x}_j}(\tilde{\mathfrak{X}}_j) \tilde{x}_{j+1}^{q_{j+1}} + \frac{\partial V_n}{\partial \tilde{x}_n}(\tilde{\mathfrak{X}}_n)\alpha(\tilde{x}) + \sum_{j=2}^{n} \sum_{i=1}^{j-1} \left|\frac{\partial V_j}{\partial \tilde{x}_i}(\tilde{\mathfrak{X}}_j)\right| |\tilde{x}_{i+1}^{q_{i+1}}|$$

is homogeneous in the bi-limit with triple $(\mathfrak{r}_p, \mathfrak{w}_p, W_p)$, $p \in \{0, \infty\}$ and there exist $\gamma_1^*, \ldots, \gamma_n^* > 0$ such that

$$W(\tilde{x}) < 0, \ W_p(\tilde{x}) < 0, \quad (36)$$

for all $\tilde{x} \neq 0$, $L \geqslant 1$ and $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$. Moreover, on account of (ii)

$$L^{-\hat{\mathfrak{r}}_i - \hat{\mathfrak{d}}_i} |\phi_i(L^{\hat{\mathfrak{r}}} \diamond \tilde{x})| \leqslant L^{-\hat{\mathfrak{z}}} |g_i(\tilde{x})|,$$

for all $\tilde{x} \in \mathbb{R}^n$, $L \geqslant L^*$ and $i \in [1, n]$. Since $\{\hat{\mathfrak{d}}_i\}_{i \in [1,n]}$ is a non-decreasing sequence (see also remark 4.2)

$$\dot{\tilde{V}}|_{(35)} \leqslant W(\tilde{x}) + L^{-\hat{\mathfrak{z}}} \sum_{j=1}^{n} \sum_{i=1}^{j} \left|\frac{\partial V_j}{\partial \tilde{x}_i}(\tilde{\mathfrak{X}}_j)\right| |g_i(\tilde{x})|. \quad (37)$$

for all $x \in \mathbb{R}^n$, $L \geqslant \max\{L^*, 1\}$ and $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$. By construction (on account of (i) and propositions 4.1 and 4.2) W and $\sum_{j=1}^n \sum_{i=1}^j \left|\frac{\partial V_j}{\partial \tilde{x}_i}\right| |g_i|$ are all homogeneous in the bi-limit with weights $\mathfrak{r}_0$ and degree $\mathfrak{w}_0$ at the origin and weights $\mathfrak{r}_\infty$ and degree $\mathfrak{w}_\infty$ at infinity. By application of proposition 2.13 of [2] to the right-hand part of (37) on account of (36), we can find $\hat{L} \geqslant \max\{L^*, 1\}$ such that $W(\tilde{x}) + L^{-\hat{\mathfrak{z}}} \sum_{j=1}^n \sum_{i=1}^j \left|\frac{\partial V_j}{\partial \tilde{x}_i}(\tilde{\mathfrak{X}}_j)\right| |g_i(\tilde{x})| < 0$ for all $\tilde{x} \neq 0$, $L \geqslant \hat{L}$ and $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$. On account of (37) and since $\tilde{V}$ is positive definite and radially unbounded, it follows that (35) and, therefore (33), is globally asymptotically stable for all $L \geqslant \hat{L}$ and $\gamma_j \geqslant \gamma_j^*$, $j \in [1, n]$. □

Remark *5.1:* Condition (ii) of theorem 5.2 is the price of relaxing assumptions (ii) and (iii) of theorem 5.1 and it is an incremental homogeneity in the upper bound property of ϕ (see [1]). For example, it can be always met for vector fields ϕ having the form (or bounded by) (6) and (7). For this reason, on account of remark III-A, theorem 5.1 finds application to systems (33) with vector fields ϕ having the form (or bounded by) (6) and (7).

VI. Conclusions

We have introduced a generalized notion of homogeneity in the bi-limit and we have seen that large classes of nonlinear systems are homogeneous in the bi-limit in this generalized sense. We have designed a state-feedback stabilizer for a chain of power integrators a homogeneous in the bi-limit state-feedback stabilizer. The closed-loop system is globally asymptotically stable robustly with respect to perturbations which are dominated by the chain of power integrators. An exhaustive analysis will study also the observer and the output feedback design issues.

