Notes on Linear Control Systems: Module VII

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***Abstract*—Controllability and observability. Eigenvalue assignment and stabilization via state-feedback. PBH controllability criterion. State Observers and detectors. PBH observability criterion. Eigenvalue assignment and stabilization via output-feedback: the separation principle.**

I. Controllability

An important problem in control theory is to find an input function which steers the state from an initial value x_0 to a final value x_f in a given time t_f. The characterization of the states which can be reached at time t_f starting from a given state x_0 is related to the notion of "reachable" states.

Definition *1.1: A state $x_f \in \mathbb{R}^n$ is said to be reachable from x_0 at time t_f if there exists an input function $\mathbf{u}$ and $t_f > 0$ such that*

$$\mathbf{x}(t_f, x_0, \mathbf{u}) = x_f \tag{1}$$

Therefore, a state x_f is reachable from x_0 if there exists an input function $\mathbf{u}$ which steers the solution $\mathbf{x}(t, x_0, \mathbf{u})$ to the point x_f in t_f sec. The set of reachable states from $x_0 := 0$ is a vector space.

Proposition *1.1: The set of reachable states from $x_0 = 0$ is a vector subspace of the state space $\mathbb{R}^n$.*

Therefore, if x_a and x_b are both reachable from $x_0 = 0$ then $c_a x_a + c_b x_b$ is reachable from $x_0 = 0$ for any $c_a, c_b \in \mathbb{R}$.

Let x_a and x_b two reachable states from $x_0 = 0$ at t_{fa} and t_{fb}. The state $c_a x_a + c_b x_b$ for reals c_a, c_b is reachable from $x_0 = 0$ at $t_f = \max\{t_{fa}, t_{fb}\}$.

Now, we want to characterize the set of reachable states in terms of the matrices A and B. To this aim, let us introduce the following time-varying $n \times n$ matrix

$$\mathbf{G}(t) := \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \tag{2}$$

This matrix is symmetric and it is known as *controllability gramian*. Also, define the *controllability matrix*

$$R := \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} \tag{3}$$

Proposition *1.2: For each $t \neq 0$: $\mathrm{Span}\{\mathbf{G}(t)\} = \mathrm{Span}\{R\}$. Moreover, the set of reachable states from $x_0 = 0$ at time t_f is $\mathrm{Span}\{\mathbf{G}(t_f)\}$.*

Note that, if $\mathrm{Span}\{\mathbf{G}(t_f)\} = \mathbb{R}^n$, any state $x_f \in \mathbb{R}^n$ is reachable from $x_0 := 0$ no matter what the final t_f is. In this case, by proposition 1.2 $\mathrm{Span}\{R\} = \mathbb{R}^n$.

Definition *1.2: If $\mathrm{Span}\{R\} = \mathbb{R}^n$ then the system is said to be controllable (or controllable).*

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Now, we are in a position to characterize any state which is reachable from a given $x_0 \in \mathbb{R}^n$. Since

$$\mathbf{x}(t, x_0, \mathbf{u}) = e^{At} x_0 + e^{A(t-\tau)} B \mathbf{u}_\tau d\tau \tag{4}$$

it follows that a state x_f is reachable from $x_0 \in \mathbb{R}^n$ at time t_f if and only if $x_f - e^{At_f} x_0$ is reachable from 0 at time t_f.

Proposition *1.3: The set of states reachable at time t_f from $x_0 \in \mathbb{R}^n$ is the set of states x_f for which $x_f - e^{At_f} x_0$ is reachable at time t_f from $x_0 := 0$ and it is equal to*

$$\{z \in \mathbb{R}^n : z = e^{At_f} x_0 + y, y \in \mathrm{Span}\{R\}\} \tag{5}$$

As a final task, we want to find the input function $\mathbf{u}$ for which a state x_f is reachable from $x_0 \in \mathbb{R}^n$ at time t_f. To this aim, first we find the input function $\mathbf{u}$ for which a state x_f is reachable is reachable from 0 at time t_f. If $x_f \in \mathrm{Span}\{\mathbf{G}(t_f)\}$ then x_f is reachable from 0 at time t_f by proposition 1.2. Therefore, there exists $w \in \mathbb{R}^n$ such that $x_f = \mathbf{G}(t_f) w$. The input function $\mathbf{u}$ defined as

$$\mathbf{u}(t) := B^\top e^{A^\top (t_f - t)} w \tag{6}$$

is such that

$$x_f = \mathbf{x}(t_f, x_0, \mathbf{u}) \tag{7}$$

Indeed,

$$\begin{aligned} \mathbf{x}(t_f, x_0, \mathbf{u}) &= \int_0^{t_f} e^{A(t_f - \tau)} B \mathbf{u}_\tau d\tau \\ &= \int_0^{t_f} e^{A(t_f - \tau)} B B^\top e^{A^\top (t_f - \tau)} w d\tau = \mathbf{G}(t_f) w = x_f \end{aligned}$$

Next, we find the input function $\mathbf{u}$ for which x_f is reachable from $x_0 \in \mathbb{R}^n$ at time t_f. If $x_f - e^{At_f} x_0 \in \mathrm{Span}\{\mathbf{G}(t_f)\}$ then $x_f - e^{At_f} x_0$ is reachable from $x_0 \in \mathbb{R}^n$ at time t_f by proposition 1.3. Therefore, if $w \in \mathbb{R}^n$ is such that $x_f - e^{At_f} x_0 = \mathbf{G}(t_f) w$, the input function $\mathbf{u}$ defined as

$$\mathbf{u}(t) := B^\top e^{A^\top (t_f - t)} w \tag{8}$$

is such that

$$x_f = \mathbf{x}(t_f, x_0, \mathbf{u}) \tag{9}$$

Proposition *1.4: If $\mathrm{Span}\{R\} = \mathbb{R}^n$, any state x_f is reachable from any x_0 within any time t_f and an input funtion which steers the state x_0 to x_f within t_f sec is*

$$\mathbf{u}(t) := B^\top e^{A^\top (t_f - t)} \mathbf{G}^{-1}(t_f)(x_f - e^{At_f} x_0) \tag{10}$$

Exercize *1.1: Consider the double integrator*

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= \mathbf{u}(t) \end{aligned} \tag{11}$$

and calculate the set of reachable states from 0. *Determine the input function* $\mathbf{u}$ *(if possible) which steers the state from* $x_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$ *to* $x_f = \begin{pmatrix} 8 & -6 \end{pmatrix}^\top$ *within* $t_f = 1$ *sec.*

In this case

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The *controllability matrix* is

$$R := \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

and $\text{Span}\{R\} = \mathbb{R}^2$, Therefore, the set of reachable states from 0 is $\mathbb{R}^2$. Also the set of reachable states from any x_0 is $\mathbb{R}^2$ (proposition 1.3).

The *controllability gramian* is

$$\begin{aligned} \mathbf{G}(t) &:= \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \\ &= \int_0^t \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} d\tau \\ &= \int_0^t \begin{pmatrix} \tau \\ 1 \end{pmatrix} \begin{pmatrix} \tau & 1 \end{pmatrix} d\tau = \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix} d\tau = \int_0^t \begin{pmatrix} \frac{1}{3}t^3 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & t \end{pmatrix} \end{aligned}$$

since

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{13}$$

Note that $\text{Span}\{R\} = \text{Span}\{\mathbf{G}(t_f)\} = \mathbb{R}^2$ (proposition 1.2) since $\mathbf{G}(t_f)$ is nonsingular for each $t_f \neq 0$.

Let us calculate the input function $\mathbf{u}$ (if possible) which steers the state from $x_0 := \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$ to $x_f = \begin{pmatrix} 8 & -6 \end{pmatrix}^\top$ (proposition 1.4). Let

$$\begin{aligned} w &:= \mathbf{G}^{-1}(t_f)(x_f - e^{At_f} x_0) \\ &= \frac{12}{t_f^4} \begin{pmatrix} t_f & -\frac{1}{2}t_f^2 \\ -\frac{1}{2}t_f^2 & \frac{1}{3}t_f^3 \end{pmatrix} \left[\begin{pmatrix} 8 \\ -6 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ &= \frac{12}{t_f^4} \begin{pmatrix} 7t_f + 3t_f^2 \\ -\frac{7}{2}t_f^2 - 2t_f^3 \end{pmatrix} = \begin{pmatrix} 120 \\ -66 \end{pmatrix} \end{aligned} \tag{14}$$

The desired input function is

$$\mathbf{u}(t) := B^\top e^{A^\top(1-t)} w = \begin{pmatrix} 1-t & 1 \end{pmatrix} \begin{pmatrix} 120 \\ -66 \end{pmatrix} = -120t + 54. \lhd$$

Exercize *1.2: Consider the model*

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= -\mathbf{x}_1(t) \\ \dot{\mathbf{x}}_2(t) &= \mathbf{x}_2(t) + \mathbf{u}(t) \end{aligned} \tag{15}$$

and calculate the set of reachable states from 0. *Determine the input function* $\mathbf{u}$ *(if possible) which steers the state from* $x_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ *to* $x_f = \begin{pmatrix} 1 & -6 \end{pmatrix}^\top$ *and, respectively, to* $x_f = \begin{pmatrix} 4 & -6 \end{pmatrix}^\top$.

In this case

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The *controllability matrix* is

$$R := \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{16}$$

and $\text{Span}\{R\} = \text{Span}\{\begin{pmatrix} 0 & 1 \end{pmatrix}^\top\}$. Therefore, the set of reachable states from 0 is $\text{Span}\{\begin{pmatrix} 0 & 1 \end{pmatrix}^\top\}$. Since

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \tag{17}$$

the set of reachable states from any $x_0 := \begin{pmatrix} x_{01} & x_{02} \end{pmatrix}^\top$ is

$$\begin{aligned} &\{z \in \mathbb{R}^n : z = e^{At_f} x_0 + y, y \in \text{Span}\{R\}\} \\ &= \{z \in \mathbb{R}^n : z = \begin{pmatrix} e^{-t_f} x_{01} \\ c \end{pmatrix}, c \in \mathbb{R}\} \end{aligned}$$

Therefore the state $x_f = \begin{pmatrix} 1 & -6 \end{pmatrix}^\top$ is reachable from $x_0 = \begin{pmatrix} 2 & 0 \end{pmatrix}^\top$ within $t_f = ln2$ sec. On the other hand, the state $x_f = \begin{pmatrix} 4 & -6 \end{pmatrix}^\top$ is not reachable from $x_0 = \begin{pmatrix} 2 & 0 \end{pmatrix}^\top$.

Using (17) the *controllability gramian* is

$$\begin{aligned} \mathbf{G}(t) &:= \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \\ &= \int_0^t \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} d\tau \\ &= \int_0^t \begin{pmatrix} 0 & 0 \\ 0 & e^{2\tau} \end{pmatrix} d\tau = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & e^{2t} - 1 \end{pmatrix} \end{aligned} \tag{18}$$

Note that for each $t_f \neq 0$

$$\text{Span}\{R\} = \text{Span}\{\mathbf{G}(t_f)\} = \text{Span}\Big\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big\}$$

Let us calculate the input function $\mathbf{u}$ which steers the state $x_0 = \begin{pmatrix} 2 & 0 \end{pmatrix}^\top$ to $x_f = \begin{pmatrix} 1 & -6 \end{pmatrix}^\top$ within $t_f = ln2$ sec. Let $w \in \mathbb{R}^2$ be such that $x_f - e^{At_f} x_0 = \mathbf{G}(t_f) w$, i.e.

$$x_f - e^{Aln2} x_0 = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \mathbf{G}(ln2) w = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} w$$

We obtain

$$w = -3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

The desired input function is

$$\mathbf{u}(t) := B^\top e^{A^\top(ln2-t)} w = -3 \begin{pmatrix} 0 & e^{ln2-t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -6e^{-t}. \lhd$$

II. Observability

Another important problem in control theory is to reconstruct the intial value x_0 of the state from the observations of the inputs and the outputs. The characterization of the states which can be reconstructed from the inputs and the outputs is related to the notion of unobservable states.

Definition *2.1: Two states* $x_a, x_b \in \mathbb{R}^n$ *are said to be indistinguishable if there exists* $t_f > 0$ *such that for any input function* $\mathbf{u}$ *defined over* $[0, t_f]$ *and for all* $t \in [0, t_f]$

$$\mathbf{y}(t, x_a, \mathbf{u}) = \mathbf{y}(t, x_b, \mathbf{u}) \tag{20}$$

Therefore, two states are indistinguishable if they produce as initial conditions the same output under the same input. If the initial state x_0 is zero, we have the following definition.

Definition *2.2: A state $x \in \mathbb{R}^n$ is said to be unobservable if there exists $t_f > 0$ such that for any input function $\mathbf{u}$ defined over $[0, t_f]$ and for all $t \in [0, t_f]$*

$$\mathbf{y}(t, x, \mathbf{u}) = \mathbf{y}(t, 0, \mathbf{u}) \tag{21}$$

The set of unobservable states is a vector space.

Proposition *2.1: The set of unobservable states is a vector subspace of the state space $\mathbb{R}^n$.*

Now, we want to characterize the set of unobservable states in terms of the matrices A and C. To this aim, let us introduce the following time-varying $n \times n$ matrix

$$\mathbf{G}_O(t) := \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau \tag{22}$$

This matrix is symmetric and it is known as *observability gramian*. At the same time, define the *observability matrix*

$$O := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{23}$$

Proposition *2.2: For each $t \neq 0$: $\mathrm{Ker}\{\mathbf{G}_O(t)\} = \mathrm{Ker}\{O\}$. Moreover, the set of unobservable states is $\mathrm{Ker}\{\mathbf{G}_O(t_f)\}$.*

Note that, if $\mathrm{Ker}\{\mathbf{G}_O(t_f)\} = \{0\}$, the only unobservable state is $x = 0$ whatever the observation interval $[0, t_f]$ is. In this case, by proposition 2.2 $\mathrm{Ker}\{O\} = \{0\}$.

Definition *2.3: If $\mathrm{Ker}\{O\} = \{0\}$ then the system is said to be observable.*

Now, we are in a position to characterize the states x_a which are indistinguishable from a given x_b. Since

$$\begin{aligned} &\mathbf{y}(t, x_a, \mathbf{u}) = \mathbf{y}(t, x_b, \mathbf{u}),\ \forall t \in [0, t_f] \\ &\Leftrightarrow \mathbf{y}(t, x_a - x_b, \mathbf{u}) = \mathbf{y}(t, 0, \mathbf{u}),\ \forall t \in [0, t_f] \end{aligned} \tag{24}$$

it follows that x_a is indistinguishable from a given x_b if and only if $x_a - x_b$ is unobservable.

Proposition *2.3: The set of states x_a is indistinguishable from a given x_b is the set of states x_a for which $x_a - x_b$ is unobservable and it is equal to*

$$\{z \in \mathbb{R}^n : z = x_b + y, y \in \mathrm{Ker}\{O\}\} \tag{25}$$

Relying on the previous characterizations, we study how to reconstruct the initial states x_0 (and, therefore, the entire solution $\mathbf{x}(t, x_0, \mathbf{u})$) from the observation of the inputs and the outputs over a time interval $[0, t_f]$. The reconstruction of x_0 can be related to the observability gramian $\mathbf{G}_O$ and the unforced output response $\mathbf{y}^{(0)}(t, x_0)$. If the system is observable we claim that x_0 can be reconstructed from

$$\int_0^{t_f} e^{A^\top \theta} C^\top \mathbf{y}^{(0)}(\theta, x_0) d\theta$$

Indeed,

$$\begin{aligned} &\int_0^{t_f} e^{A^\top \theta} C^\top \mathbf{y}^{(0)}(\theta, x_0) d\theta \\ &= \int_0^{t_f} e^{A^\top \theta} C^\top C e^{A\theta} x_0 d\theta = \mathbf{G}_O(t_f) x_0 \end{aligned}$$

and therefore

$$x_0 = \mathbf{G}_O^{-1}(t_f) \int_0^{t_f} e^{A^\top \theta} C^\top \mathbf{y}^{(0)}(\theta, x_0)(x_0) d\theta$$

Proposition *2.4: If $\mathrm{Ker}\{O\} = \{0\}$, the only unobservable state is 0 and the initial state x_0 can be reconstructed from the observation of the input $\mathbf{u}$ and the ensuing output $\mathbf{y}(t)(x_0, \mathbf{u})$ over the time interval $[0, t_f]$, $t_f > 0$, as*

$$x_0 = \mathbf{G}_O^{-1}(t_f) \int_0^{t_f} e^{A^\top \theta} C^\top \mathbf{y}(\theta, x_0) d\theta$$

Exercize *2.1: Consider the double integrator*

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ \dot{\mathbf{x}}_2(t) &= \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{x}_1(t) \end{aligned}$$

and calculate the set of unobservable states. Reconstruct the initial value of the state x_0 from the unforced output response $1 + t$ over the time interval $[0, t_f]$ with $t_f = 1$ sec.

In this case

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

The *observability matrix* is

$$O := \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

and $\mathrm{Ker}\{O\} = \{0\}$. Therefore, the set of unobservable states is $\{0\}$. Also the set of indistinguishable states from any x_a is $\{x_a\}$.