VII. Appendix

A. Proofs of the main and auxiliary results

Since a vector field is homogeneous in the generalized sense if and only if each one of its coordinate functions is homogeneous in the classical sense, it is clear that many properties and results on homogeneity in the bi-limit in the classical sense can be used in our generalized framework and for such properties we refer to properties P1-P3 and propositions 2.10-2.13 of [2].

In this paper we extensively use the procedure of obtaining a homogeneous in the bi-limit function from the combination of two functions, the first one ϕ homogeneous in the ∞-limit and the other one ψ homogeneous in the 0-limit. We adopt a different procedure from the one pointed out in [2] and we state it here without proof. In [2] the two functions ϕ and ψ are combined as $\frac{\psi}{1+\psi}(1+\phi)$ (the combination has a fixed structure), we propose a combination $\phi\alpha+\psi\beta$, where α and β can be designed to take into account additional homogeneity constraints (for example, on the derivative or the integral of the combination).

Proposition *7.1: Assume that*
(i) $\phi : \mathbb{R}^n \to \mathbb{R}$ *is homogeneous in the* ∞*-limit with triple* $(\mathfrak{r}_\infty, \mathfrak{h}_\infty, \phi_\infty)$, $\mathfrak{h}_\infty \geqslant 0$, *and* $\psi : \mathbb{R}^n \to \mathbb{R}$ *is homogeneous in the* 0*-limit with triple* $(\mathfrak{r}_0, \mathfrak{h}_0, \psi_0)$, ϕ *and* ψ *all vanishing at the origin,*
and $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$ *are continuous functions such that*
(ii) $\phi\alpha$ *is homogeneous in the 0-limit with triple* $(\mathfrak{r}_0, \mathfrak{h}_0, 0)$,
(iii) $\sup_{x\in\mathbb{R}^n} |\alpha(x)| < +\infty$, $\sup_{x\in\mathbb{R}^n} |\beta(x)| < +\infty$ *and* $\lim_{\|x\|\to+\infty} |\psi(x)\beta(x)| = 0$,
(iv) $\lim_{\|x\|\to+\infty} \alpha(x) = 1$ *and* $\beta(0) = 1$.
The function $x \mapsto \phi(x)\alpha(x) + \psi(x)\beta(x)$ *is homogeneous in the bi-limit with triples* $(\mathfrak{r}_\infty, \mathfrak{h}_\infty, \phi_\infty)$ *and* $(\mathfrak{r}_0, \mathfrak{h}_0, \psi_0)$.

Remark *7.1:* Let ϕ, ψ be as in (i) of proposition 7.1 and consider the function

$$(z, \phi, \psi) \mapsto \mathfrak{J}(z, \phi, \psi) := (1 - e^{-z})\phi + e^{-z}\psi \qquad (38)$$

with a homogeneous in the 0-limit vector field $\zeta(x) := \sum_{i=1}^n \zeta_i(x)\frac{\partial}{\partial x_i}$ with triple $(\mathfrak{r}_0, 0, \zeta_0)$ such that $\|\zeta(0)\|_{\mathfrak{r}_0} = 0$ and $\lim_{\|x\|\to+\infty} \|\zeta(x)\|_{\mathfrak{r}_0} = +\infty$ ($\|\cdot\|_{\mathfrak{r}_0}$ denotes the weighted homogeneous norm with weigths $\mathfrak{r}_0$). Notice that if ψ is norm-bounded by a multi-power function of $\zeta_1, \ldots, \zeta_n$, i.e. $|\psi(x)| \leqslant c\sum_{k\in[1,N]}\prod_{i\in[1,n]} |\zeta_i^{h_i^{(k)}}(x)|$ for all x and for some $c > 0$, integer $N \geqslant 1$ and reals $h_i^{(k)} \geqslant 0$, then $\lim_{\|x\|\to+\infty} |\psi(x)e^{-\|\zeta(x)\|_{\mathfrak{r}_0}^{\mathfrak{m}}}| = 0$ for each $\mathfrak{m} > 0$. Moreover, if $\mathfrak{m} \geqslant \mathfrak{h}_0$ then $\phi(x)(1 - e^{-\|\zeta(x)\|_{\mathfrak{r}_0}^{\mathfrak{m}}})$ is homogeneous in the 0-limit with triple $(\mathfrak{r}_0, \mathfrak{h}_0, 0)$ (since $\phi(0) = 0$ and $(1 - e^{-z})/z \to 1$ as $z \to 0$). Under these conditions, the function $\mathfrak{J}(\|\zeta\|_{\mathfrak{r}_0}^{\mathfrak{m}}, \phi, \psi)$ is homogeneous in the bi-limit with triples $(\mathfrak{r}_\infty, \mathfrak{h}_\infty, \phi_\infty)$ and $(\mathfrak{r}_0, \mathfrak{h}_0, \psi_0)$ by application of proposition 7.1.□