The *observability gramian*

$$\begin{aligned} \mathbf{G}_O(t) &:= \int_0^t e^{A^\top \tau} C^\top C e^{A\tau} d\tau \\ &= \int_0^t \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} d\tau \\ &= \int_0^t \begin{pmatrix} 1 \\ \tau \end{pmatrix} \begin{pmatrix} 1 & \tau \end{pmatrix} d\tau = \int_0^t \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix} d\tau = \begin{pmatrix} t & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 \end{pmatrix} \end{aligned}$$

since

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{27}$$

Note that $\mathrm{Ker}\{O\} = \mathrm{Ker}\{\mathbf{G}_O(t_f)\} = \{0\}$ since $\mathbf{G}_O(t_f)$ is nonsingular for each $t_f \neq 0$. Next, we see how to reconstruct x_0 from the unforced output response $1 + t$ over the time interval $[0, t_f]$ with $t_f = 1$ sec. Then

$$\begin{aligned} \mathbf{G}_O^{-1}(t_f) e^{A^\top t} C^\top &= \frac{12}{t_f^4} \begin{pmatrix} \frac{1}{3}t_f^3 & -\frac{1}{2}t_f^2 \\ -\frac{1}{2}t_f^2 & t_f \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} \\ &= \frac{12}{t_f^4} \begin{pmatrix} \frac{1}{3}t_f^3 - \frac{1}{2}t_f^2 t \\ -\frac{1}{2}t_f^2 + t_f t \end{pmatrix} = 12 \begin{pmatrix} \frac{1}{3} - \frac{1}{2}t \\ -\frac{1}{2} + t \end{pmatrix} \end{aligned} \tag{28}$$

The initial state x_0 is reconstructed from the unforced response $1 + t$ as

$$\begin{aligned} x_0 &= \mathbf{G}_O^{-1}(t_f) \int_0^{t_f} e^{A^\top \theta} C^\top \mathbf{y}_\theta^{(unforced)}(x_0) d\theta \\ &= 12 \int_0^1 \begin{pmatrix} \frac{1}{3} - \frac{1}{2}\theta \\ -\frac{1}{2} + \theta \end{pmatrix} (1 + \theta) d\theta \\ &= 12 \int_0^1 \begin{pmatrix} \frac{1}{3} - \frac{1}{2}\theta \\ -\frac{1}{2} + \theta \end{pmatrix} (1 + \theta) d\theta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \triangleleft \end{aligned}$$

Exercize *2.2: Consider the model*

$$\begin{aligned}\dot{\mathbf{x}}_1(t) &= -\mathbf{x}_1(t)\\ \dot{\mathbf{x}}_2(t) &= \mathbf{x}_2(t) + \mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{x}_2(t)\end{aligned} \tag{29}$$

and calculate the set of indistinguishable states from $x = \begin{pmatrix}1 & 1\end{pmatrix}^\top$.

In this case

$$A = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\ B = \begin{pmatrix}0\\ 1\end{pmatrix},\ C = \begin{pmatrix}0 & 1\end{pmatrix}$$

The *observability matrix* is

$$O := \begin{pmatrix}C\\ CA\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \tag{30}$$

and $\mathrm{Ker}\{O\} = \mathrm{Span}\{\begin{pmatrix}1 & 0\end{pmatrix}^\top\}$, Therefore, the set of unobservable states is

$$\mathrm{Ker}\{O\} = \mathrm{Span}\Big\{ \begin{pmatrix}1\\ 0\end{pmatrix} \Big\}$$

(propositions 2.2). The set of indistinguishable states from $x = \begin{pmatrix}1 & 1\end{pmatrix}^\top$ is

$$\begin{aligned}&\{z \in \mathbb{R}^n : z = x + y, y \in \mathrm{Ker}\{O\}\}\\ &= \{z \in \mathbb{R}^n : z = \begin{pmatrix}c\\ 1\end{pmatrix}, c \in \mathbb{R}\}\end{aligned}$$

Therefore, all the initial states $\begin{pmatrix}c & 1\end{pmatrix}^\top$, $c \in \mathbb{R}$, cannot be reconstructed from the observation of the input $\mathbf{u}$ and the ensuing output $\mathbf{y}(t, x_0, \mathbf{u})$ over any time interval $[0, t_f]$, $t_f > 0$. ◁

III. Eigenvalues assignment and stabilization

We have seen that for controllable systems it is possible to drive the state from any initial state x_0 to 0 within any given time t_f with an input function

$$\mathbf{u}(t) := -B^\top e^{A^\top (t_f - t)} \mathbf{G}^{-1}(t_f) e^{A t_f} x_0$$

where $\mathbf{G}(t_f)$ is the controllability gramian. We have also seen that this control input lacks in robustness. In this section, we want to study the problem of steering all the states from any initial state x_0 to 0 within an *infinite time interval* (i.e. $t_f = +\infty$) with a given convergence rate. This can be formulated as a problem of "assigning" the eigenvalues of the matrix A in such a way that the natural modes are all convergent with the given convergence rates.

A. Eigenvalues assignment via state feedback

Consider the class of control laws

$$\mathbf{u}(t) = F\mathbf{x}(t) + \mathbf{v}(t) \tag{31}$$

with matrix $F(1 \times n)$ and v is the new control input. These control laws are commonly referred to as *static state feedback* laws, in the sense that the state information is used to implement the control law and the relation between x and v on one side and u on the other is instantaneous, i.e. no dynamics. The system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$, subject to the control input (31), is represented by the new equations

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t) + B\mathbf{v}(t) \tag{32}$$

In other words, the matrix A has been changed into $A + BF$ (see Figure 1). If we are able to find a matrix F in such a way that the eigenvalues of $A + BF$ are equal to a given set $\{\lambda_1^*, \dots, \lambda_n^*\} \subset \mathbb{C}$, then the natural modes are all convergent with rate convergence corresponding to the given negative real parts. On the other hand, this guarantees also asymptotic stability of (32). This problem can be formulated as follows. Let $\sigma(N)$ denote the *spectrum* of a square matrix N, i.e. the set of its eigenvalues.

Definition *3.1: (Spectrum assignment by state feedback). Given numbers* $\{\lambda_1^*, \dots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ *for all* $i = 1, \dots, n$, *either real or complex conjugate, find a matrix* $F(1 \times n)$ *such that* $\sigma(A + BF) = \{\lambda_1^*, \dots, \lambda_n^*\}$.

A necessary and sufficient condition for the existence of F is the following.

Proposition *3.1: The Spectrum assignment by state feedback problem is solvable if and only if the system is controllable, i.e. the controllability matrix* R *is nonsingular. In particular, if* R *is nonsingular then for given numbers* $\{\lambda_1^*, \dots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ *for all* $i = 1, \dots, n$, *the Spectrum assignment problem is solvable with*

$$F = -\gamma p^*(A) \tag{33}$$

where γ *is the last row of* R^{-1} *and* $p^*(\lambda) := \prod_{j=1}^n (\lambda - \lambda_j^*)$.

1) Controllability as a necessary condition for the solvability of the Spectrum assignment problem: We want to show that a necessary condition for the solvability of the Spectrum assignment by state feedback problem is the controllability of the system. To this aim, we will assume to have a matrix F which solves the Spectrum assignment problem. If the system is not controlable, we will come to a contradiction. Indeed, if the system not controlable, R is not nonsingular and say

$$n > r := \mathrm{rank}_{\mathbb{R}}\{R\}$$

Let $v_1, \dots, v_r \in \mathbb{R}^n$ be a basis of $\mathrm{Span}\{R\}$ (we may assume that $v_i := A^{i-1}B$, $i = 1, \dots, r$, i.e. the first r columns of R) and define

$$T := \begin{pmatrix}v_1 & \cdots & v_r & w_1 & \cdots & w_{n-r}\end{pmatrix}^{-1} \tag{34}$$

where $w_1, \dots, w_{n-r} \in \mathbb{R}^n$ are such that $v_1, \dots, v_r, w_1, \dots, w_{n-r}$ altogether are a basis of $\mathbb{R}^n$, i.e. the matrix

$$\begin{pmatrix}v_1 & \cdots & v_r & w_1 & \cdots & w_{n-r}\end{pmatrix}$$

is nonsingular. If we transform the state as $z = Tx$ this will induce a transformation on the matrices A, B, F as follows

$$\tilde{A} = TAT^{-1},\ ;\tilde{B} = TB,\ \tilde{F} = FT. \tag{35}$$

It can be seen that there exist matrices $\tilde{A}_{11}(r \times r), \tilde{A}_{12}(r \times (n-r)), \tilde{A}_{22}((n-r) \times (n-r))$ and $\tilde{B}_1(r \times 1)$ such that

$$\tilde{A} = TAT^{-1} = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ 0 & \tilde{A}_{22}\end{pmatrix},\ \tilde{B} = TB = \begin{pmatrix}\tilde{B}_1\\ 0\end{pmatrix} \tag{36}$$

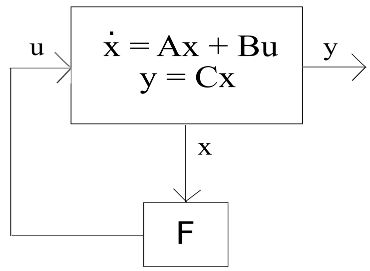

Figure 1. Control scheme for eigenvalue assignment by state feedback.

(this follows from $A\text{Span}\{R\} \subset \text{Span}\{R\}$ and $B \in \text{Span}\{R\}$).

Note that for all $\lambda \in \mathbb{C}^n$

$$\begin{aligned}&\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F})) = \det(\lambda I - T(A + BF)T^{-1})\\&= \det(T(\lambda I - (A + BF))T^{-1})\\&= \det T \det(\lambda I - (A + BF)) \det T^{-1}\end{aligned}$$

and, therefore, the roots of $\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F}))$ and $\det(\lambda I - (A + BF))$ are the same, i.e.

$$\sigma(\tilde{A} + \tilde{B}\tilde{F}) = \sigma(A + BF) \tag{37}$$

From (36) and writing $\tilde{F} := \begin{pmatrix} \tilde{F}_1 & \tilde{F}_2 \end{pmatrix}$ for some matrices $\tilde{F}_1(1 \times r)$ and $\tilde{F}_2(1 \times (n - r))$, then

$$\begin{aligned}&\sigma(A + BF) = \sigma(\tilde{A} + \tilde{B}\tilde{F})\\&= \sigma\begin{pmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{F}_1 & \tilde{A}_{12} + \tilde{B}_1\tilde{F}_2 \\ 0 & \tilde{A}_{22} \end{pmatrix}\\&= \sigma(\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1) \cup \sigma(\tilde{A}_{22})\end{aligned}$$

Moreover, $\sigma(\tilde{A}_{22}) \subset \sigma(A)$ since $\sigma(\tilde{A}) = \sigma(A)$. It follows that the eigenvalues of A which correspond to $\sigma(\tilde{A}_{22})$ cannot be changed into any given subset of $\{\lambda_1^*, \ldots, \lambda_n^*\}$ and this contradicts the existence of F which solves the Spectrum assignment problem.

2) Controllability as a sufficient condition for the solvability of the Spectrum assignment problem: Ackermann formula for spectrum assignment: Next, we want to show that a sufficient condition for the solvability of the Spectrum assignment by state feedback problem is the controllability of the system. This is the constructive part of our result and gives a matrix F, defined in (33), which assigns the given spectrum to the matrix A (i.e. solves the Spectrum assignment problem).

Assume that the system is controllable, i.e. R is nonsingular. Let γ be the last row of R^{-1} and define the reals $a_0^*, \ldots, a_{n-1}^*$ in such a way that

$$\begin{aligned}p^*(\lambda) &:= \prod_{j=1}^{n}(\lambda - \lambda_j^*)\\&:= a_0^* + a_1^*\lambda + \cdots + a_{n-1}^*\lambda^{n-1} + \lambda^n\end{aligned} \tag{38}$$

Note that the roots of $p^*(\lambda)$ are exactly the given $\{\lambda_1^*, \ldots, \lambda_n^*\}$.

We will outline the procedure for obtaining the matrix F in suitable new coordinates $\tilde{F}$ (for which the matrices A and B have *ad hoc* expressions) and then back to F in the original coordinates. Define

$$T := \begin{pmatrix} \gamma \\ \gamma A \\ \vdots \\ \gamma A^{n-1} \end{pmatrix}$$

It can be shown that T is nonsingular (this follows from the invertibility of R). It can be also seen that

$$\begin{aligned}&\tilde{A} = TAT^{-1}\\&= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}\end{aligned} \tag{39}$$

and

$$\tilde{B} = TB = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{40}$$

where $a_0, \ldots, a_{n-1}$ are the coefficients of the characteristic polynomial $p(\lambda)$ of A:

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n \tag{41}$$

Define

$$\begin{aligned}&\tilde{F}\\&:= \begin{pmatrix} a_0 - a_0^* & a_1 - a_1^* & \cdots & a_{n-2} - a_{n-2}^* & a_{n-1} - a_{n-1}^* \end{pmatrix}\end{aligned}$$

then

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0^* & -a_1^* & -a_2^* & \cdots & -a_{n-2}^* & -a_{n-1}^* \end{pmatrix} \tag{42}$$

and

$$\begin{aligned}&\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F}))\\&= a_0^* + a_1^*\lambda + \cdots + a_{n-1}^*\lambda^{n-1} + \lambda^n = p^*(\lambda)\end{aligned} \tag{43}$$

Therefore, $\{\lambda_1^*, \dots, \lambda_n^*\}$ are the eigenvalues of $\tilde{A} + \tilde{B}\tilde{F}$. Getting back in original coordinates

$$F := \tilde{F}T \tag{44}$$

But $\sigma(\tilde{A} + \tilde{B}\tilde{F}) = \sigma(A + BF)$. Indeed,

$$\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F})) = \det(\lambda I - (TAT^{-1} + TB\tilde{F}TT^{-1}))$$
$$= \det(T(\lambda I - (A + BF))T^{-1})$$
$$= \det T \det(\lambda I - (A + BF)) \det T^{-1}$$

We conclude that the Spectrum assignment problem by state feedback is solved by F in (44). Moreover, it is easy to see, after some manipulations, that

$$F = -\gamma p^*(A) \tag{45}$$

where γ is the last row of R^{-1} and

$$p^*(A) = a_0 I + a_1 A + \dots + a_{n-1} A^{n-1} + A^n. \tag{46}$$

The above formula (45) for F is known as Ackermann formula for spectrum assignment.

B. Stabilization via state feedback

If the system is not controllable there is a subset of the eigenvalues of $A + BF$ that are invariant under any choice of F (*invariant spectrum*). This subset is exactly the spectrum of the matrix $\tilde{A}_{22}$ (see (36)) which is a subset of the spectrum of A. The matrix $\tilde{A}_{22}$ can be calculated from TAT^{-1} where T is defined as in (34). Even if the system is not controlable, it is possible to find a F such that $\sigma(A + BF) = \{\lambda_1^*, \dots, \lambda_n^*\}$ as long as the invariant spectrum of $A + BF$ is a subset of the given set $\{\lambda_1^*, \dots, \lambda_n^*\}$. Denote by $\mathfrak{F}_R$ this invariant spectrum of $A + BF$.

Proposition *3.2: Given numbers $\{\lambda_1^*, \dots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ for all $i = 1, \dots, n$, either real or complex conjugate, there exists a matrix $F(1 \times n)$ such that $\sigma(A + BF) = \{\lambda_1^*, \dots, \lambda_n^*\}$ if and only if $\mathfrak{F}_R \subset \{\lambda_1^*, \dots, \lambda_n^*\}$.*

1) Design of stabilizing state feedback controllers: We show how to design F when $\mathfrak{F}_R \subset \mathfrak{L} = \{\lambda_1^*, \dots, \lambda_n^*\}$. Let $r := \operatorname{rank}_{\mathbb{R}}\{R\}$. Under the coordinate transformation $z = Tx$, where T is defined as in (34), the matrices A and B are trasformed into

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \tag{47}$$

with $\tilde{A}_{11}(r \times r), \tilde{A}_{12}(r \times (n-r)), \tilde{A}_{22}((n-r) \times (n-r))$ and $\tilde{B}_1(r \times 1)$. We want to show that

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{r-1}\tilde{B}_1 \end{pmatrix} = r \tag{48}$$

This means that the Eigenvalues assignment problem is solvable with matrices $\tilde{A}_{11}$ and $\tilde{B}_1$. As a matter of fact, since $TA^jT^{-1} = (TAT^{-1})^j = \tilde{A}^j$ for all integer j, we have

$$r = \operatorname{rank}_{\mathbb{R}}\{R\} = \operatorname{rank}_{\mathbb{R}}\{\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}\}$$
$$= \operatorname{rank}_{\mathbb{R}}\{T\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}\}$$
$$= \operatorname{rank}_{\mathbb{R}}\{\begin{pmatrix} TB & TAT^{-1}TB & \cdots & TA^{n-1}T^{-1}TB \end{pmatrix}\}$$
$$= \operatorname{rank}_{\mathbb{R}}\{\begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix}\}$$
$$= \operatorname{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{r-1}\tilde{B}_1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right\}$$
$$= \operatorname{rank}_{\mathbb{R}}\{\begin{pmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \cdots & \tilde{A}_{11}^{r-1}\tilde{B}_1 \end{pmatrix}\} \tag{49}$$

i.e. (48). Define

$$\tilde{F} := \begin{pmatrix} \tilde{F}_1 & 0 \end{pmatrix} \tag{50}$$

with $\tilde{F}_1(1 \times r)$ such that

$$\sigma(\tilde{A}_{11} + \tilde{F}_1\tilde{B}_1) = \mathfrak{L} \backslash \mathfrak{F}_R$$

($\tilde{F}_1$ exists by virtue of (48) and proposition 3.1) and

$$F := \tilde{F}T \tag{51}$$

Note that for all $\lambda \in \mathbb{C}^n$

$$\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F})) = \det(\lambda I - T(A + BF)T^{-1})$$
$$= \det(T(\lambda I - (A + BF))T^{-1})$$
$$= \det T \det(\lambda I - (A + BF)) \det T^{-1}$$

and, therefore, the roots of $\det(\lambda I - (\tilde{A} + \tilde{B}\tilde{F}))$ and $\det(\lambda I - (A + BF))$ are the same, i.e.

$$\sigma(\tilde{A} + \tilde{B}\tilde{F}) = \sigma(A + BF) \tag{52}$$

Moreover,

$$\sigma(A + BF) = \sigma(\tilde{A} + \tilde{B}\tilde{F}) = \sigma(\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1) \cup \sigma(\tilde{A}_{22})$$
$$= \sigma(\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1) \cup \mathfrak{F}_R = \mathfrak{L}$$

The construction of the matrix F can be summed up as follows:

Step procedure for the design of stabilizing state-feedback controllers

(i) Let $r := \operatorname{rank}_{\mathbb{R}}\{R\}$. Find $w_1, \dots, w_{n-r}$ such that $B, AB, \cdots, A^{r-1}B, w_1, \dots, w_{n-r}$ is a basis of $\mathbb{R}^n$ and define T as

$$T := \begin{pmatrix} B & AB & \cdots & A^{r-1}B & w_1 & \cdots & w_{n-r} \end{pmatrix}^{-1} \tag{53}$$

(ii) Find the matrices $\tilde{A}_{11}(r \times r), \tilde{A}_{12}(r \times (n-r)), \tilde{A}_{22}((n-r) \times (n-r))$ and $\tilde{B}_1(r \times 1)$ for which