Proof of proposition 4.2. We proceed by induction.

Initial step. Recall that $w^{\mathfrak{r}} = \mathrm{sgn}(w)|w|^{\mathfrak{r}}$ and $\frac{dw^{\mathfrak{r}}}{dw} = \mathfrak{r}|w|^{\mathfrak{r}-1}$ if $\mathfrak{r} \geqslant 1$ and $\frac{d|w|^{\mathfrak{r}}}{dw} = \mathfrak{r}w^{\mathfrak{r}-1}$ if $\mathfrak{r} > 1$. First, consider the function $V_1^{(p)}$. It is continuous over $\mathbb{R}$ with $V_1^{(p)}(0) = 0$ (since $\mathfrak{f}_{p,1} + \mathfrak{a}q_1\mathfrak{r}_{p,1} > 0$ from (21)), positive definite and radially unbounded over $\mathbb{R}$. In addition, since $\mathfrak{f}_{p,1} > \mathfrak{r}_{p,1}$ (from (20)), $\frac{dV_1^{(p)}}{dx_1}$ is continuous over $\mathbb{R}$ with $\frac{dV_1^{(p)}}{dx_1}(0) = 0$. On account of the definition (19) of $\mathfrak{f}_{p,1}$, since $\mathfrak{f}_{\infty,1} + \mathfrak{a}q_1\mathfrak{r}_{\infty,1} > \mathfrak{f}_{0,1} + \mathfrak{a}q_1\mathfrak{r}_{0,1} > 0$ (from (21)) and on account of properties P1-P3 and propositions 2.10-2.12 of [2], $\frac{dV_1}{dx_1}$ is homogeneous in the bi-limit with triple $(\mathfrak{r}_{p,1}, \mathfrak{w}_p - \mathfrak{d}_{p,1} - \mathfrak{r}_{p,1}, v_{p,1,1})$, $p \in \{0, \infty\}$, where $v_{p,1,1}$ is defined in (24).

Let $v_{p,1,j-1}, \cdots, v_{p,j-1,j-1}$, $j \in [2, n]$, and $w_{p,1,i}, \cdots, w_{p,i,i}$, $i \in [1, j-1]$, be defined as in (24)-(25). Also let $\mathfrak{R}_{p,j} := (\mathfrak{r}_1, \ldots, \mathfrak{r}_j)^T$, $\mathfrak{D}_{p,j} := (\mathfrak{d}_{p,1}, \ldots, \mathfrak{d}_{p,j})^T$ and $\mathcal{X}_j := (x_1, \ldots, x_j)^T$ with $\left(\frac{\partial V_{j-1}}{\partial \mathcal{X}_{j-1}}\right)_p^T := (v_{p,1,j-1}, \ldots, v_{p,j-1,j-1})^T$ and $\left(\frac{\partial \alpha_i^{\mathfrak{a}_{i+1}}}{\partial \mathcal{X}_i}\right)_p^T := (w_{p,1,i}, \ldots, w_{p,i,i})^T$, $i \in [1, j-1]$, $p \in \{0, \infty\}$.

Induction Step $(j-1)$. V_{j-1} *is positive definite and radially unbounded over* $\mathbb{R}^{j-1}$, *each* $\left(\frac{\partial V_{j-1}}{\partial \mathcal{X}_{j-1}}\right)^T$ *is continuous over* $\mathbb{R}^{j-1}$, *with* $\left(\frac{\partial V_{j-1}}{\partial \mathcal{X}_{j-1}}\right)^T(0) = 0$, *and homogeneous (in the generalized sense) in the bi-limit with triple* $(\mathfrak{R}_{p,j-1}, \mathbf{1}_{j-1} \cdot \mathfrak{w}_p - \mathfrak{R}_{p,j-1} - \mathcal{D}_{p,j-1}, \left(\frac{\partial V_{j-1}}{\partial \mathcal{X}_{j-1}}\right)_p^T)$, $p \in \{0, \infty\}$ *and* $\left(\frac{\partial \alpha_i^{\mathfrak{a}_{i+1}}}{\partial \mathcal{X}_i}\right)^T$, $i \in [1, j-1]$, *is continuous over* $\mathbb{R}^i$, *with* $\left(\frac{\partial \alpha_i^{\mathfrak{a}_{i+1}}}{\partial \mathcal{X}_i}\right)^T(0) = 0$, *and homogeneous in the bi-limit with triple* $(\mathfrak{R}_{p,i}, (\mathbf{1}_i \cdot \mathfrak{a}_{i+1}q_{i+1}\mathfrak{r}_{p,i+1} - \mathfrak{R}_{p,i}, \left(\frac{\partial \alpha_i^{\mathfrak{a}_{i+1}}}{\partial \mathcal{X}_i}\right)_p^T)$, $p \in \{0, \infty\}$. We show that the induction step holds for j.

(V_j *is continuous, positive definite and radially unbounded over* $\mathbb{R}^j$). Since $V_j^{(p)}$ is continuous, positive definite and radially unbounded over $\mathbb{R}^j$ and using the induction hypothesis (V_{j-1} is positive definite and radially unbounded over $\mathbb{R}^{j-1}$), V_j is positive definite and radially unbounded over $\mathbb{R}^{(j)}$.

($(\frac{\partial V_j}{\partial \mathcal{X}_j})^T$ *is continuous over* $\mathbb{R}^j$). Let $k \in [1, j]$ and $A_{k,i} := -\frac{\partial \alpha_i^{\mathfrak{a}_{i+1}}}{\partial x_k}$ for $i \in [1, j]$. Notice that $A_{k,i} \equiv 0$ for all $i \in [1, k-1]$. Notice also that $\frac{\partial}{\partial x_k}|x_i^{\mathfrak{a}_i q_i} - \alpha_{i-1}^{\mathfrak{a}_i}|^{\frac{1}{\mathfrak{a}_i q_i \mathfrak{r}_{p,i}}}$, $i \in [k, j]$, is continuous over $\mathbb{R}^i$ (it is zero for $i \in [1, k-1]$) with $\left(\frac{\partial}{\partial x_k}|x_i^{\mathfrak{a}_i q_i} - \alpha_{i-1}^{\mathfrak{a}_i}|^{\frac{\mathfrak{n}}{\mathfrak{a}_i q_i \mathfrak{r}_{p,i}}}\right)(0) = 0$, by continuity of $A_{k,i-1}$ over $\mathbb{R}^{i-1}$ (induction step) and since $1 - \mathfrak{a}_i q_i \mathfrak{r}_{p,i} > 0$ for each $i \in [k, j]$ (from (17)) and $\mathfrak{a}_k q_k - 1 \geqslant 0$ (from (13)). From this with the induction hypothesis on $\frac{\partial V_{j-1}}{\partial x_k}$, $k \in [1, j-1]$, and $A_{k,j-1}$ and since $\mathfrak{f}_{p,i} \geqslant 1$ for each $i \in [k, j]$ (from (20)), it follows that $\frac{\partial V_j}{\partial x_k}$, $k \in [1, j]$, is continuous over $\mathbb{R}^j$ with $\left(\frac{\partial V_j}{\partial x_k}\right)(0) = 0$.