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \ \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \tag{54}$$

(iii) Find $\tilde{F}_1(1 \times r)$ such that $\sigma(\tilde{A}_{11} + \tilde{F}_1\tilde{B}_1) = \mathfrak{L} \backslash \mathfrak{F}_R$. In particular,

$$\tilde{F}_1 = -\gamma_s p_s^*(\tilde{A}_{11}) \tag{55}$$

where $p_s^*(\lambda) := \alpha_0^* + \alpha_1^*\lambda + \dots + \alpha_{r-1}^*\lambda^{r-1} + \lambda^r$ is the polynomial which has the roots in $\mathfrak{L} \backslash \mathfrak{F}_R$ and

$$p_s^*(\tilde{A}_{11}) := \alpha_0^* I + \alpha_1^* \tilde{A}_{11} + \dots + \alpha_{r-1}^* \tilde{A}_{11}^{r-1} + \tilde{A}_{11}^r \tag{56}$$

and γ_s is the last row of the inverse of

$$R_s := \begin{pmatrix} \tilde{B}_1 & \tilde{A}_1\tilde{B}_1 & \cdots & \tilde{A}_1^{r-1}\tilde{B}_1 \end{pmatrix} \tag{57}$$

(iv) Define

$$\tilde{F} := \begin{pmatrix} \tilde{F}_1 & 0 \end{pmatrix} \tag{58}$$

and, finally, set

$$F := \tilde{F}T \tag{59}$$

Exercize *3.1: Given*

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \ \beta \in \mathbb{R} \tag{60}$$

find, if possible, F such that $\sigma(A+BF)=\{-2,-2\}$.

The controllability matrix R is

$$R = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \beta & 1-2\beta \end{pmatrix}$$

Therefore, the system is controllable if and only if $\beta \neq 1$.

Case $\beta = 1$. The system is not controllable. We use proposition 3.2. In this case we have to check if the invariant spectrum of A is a subset of $\{-2,-2\}$. Note that

$$\begin{aligned} r &:= \mathrm{rank}_{\mathbb{R}}\{R\} = \mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix} B & AB \end{pmatrix}\} \\ &= \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right\} = 1 \end{aligned} \tag{61}$$

For calculating the invariant spectrum of A, we change the coordinates as follows. A basis of $\mathrm{Span}\{R\}$ is

$$v_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(we can take the first column of R since $r=1$). Define

$$T = \begin{pmatrix} v_1 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

In the new coordinates $z = Tx$

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix},\ \tilde{B} = B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{62}$$

Since $r=1$, $\tilde{A}_{11}=-1$, $\tilde{A}_{12}=1$, $\tilde{A}_{22}=-2$ and $\tilde{B}_1=1$ and we conclude that the invariant spectrum is $\mathfrak{F}_R := \sigma(\tilde{A}_{22}) = \{-2\}$. Since $\mathfrak{F}_R := \{-2\} \subset \mathfrak{L} := \{-2,-2\}$, by proposition 3.2 there exists F such that $\sigma(A+BF)=\{-2,-2\}$. Let construct this matrix F. We have

$$\tilde{F} := \begin{pmatrix} \tilde{F}_1 & 0 \end{pmatrix} \tag{63}$$

where $\tilde{F}_1$ is such that $\sigma(\tilde{A}_{11}+\tilde{B}_1\tilde{F}_1) = \mathfrak{L}\backslash\mathfrak{F}_R = \{-2\}$. Such $\tilde{F}_1$ exists since $\tilde{A}_{11}$ and $\tilde{B}_1$ represent a controllable system, indeed

$$\mathrm{rank}_{\mathbb{R}}\{\tilde{B}_1\} = 1 = r \tag{64}$$

On the hand, $\sigma(\tilde{A}_{11}+\tilde{B}_1\tilde{F}_1)=\{-2\}$ if and only if $\tilde{F}_1=-1$. Finally, define

$$F := \tilde{F}T = \begin{pmatrix} -1 & 0 \end{pmatrix} T = -\frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix} \tag{65}$$

We can check that

$$\begin{aligned} \sigma(A+BF) &= \sigma\Big(\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} \Big) \\ &= \sigma\begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix} = \{-2,-2\} \end{aligned} \tag{66}$$

Case $\beta \neq 1$. The system is controllable. In this case we use proposition 3.1. The spectrum to be assigned is $\{-2,-2\}$ and, therefore,

$$p^*(\lambda) := (\lambda+2)^2$$

The matrix F which solves the Spectrum assignment problem by state feedback with $\mathfrak{L} := \{-2,-2\}$ is

$$F = -\gamma p^*(A) \tag{67}$$

where γ is the last row of R^{-1}. Since

$$R = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \beta & 1-2\beta \end{pmatrix}$$

then

$$R^{-1} = \frac{1}{1-\beta}\begin{pmatrix} 1-2\beta & 1 \\ -\beta & 1 \end{pmatrix}$$

and

$$\gamma := \frac{1}{1-\beta}\begin{pmatrix} -\beta & 1 \end{pmatrix}$$

Moreover

$$\begin{aligned} p^*(A) &:= (A+2I)^2 = A^2+4A+4I \\ &= \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}^2 + 4\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Finally,

$$F = -\gamma p^*(A) = -\frac{1}{1-\beta}\begin{pmatrix} -\beta & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \end{pmatrix} \tag{68}$$

We can check that

$$\begin{aligned} \sigma(A+BF) &= \sigma\Big(\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 \\ \beta \end{pmatrix}\begin{pmatrix} -1 & 0 \end{pmatrix} \Big) \\ &= \sigma\begin{pmatrix} -2 & 0 \\ 1-\beta & -2 \end{pmatrix} = \{-2,-2\}.\triangleleft \end{aligned} \tag{69}$$

C. The PBH controllability criterion

The invariant spectrum $\mathfrak{F}_R$ of $A+BF$ can be determined without a coordinate transformation by calling upon the so-called *PBH controllability criterion* (the acronym PBH is given by the initials of the researchers Popov, Belevitch and Hautus who introduced the criterion).

Proposition *3.3: (PBH controllability criterion). A necessary and sufficient condition for reachability, i.e. R nonsingular, is*

$$\mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix} \lambda I - A & B \end{pmatrix}\} = n$$

for each $\lambda \in \sigma(A)$. Moreover,

$$\mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix} \lambda I - A & B \end{pmatrix}\} \quad \begin{cases} < n & \Rightarrow \lambda \in \mathfrak{F}_R \\ = n & \Rightarrow \lambda \notin \mathfrak{F}_R \end{cases} \tag{70}$$

Proposition *3.4: (PBH controllability criterion). A necessary and sufficient condition for reachability, i.e. R nonsingular, is that*

$$\mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix} \lambda I - A & B \end{pmatrix}\} = n$$

for each $\lambda \in \mathbb{C}^n$.

Exercize *3.2: We want to revisit the results of example 3.1 through the PBH controllability criterion.*

Case $\beta = 1$. The system is not controllable. It is easily seen that $\sigma(A)=\{-1,-2\}$. By proposition 3.3

$$\begin{aligned} \lambda = -2 &\Rightarrow \mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix} \lambda I - A & B \end{pmatrix}\} \\ &= \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \right\} = 1 < n = 2 \Rightarrow \{-2\} \in \mathfrak{F}_R \end{aligned}$$

and

$$\lambda = -1 \Rightarrow \text{rank}_{\mathbb{R}}\{\begin{pmatrix}\lambda I - A & B\end{pmatrix}\}$$
$$= \text{rank}_{\mathbb{R}}\Big\{\begin{pmatrix}0 & 0 & 1\\ -1 & 1 & 1\end{pmatrix}\Big\} = 2 = n \Rightarrow \{-1\} \notin \mathfrak{F}_R$$

Therefore, there exists F such that $\sigma(A+BF) = \{-2,-2\}$.
Case $\beta \neq 1$. The system is controllable and $\mathfrak{F}_R$ is empty. By proposition 3.3

$$\lambda = -2 \Rightarrow \text{rank}_{\mathbb{R}}\{\begin{pmatrix}\lambda I - A & B\end{pmatrix}\}$$
$$= \text{rank}_{\mathbb{R}}\Big\{\begin{pmatrix}-1 & 0 & 1\\ -1 & 0 & \beta\end{pmatrix}\Big\} = 2 = n \Rightarrow \{-2\} \notin \mathfrak{F}_R$$

and

$$\lambda = -1 \Rightarrow \text{rank}_{\mathbb{R}}\{\begin{pmatrix}\lambda I - A & B\end{pmatrix}\}$$
$$= \text{rank}_{\mathbb{R}}\Big\{\begin{pmatrix}0 & 0 & 1\\ -1 & 1 & \beta\end{pmatrix}\Big\} = 2 = n \Rightarrow \{-1\} \notin \mathfrak{F}_R. \lhd$$