We are left with proving the continuity of $A_{k,j}$ over $\mathbb{R}^j$. Since $\mathfrak{a}_{j+1}(\mathfrak{r}_{p,j} + \mathfrak{d}_{p,j}) - \mathfrak{a}_j q_j \mathfrak{r}_{p,j} > 0$ (from (14)), $\mathfrak{a}_j q_j - 1 \geqslant 0$ (from (13)) and by continuity of $A_{k,j-1}$ over $\mathbb{R}^{j-1}$ (induction step), $\dfrac{\partial \alpha_j^{(p)}}{\partial x_k}$ and $\dfrac{\partial}{\partial x_k}\|\zeta_j\|_{\mathfrak{R}_{0,j}}^{\mathfrak{m}}$ are both continuous over $\mathbb{R}^j$, with $\dfrac{\partial \alpha_j^{(p)}}{\partial x_k}(0) = 0$ and $(\dfrac{\partial}{\partial x_k}\|\zeta_j\|_{\mathfrak{R}_{p,j}}^{\mathfrak{m}})(0) = 0$. It follows that $A_{k,j}$ is continuous over $\mathbb{R}^j$ with $A_{k,j}(0) = 0$.

(*Homogeneity of* $(\frac{\partial V_j}{\partial \mathfrak{X}_j})^T$). Since $q_j \mathfrak{r}_{p,j} = \mathfrak{r}_{p,j-1} + \mathfrak{d}_{p,j-1}$) and using properties P1-P3 and propositions 2.10-2.12 of [2], $\int_{\alpha_{j-1}^{\frac{1}{q_j}}}^{x_j} [s^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}] ds$ is homogeneous in the bi-limit with triple

$$\left(\mathfrak{R}_{j,p}, \mathfrak{a}_j q_j \mathfrak{r}_{p,j} + \mathfrak{r}_{p,j}, \int_{\alpha_{p,j-1}^{\frac{1}{q_j}}}^{x_j} [s^{\mathfrak{a} q_j} - \alpha_{p,j-1}^{\mathfrak{a}}] ds\right),\ p \in \{0, \infty\}.$$

With the induction hypothesis on $A_{j-1,k}$, $\dfrac{\partial}{\partial x_k}|x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}$ is homogeneous in the bi-limit with triple

$$\left(\mathfrak{R}_{j,p}, \mathfrak{f}_{p,j} - \mathfrak{r}_{p,k}, \left(\frac{\partial}{\partial x_k}|x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}\right)_p\right),\ p \in \{0, \infty\},$$

where

$$\left(\frac{\partial}{\partial x_k}|x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}\right)_p := \mathfrak{f}_{p,j}|x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}-1}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}} \cdot$$
$$\cdot\Bigg[(1 - \delta_{j,k}) \frac{w_{p,j-1,k}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}} \left(x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}\right)^{\frac{1 - \mathfrak{a}_j q_j \mathfrak{r}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}}$$
$$+ \delta_{j,k} \frac{|x_k|^{\mathfrak{a}_k q_k - 1}}{\mathfrak{r}_{p,k}} \left(x_k^{\mathfrak{a}_k q_k} - \alpha_{p,k-1}^{\mathfrak{a}_k}\right)^{\frac{1 - \mathfrak{a}_k q_k \mathfrak{r}_{p,k}}{\mathfrak{a}_k q_k \mathfrak{r}_{p,k}}}\Bigg] \qquad (39)$$

and

$$|x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}} \left(x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}\right)$$
$$(\text{resp. } |x_j^{\mathfrak{a}_j q_j} - \alpha_{j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}} \left(x_j - \alpha_{j-1}^{\frac{1}{q_j}}\right) A_{k,j-1}) \qquad (40)$$

is homogeneous in the bi-limit with triple

$$\Big(\mathfrak{R}_{j,p}, \mathfrak{f}_{p,j} + \mathfrak{a}_j q_j \mathfrak{r}_{p,j}, |x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}} \left(x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}\right)$$
$$\Big(\text{resp. } (\mathfrak{R}_{j,p}, \mathfrak{f}_{p,j} + \mathfrak{a}_j q_j \mathfrak{r}_{p,j} + \mathfrak{r}_{p,j} - \mathfrak{r}_{p,k},$$
$$|x_j^{\mathfrak{a}_j q_j} - \alpha_{p,j-1}^{\mathfrak{a}_j}|^{\frac{\mathfrak{f}_{p,j}}{\mathfrak{a}_j q_j \mathfrak{r}_{p,j}}} \left(x_j - \alpha_{p,j-1}^{\frac{1}{q_j}}\right) w_{p,j-1,k})\Big),\ p \in \{0, \infty\}.$$

Therefore, using the definition of $\mathfrak{f}_{p,j}$ and since $\{\mathfrak{d}_\infty\}_{i \in [1,n]}$ is non-decreasing (resp. $\{\mathfrak{d}_0\}_{i \in [1,n]}$ is non-increasing), we see that $\frac{\partial V_j^{(p)}}{\partial x_k}$, $p \in \{0, \infty\}$, is homogeneous in the p-limit with triple $(\mathfrak{R}_{j,p}, \mathfrak{w}_p - \mathfrak{r}_{p,k} - \mathfrak{d}_{p,k}, v_{p,k,j})$, where $v_{p,k,j}$ is defined in (24). On account of $\mathfrak{f}_{\infty,1} + \mathfrak{a} q_k \mathfrak{r}_{\infty,k} := \mathfrak{w}_\infty - \mathfrak{r}_{\infty,k} - \mathfrak{d}_{\infty,k} > \mathfrak{f}_{0,k} + \mathfrak{a} q_k \mathfrak{r}_{0,k} := \mathfrak{w}_0 - \mathfrak{r}_{0,k} - \mathfrak{d}_{0,k}$ (from (21)) and using the induction hypothesis on $\frac{\partial V_{j-1}}{\partial x_k}$, by application of property P2 of [2] we obtain that $\frac{\partial V_j}{\partial x_k}$ is homogeneous in the bi-limit with triple $(\mathfrak{R}_{j,p}, \mathfrak{w}_p - \mathfrak{r}_{p,k} - \mathfrak{d}_{p,k}, v_{p,k,j})$.

We are left with proving that $A_{k,j}$, $k \in [1, j-1]$, is homogeneous in the bi-limit with triple $(\mathfrak{R}_{j,p}, \mathfrak{a} q_{j+1} \mathfrak{r}_{p,j+1} - \mathfrak{r}_{p,k}, w_{p,k,j})$, $p \in \{0, \infty\}$. By the induction hypothesis on $A_{k,j-1}$ it is seen that $-\dfrac{\partial \alpha_j^{(p)}}{\partial x_k}$ is homogeneous in the p-limit with triple $(\mathfrak{R}_{j,p}, \mathfrak{a} q_{j+1} \mathfrak{r}_{p,j+1} - \mathfrak{r}_{p,k}, w_{p,k,j})$, $p \in \{0, \infty\}$, where $w_{p,k,j}$ is defined in (25). Also, since $\mathfrak{m} \geqslant \mathfrak{a} q_{j+1} \mathfrak{r}_{0,j+1} - \mathfrak{r}_{0,k} + 1$ (from (15)), $1 - \mathfrak{a} q_i \mathfrak{r}_{0,i} > 0$ for $i \in [1, n]$ (from (17)) and $A_{k,i}(0) = 0$ for $i \in [1, j]$ with $\alpha_j^{(0)}(0) = \alpha_j^{(\infty)}(0) = 0$, $-\dfrac{\partial \alpha_j^{(0)}}{\partial x_k} + \left(\dfrac{\partial}{\partial x_k}\|\zeta_j\|_{\mathfrak{R}_{0,j}}^{\mathfrak{m}}\right)[\alpha_j^{(0)} - \alpha_j^{(\infty)}]$ is homogeneous in the 0-limit with triple $(\mathfrak{R}_{j,0}, \mathfrak{a} q_{j+1} \mathfrak{r}_{0,j+1} - \mathfrak{r}_{0,k}, w_{0,k,j})$. Application of proposition 7.1 gives that $A_{k,j}$, $k \in [1, j-1]$, is homogeneous in the bi-limit with triple $(\mathfrak{R}_{j,p}, \mathfrak{a} q_{j+1} \mathfrak{r}_{p,j+1} - \mathfrak{r}_{p,k}, w_{p,k,j})$, $p \in \{0, \infty\}$. This concludes the proof of the induction step for j and the proof of the proposition by induction.□

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