D. Design of asymptotic observers

We have seen that for observable systems it is possible to reconstruct the initial state x_0, and therefore, the state $\mathbf{x}(t, x_0, \mathbf{u})$, through the observations of the unforced output response $\mathbf{y}^{(0)}(t, x_0)$ over a time interval $[0, t_f]$ as

$$x_0 = \mathbf{G}_O^{-1}(t_f)\int_0^{t_f} e^{A^\top\theta}C^\top\mathbf{y}(t)^{(unforced)}(x_0)d\theta$$

where $\mathbf{G}_O(\cdot)$ is the observability gramian. In this section, we want to study the problem of reconstructing or estimating the state $\mathbf{x}(t, x_0, \mathbf{u})$ through observations of $\mathbf{y}$ and $\mathbf{u}$ over an *infinite time interval* (i.e. $T = +\infty$) with a given convergence rate. This can be formulated as a problem of "assigning" the convergence rate of the error between the state and its reconstruction. Consider the class of "state estimators"

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K(\mathbf{y}(t) - C\hat{\mathbf{x}}(t)). \tag{71}$$

If $e := x - \hat{x}$ is the *estimation error*, then the error dynamics is described by the equations

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\\ &= A\hat{\mathbf{x}} + B\mathbf{u}(t) + K(\mathbf{y}(t) - C\hat{\mathbf{x}}(t)) - A\mathbf{x}(t) - B\mathbf{u}(t)\\ &= (A - KC)\mathbf{e}(t),\end{aligned} \tag{72}$$

with matrix $K(n\times 1)$. If we are able to find a matrix K in such a way that the eigenvalues of $A - KC$ are equal to a given set $\{\lambda_1^*, \ldots, \lambda_n^*\}$ with negative real part, then the natural modes of (72) are all convergent with rate convergence corresponding to the given negative real parts and the state is reconstructed from the output with the assigned rate. The system (71) is known as *asymptotic state observer* and dynamically and asymptotically reconstructs the state $\mathbf{x}(t)$ with $\hat{\mathbf{x}}(t)$.

Our problem can be formulated as follows.

Definition *3.2: (Asymptotic state observation). Given numbers* $\{\lambda_1^*, \ldots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ *for all* $i = 1, \ldots, n$, *either real or complex conjugate, find a matrix* $K(n \times 1)$ *such that* $\sigma(A - KC) = \{\lambda_1^*, \ldots, \lambda_n^*\}$.

Note that

$$\sigma(A - KC) = \sigma((A - KC)^\top) = \sigma(A^\top - C^\top K^\top) \tag{73}$$

If we establish the following equivalences

$$\begin{aligned}A &\leftrightarrow A^\top\\ B &\leftrightarrow C^\top\\ F &\leftrightarrow -K^\top\end{aligned} \tag{74}$$

we find out that it is possible to assign the eigenvalues of $A - KC$ with some matrix K if and only it is possible to assign the eigenvalues of $A + BF$ with some matrix F. Therefore, a necessary and sufficient condition for the existence of K comes directly from proposition 3.5.

Proposition *3.5: The Asymptotic state observation problem is solvable if and only if the system is observable, i.e. the observability matrix* O *is nonsingular. In particular, if* O *is nonsingular then for any given numbers* $\{\lambda_1^*, \ldots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ *for all* $i = 1, \ldots, n$, *the Asymptotic state observation problem is solvable with*

$$K = p^*(A)\gamma \tag{75}$$

where γ *is the last column of* O^{-1} *and* $p^*(\lambda) := \prod_{j=1}^n(\lambda - \lambda_j^*)$.

1) Observability as a sufficient condition for the solvability of the asymptotic state observation problem: dual Ackermann formula: Since the assignment of the eigenvalues of $A - KC$ with some matrix K is equivalent to the assignment of the eigenvalues of $A + BF$ with some matrix F under the equivalences (74), by proposition 3.1 the Asymptotic state observation problem is solvable if and only the controllability matrix defined with $A \leftrightarrow A^\top$ and $B \leftrightarrow C^\top$, i.e. the matrix

$$R := \begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix} \tag{76}$$

is nonsingular. Since for all $j \in \mathbb{N}$

$$(A^\top)^j = (A^j)^\top \tag{77}$$

and

$$R := \begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix} = \begin{pmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{pmatrix}^\top = O^\top \tag{78}$$

it follows that the Asymptotic state observation problem is solvable if and only the observability matrix is nonsingular.

Moreover, by proposition 3.1 if O is nonsingular and under the equivalences (74), for any given numbers $\{\lambda_1^*, \ldots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ for all $i = 1, \ldots, n$, the Asymptotic state observation problem is solvable with

$$-K^\top := F = -\gamma p^*(A^\top) \tag{79}$$

where γ denotes the last row of

$$R^{-1} = \begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix}^{-1} \tag{80}$$

We have by changing signs and transposing (79)

$$K = (p^*(A^\top))^\top\gamma^\top \tag{81}$$

On account of (77)

$$\begin{aligned}&(p^*(A^\top))^\top = (a_0^* I + a_1^* A^\top + a_2^*(A^\top)^2 + \dots\\&+a_{n-1}^*(A^\top)^{n-1} + (A^\top)^n)^\top\\&(p^*(A^\top))^\top = (a_0^* I + a_1^* A^\top + a_2^*(A^2)^\top + \dots\\&+a_{n-1}^*(A^{n-1})^\top + (A^n)^\top)^\top\\&= a_0^* I + a_1^* A + a_2^* A^2 + \dots + a_{n-1}^* A^{n-1} + A^n = p^*(A)\end{aligned}$$

Also, note that since transpose and inverse commute

$$\begin{aligned}&\left(\begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix}^{-1}\right)^\top\\&= \left(\left(\begin{pmatrix}C\\CA\\\vdots\\CA^{n-1}\end{pmatrix}^\top\right)^{-1}\right)^\top = \left(\left(\begin{pmatrix}C\\CA\\\vdots\\CA^{n-1}\end{pmatrix}^\top\right)^\top\right)^{-1}\\&= O^{-1}\end{aligned}$$

and since γ is the last row of the matrix inside the transpose on the left of the first equality, it follows that $\gamma := \gamma^\top$ is the last column of O^{-1}. This gives (75) back.

2) Observability as a necessary condition for the solvability of the asymptotic state observation problem: Let's see that observability is a necessary condition for the solvability of the Asymptotic state observation problem. If the system is not observable, under the equivalences (74) there exist nonsingular $T(n \times n)$ and matrices $\tilde{A}_{11}^\top((n-s)\times(n-s))$, $\tilde{A}_{12}^\top((n-s)\times s)$, $\tilde{A}_{22}^\top(s \times s)$ and $\tilde{C}_1^\top((n-s)\times 1)$, with

$$\begin{aligned}n - s &:= \mathrm{rank}_{\mathbb{R}}\{\begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix}\}\\&= \mathrm{rank}_{\mathbb{R}}\{O^\top\} = \mathrm{rank}_{\mathbb{R}}\{O\}\end{aligned} \tag{82}$$

such that

$$TA^\top T^{-1} = \begin{pmatrix}\tilde{A}_{11}^\top & \tilde{A}_{12}^\top\\ 0 & \tilde{A}_{22}^\top\end{pmatrix},\; TC^\top = \begin{pmatrix}\tilde{C}_1^\top\\0\end{pmatrix} \tag{83}$$

Indeed, T is defined as follows. Let $v_1, \dots, v_{n-s} \in \mathbb{R}^n$ be a basis of

$$\mathrm{Span}\{\begin{pmatrix}C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top\end{pmatrix}\} = \mathrm{Span}\{O^\top\}$$

(we may assume $v_i := (A^\top)^{i-1}C^\top$) and

$$T := \begin{pmatrix}v_1 & \cdots & v_{n-s} & w_1 & \cdots & w_s\end{pmatrix}^{-1} \tag{84}$$

where $w_1, \dots, w_s \in \mathbb{R}^n$ are chosen independent each other and from $v_1, \cdots, v_s$, i.e. $v_1, \cdots, v_{n-s}, w_1, \cdots, w_s$ are a basis of $\mathbb{R}^n$. Taking transposes in (83), we conclude that there exist matrices $\tilde{A}_{11}((n-s)\times(n-s))$, $\tilde{A}_{12}(s\times(n-s))$, $\tilde{A}_{22}(s\times s)$ and $\tilde{C}_1(1\times s)$ such that

$$SAS^{-1} = \begin{pmatrix}\tilde{A}_{11} & 0\\ \tilde{A}_{12} & \tilde{A}_{22}\end{pmatrix},\; CS^{-1} = \begin{pmatrix}\tilde{C}_1 & 0\end{pmatrix} \tag{85}$$

where $S := (T^\top)^{-1} = (T^{-1})^\top$. Note that, since inverse and transpose commute so that

$$S := (T^\top)^{-1} = \begin{pmatrix}C\\CA\\\vdots\\CA^{n-s-1}\\w_1^\top\\\vdots\\w_s^\top\end{pmatrix} \tag{86}$$

E. Design of state detectors

It is clear from (85) that if the system is not observable there is a non-empty subset of the eigenvalues of $A - KC$ that is invariant under any choice of K (*invariant spectrum*). In this case, the system (71) is known as *state detector* and dynamically and asymptotically reconstructs the state $\mathbf{x}(t)$ with $\hat{\mathbf{x}}(t)$ but not with guaranteed rate (since some of the eigenvalues of $A - KC$ are not assignable). The invariant subset under any choice of K is exactly the spectrum of the matrix $\tilde{A}_{22}$ (see 85) which is a subset of the spectrum of A. Indeed, with the coordinate transformation $z = Sx$ the matrices $A - KC$ is transformed into $\tilde{A} - \tilde{K}\tilde{C}$ with

$$\tilde{A} = SAS^{-1} = \begin{pmatrix}\tilde{A}_{11} & 0\\ \tilde{A}_{12} & \tilde{A}_{22}\end{pmatrix},\; \tilde{C} = CS^{-1} = \begin{pmatrix}\tilde{C}_1 & 0\end{pmatrix} \tag{87}$$

and

$$\tilde{K} = SK \tag{88}$$

Note that for all $\lambda \in \mathbb{C}^n$

$$\begin{aligned}&\det(\lambda I - (\tilde{A} - \tilde{K}\tilde{C})) = \det(\lambda I - S(A - KC)S^{-1})\\&= \det(S(\lambda I - (A - KC))S^{-1})\\&= \det S \det(\lambda I - (A - KC)) \det S^{-1}\end{aligned}$$

and, therefore, the roots of $\det(\lambda I - (\tilde{A} - \tilde{K}\tilde{C}))$ and of $\det(\lambda I - (A - KC))$ are the same, i.e.

$$\sigma(\tilde{A} - \tilde{K}\tilde{C}) = \sigma(A - KC) \tag{89}$$

Assuming that

$$\tilde{K} := \begin{pmatrix}\tilde{K}_1\\\tilde{K}_2\end{pmatrix}$$

for some matrices $\tilde{K}_1((n-s)\times 1)$ and $\tilde{K}_2(s\times 1)$, then

$$\begin{aligned}&\sigma(A - KC) = \sigma(\tilde{A} - \tilde{K}\tilde{C}) = \sigma\begin{pmatrix}\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1 & 0\\ \tilde{A}_{12} - \tilde{K}_2\tilde{C}_1 & \tilde{A}_{22}\end{pmatrix}\\&= \sigma(\tilde{A}_{11} - \tilde{K}_1\tilde{C}_1) \cup \sigma(\tilde{A}_{22})\end{aligned} \tag{90}$$

Moreover, $\sigma(\tilde{A}_{22}) \subset \sigma(A)$ since $\sigma(\tilde{A}) = \sigma(A)$. It follows that the eigenvalues of A which correspond to $\sigma(\tilde{A}_{22})$ are invariant under any choice of K. The matrix $\tilde{A}_{22}$ can be calculated from SAS^{-1} where S is defined as in (86).

Even if the system is not observable, it is possible to find a K such that $\sigma(A - KC) = \{\lambda_1^*, \dots, \lambda_n^*\}$ as long as the invariant spectrum of $A - KC$ is a subset of the given set $\{\lambda_1^*, \dots, \lambda_n^*\}$. Denote by $\mathfrak{F}_O$ this invariant spectrum of $A - KC$.

Proposition *3.6: Given numbers* $\{\lambda_1^*, \ldots, \lambda_n^*\}$, $\lambda_i^* \in \mathbb{C}^-$ *for all* $i = 1, \ldots, n$, *either real or complex conjugate, there exists a matrix* $K(n \times 1)$ *such that* $\sigma(A - KC) = \{\lambda_1^*, \ldots, \lambda_n^*\}$ *if and only if* $\mathfrak{F}_O \subset \{\lambda_1^*, \ldots, \lambda_n^*\}$.

Let's see how to construct K if $\mathfrak{F}_O \subset \mathfrak{L} := \{\lambda_1^*, \ldots, \lambda_n^*\}$. Assume that $\mathfrak{F}_O \subset \mathfrak{L}$. Let $n - s := \mathrm{rank}_{\mathbb{R}}\{O\}$. Under the coordinate transformation $z = Tx$ the matrices A and C are transformed into

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix}, \; \tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} \tag{91}$$

with $\tilde{A}_{11}((n-s) \times (n-s)), \tilde{A}_{12}(s \times (n-s)), \tilde{A}_{22}(s \times s)$ and $\tilde{C}_1(1 \times (n-s))$. We want to show that

$$\mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_1\tilde{A}_{11} \\ \vdots \\ \tilde{C}_{11}\tilde{A}_{11}^{n-s-1} \end{pmatrix} \right\} = n - s \tag{92}$$

This means that the Asymptotic state observation problem is solvable with matrices $\tilde{C}_1$ and $\tilde{A}_{11}$ (proposition 3.5). As a matter of fact, since $SA^jS^{-1} = (SAS^{-1})^j = \tilde{A}^j$ for all integer j, we have

$$\begin{aligned} s &= \mathrm{rank}_{\mathbb{R}}\{O\} \\ &= \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} T^{-1} \right\} \\ &= \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \\ \cdots \\ CT^{-1}TA^{n-1}T^{-1} \} \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \cdots \\ \tilde{C}\tilde{A}^{n-1} \end{pmatrix} \right\} \\ &= \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \tilde{C}_1 & 0 \\ \tilde{C}_1\tilde{A}_{11} & 0 \\ \vdots & \vdots \\ \tilde{C}_1\tilde{A}_{11}^{n-s-1} & 0 \\ \tilde{C}_1\tilde{A}_{11}^{n-s} & 0 \\ \vdots & \vdots \\ \tilde{C}_1\tilde{A}_{11}^{n-1} & 0 \end{pmatrix} \right\} = \mathrm{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_1\tilde{A}_{11} \\ \vdots \\ \tilde{C}_1\tilde{A}_{11}^{n-s-1} \end{pmatrix} \right\} \end{aligned} \tag{93}$$

Define

$$\tilde{K} := \begin{pmatrix} \tilde{K}_1 \\ 0 \end{pmatrix} \tag{94}$$

with $\tilde{K}_1((n-s) \times 1)$ such that

$$\sigma(\tilde{A}_{11} - \tilde{C}_1\tilde{K}_1) = \mathfrak{L} \backslash \mathfrak{F}_O$$

($\tilde{K}_1$ exists by virtue of (92) and proposition 3.5) and

$$K := S^{-1}\tilde{K} \tag{95}$$

Note that for all $\lambda \in \mathbb{C}^n$

$$\begin{aligned} &\det(\lambda I - (\tilde{A} - \tilde{K}\tilde{C})) = \det(\lambda I - S(A - KC)S^{-1}) \\ &= \det(S(\lambda I - (A - KC))S^{-1}) \\ &= \det S \det(\lambda I - (A - KC)) \det S^{-1} \end{aligned}$$

and, therefore, the roots of $\det(\lambda I - (\tilde{A} - \tilde{K}\tilde{C}))$ and of $\det(\lambda I - (A - KC))$ are the same, i.e.

$$\sigma(\tilde{A} - \tilde{K}\tilde{C}) = \sigma(A - KC) \tag{96}$$

Assuming that

$$\tilde{K} := \begin{pmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{pmatrix}$$

for some matrices $\tilde{K}_1((n-s) \times 1)$ and $\tilde{K}_2(s \times 1)$, then

$$\begin{aligned} &\sigma(A - KC) = \sigma(\tilde{A} - \tilde{K}\tilde{C}) = \sigma(\tilde{A}_{11} + \tilde{C}_1\tilde{K}_1) \cup \sigma(\tilde{A}_{22}) \\ &= \sigma(\tilde{A}_{11} + \tilde{C}_1\tilde{K}_1) \cup \mathfrak{F}_O = \mathfrak{L} \end{aligned}$$

The construction of the matrix K can be summed up as follows:

Step procedure for the design of state detectors:

(i) Let $n - s := \mathrm{rank}_{\mathbb{R}}\{O\}$. Find $w_1, \ldots, w_s \in \mathbb{R}^n$ such that $C^\top, (CA)^\top, \cdots, (CA^{n-s-1})^\top, w_1, \cdots, w_s$ are a basis of $\mathbb{R}^n$ and define S as

$$S = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-s-1} \\ w_1^\top \\ \vdots \\ w_s^\top \end{pmatrix} \tag{97}$$

(ii) Find the matrices $\tilde{A}_{11}((n-s) \times (n-s)), \tilde{A}_{12}(s \times (n-s)), \tilde{A}_{22}(s \times s)$ and $\tilde{C}_1(1 \times s)$ for which

$$\tilde{A} = SAS^{-1} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix}, \; \tilde{C} = CS^{-1} = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} \tag{98}$$

(iii) Find $\tilde{K}_1((n-s) \times 1)$ such that $\sigma(\tilde{A}_{11} - \tilde{C}_1\tilde{K}_1) = \mathfrak{L} \backslash \mathfrak{F}_O$. In particular,

$$\tilde{K}_1 = p_o^*(\tilde{A}_{11})\gamma_o \tag{99}$$

where $p_o^*(\lambda) := \beta_0^* + \beta_1^*\lambda + \cdots + \beta_{n-s-1}^*\lambda^{n-s-1} + \lambda^{n-s}$ is the polynomial with roots in $\mathfrak{L} \backslash \mathfrak{F}_O$ and

$$p_o^*(\tilde{A}_{11}) := \beta_0^* I + \beta_1^*\tilde{A}_{11} + \cdots + \beta_{n-s-1}^*\tilde{A}_{11}^{n-s-1} + \tilde{A}_{11}^{n-s}$$

and γ_o denotes the last column of the inverse of

$$O := \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_1\tilde{A}_{11} \\ \vdots \\ \tilde{C}_1\tilde{A}_{11}^{n-s-1} \end{pmatrix} \tag{100}$$

(iv) Define

$$\tilde{K} := \begin{pmatrix} \tilde{K}_1 \\ 0 \end{pmatrix} \tag{101}$$

and

$$K := S^{-1}\tilde{K} \tag{102}$$

Exercize *3.3: Given*

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \; C = \begin{pmatrix} 1 & \alpha \end{pmatrix}, \; \alpha \in \mathbb{R} \tag{103}$$

find, if possible, K *such that* $\sigma(A-KC)=\{-2,-3\}$.

The observability matrix O is

$$O=\begin{pmatrix} C \\ CA \end{pmatrix}=\begin{pmatrix} 1 & \alpha \\ \alpha-1 & -2\alpha \end{pmatrix}$$

Since det$O=-\alpha(\alpha+1)$, the system is observable if and only if $\alpha\neq -1$ and $\alpha\neq 0$.

Case $\alpha=0$. The system is not observable. We use proposition 3.6. In this case we have to check if the invariant spectrum of A is a subset of $\{-2,-3\}$. We have $n-s:=\mathrm{rank}_{\mathbb{R}}\{O\}=1$. For calculating the invariant spectrum of A, we change the coordinates as follows. A basis of $\mathrm{Span}\{O^\top\}$ is

$$v_1:=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(we can take the first column $C^\top$ of $O^\top$ since $s=1$). Define

$$S:=\begin{pmatrix} v_1^\top \\ w_1^\top \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I$$

Therefore, in the new coordinates $z=Sx$

$$\begin{aligned}\tilde{A}&=SAS^{-1}=A=\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix},\\ \tilde{C}&=CS^{-1}=C=\begin{pmatrix} 1 & 0 \end{pmatrix}\end{aligned} \tag{104}$$

Since $s=1$, $\tilde{A}_{11}=-1$, $\tilde{A}_{12}=1$, $\tilde{A}_{22}=-2$ and $\tilde{C}_1=1$ and we conclude that the invariant spectrum is $\mathfrak{F}_O:=\sigma(\tilde{A}_{22})=\{-2\}$. Since $\mathfrak{F}_O:=\{-2\}\subset\mathfrak{L}:=\{-2,-3\}$, by proposition 3.6 there exists K such that $\sigma(A-KC)=\{-2,-3\}$. Let construct the matrix K. We have

$$\tilde{K}:=\begin{pmatrix} \tilde{K}_1 \\ 0 \end{pmatrix} \tag{105}$$

where $\tilde{K}_1$ is such that $\sigma(\tilde{A}_{11}-\tilde{K}_1\tilde{C}_1)=\mathfrak{L}\backslash\mathfrak{F}_O=\{-3\}$. Such $\tilde{K}_1$ exists since

$$\mathrm{rank}_{\mathbb{R}}\{\tilde{C}_1\}=1=s \tag{106}$$

i.e. $\tilde{A}_{11}$ and $\tilde{C}_1$ represent an observable system. On the hand, $\sigma(\tilde{A}_{11}-\tilde{K}_1\tilde{C}_1)=\{-3\}$ if and only if $\tilde{K}_1=2$. Finally, define

$$K:=S^{-1}\tilde{K}=\begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{107}$$

We can check that

$$\begin{aligned}\sigma(A-KC)&=\sigma\Big(\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}-\begin{pmatrix} 2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\Big)\\ &=\sigma\begin{pmatrix} -3 & 0 \\ 1 & -2 \end{pmatrix}=\{-2,-3\}\end{aligned} \tag{108}$$

Case $\alpha=-1$. The system is not observable. We have to check if the invariant spectrum of A is a subset of $\{-2,-3\}$. We have $s:=\mathrm{rank}_{\mathbb{R}}\{O\}=1$. For calculating the invariant spectrum of A, we change the coordinates as follows. A basis of $\mathrm{Span}\{O^\top\}$ is

$$v_1:=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(we can take the first column $C^\top$ of $O^\top$ since $s=1$). Define

$$S=\begin{pmatrix} v_1^\top \\ w_1^\top \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

By direct calculations

$$S^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Therefore, in the new coordinates $z=Sx$

$$\tilde{A}=SAS^{-1}=\begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix},\ \tilde{C}=CS^{-1}=\begin{pmatrix} 2 & 1 \end{pmatrix} \tag{109}$$

Since $s=1$, $\tilde{A}_{11}=-2$, $\tilde{A}_{12}=1$, $\tilde{A}_{22}=-1$ and $\tilde{C}_1=2$ and we conclude that the invariant spectrum is $\mathfrak{F}_O:=\sigma(\tilde{A}_{22})=\{-1\}$. Since $\mathfrak{F}_O:=\{-1\}\notin\mathfrak{L}:=\{-2,-3\}$, by proposition 3.6 there does not exist K such that $\sigma(A-KC)=\{-2,-3\}$.

Case $\alpha\neq 1,\alpha\neq 0$. The system is observable. In this case we use proposition 3.5. The spectrum to be assigned is $\{-2,-3\}$ and, therefore,

$$p^*(\lambda):=(\lambda+2)(\lambda+3)=\lambda^2+5\lambda+6$$

The matrix K which solves the Asymptotic state observation problem with $\mathfrak{L}:=\{-2,-3\}$ is

$$K=p^*(A)\gamma \tag{110}$$

where γ is the last column of O^{-1}. Since

$$O=\begin{pmatrix} C \\ CA \end{pmatrix}=\begin{pmatrix} 1 & \alpha \\ \alpha-1 & -2\alpha \end{pmatrix}$$

then

$$O^{-1}=-\frac{1}{\alpha(\alpha+1)}\begin{pmatrix} -2\alpha & -\alpha \\ 1-\alpha & 1 \end{pmatrix}$$

and

$$\gamma:=-\frac{1}{\alpha(\alpha+1)}\begin{pmatrix} -\alpha \\ 1 \end{pmatrix}$$

Moreover

$$\begin{aligned}p^*(A)&:=(A+2I)(A+3I)=A^2+5A+6I\\ &=\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}^2+5\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}+6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\end{aligned}$$

Finally,

$$\begin{aligned}K&=p^*(A)\gamma=-\frac{1}{\alpha(\alpha+1)}\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -\alpha \\ 1 \end{pmatrix}\\ &=\frac{1}{(\alpha+1)}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\end{aligned} \tag{111}$$

We can check that

$$\begin{aligned}\sigma(A-KC)&=\sigma\Big(\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}-\frac{1}{(\alpha+1)}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & \alpha \end{pmatrix}\Big)\\ &=\sigma\begin{pmatrix} -1-\frac{2}{\alpha+1} & -\frac{2\alpha}{\alpha+1} \\ 1-\frac{2}{\alpha+1} & -2-\frac{2\alpha}{\alpha+1} \end{pmatrix}=\{-2,-3\}.\lhd\end{aligned} \tag{112}$$

F. The PBH observability criterion

The invariant spectrum of $A-KC$ can be determined without a coordinate transformation by calling upon the so-called *PBH observability criterion.* Denote by $\mathfrak{F}_O$ this invariant spectrum.

Proposition *3.7: (PBH observability criterion). A necessary and sufficient condition for observability, i.e. O nonsingular, is that*

$$\text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} = n$$

for each $\lambda \in \sigma(A)$. Moreover,

$$\text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \quad \begin{cases} < n & \Rightarrow \lambda \in \mathfrak{F}_O \\ = n & \Rightarrow \lambda \notin \mathfrak{F}_O \end{cases} \tag{113}$$

Since the matrix

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}$$

has rank n for each $\lambda \notin \sigma(A)$ (remember that $\det(\lambda I - A) = 0$ only for $\lambda \in \sigma(A)$ by definition of eigenvalues), we have the following equivalent form of the PBH observability criterion.

Proposition *3.8: (PBH observability criterion). A necessary and sufficient condition for observability, i.e. O nonsingular, is that*

$$\text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} = n$$

for each $\lambda \in \mathbb{C}^n$.

Exercize *3.4: Revisit the results of example 3.3 by using the PBH observability criterion.*

Case $\alpha = 0$. The system is not observable. It is easily seen that $\sigma(A) = \{-1, -2\}$. We have

$$\begin{aligned} \lambda = -2 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \right\} = 1 < n = 2 \Rightarrow \{-2\} \in \mathfrak{F}_O \end{aligned}$$

and

$$\begin{aligned} \lambda = -1 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \right\} = 2 = n \Rightarrow \{-1\} \notin \mathfrak{F}_O \end{aligned}$$

By proposition 3.7 there exists K such that $\sigma(A - KC) = \{-2, -3\}$.

Case $\alpha = -1$. The system is not observable. We have

$$\begin{aligned} \lambda = -2 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \right\} = n = 2 \Rightarrow \{-2\} \notin \mathfrak{F}_O \end{aligned}$$

and

$$\begin{aligned} \lambda = -1 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \right\} = 1 < 2 = n \Rightarrow \{-1\} \in \mathfrak{F}_O \end{aligned}$$

By proposition 3.7 there does not exist K such that $\sigma(A - KC) = \{-2, -3\}$.

Case $\alpha \neq 0, \alpha \neq -1$. The system is observable. We have

$$\begin{aligned} \lambda = -2 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 1 & \alpha \end{pmatrix} \right\} = n = 2 \Rightarrow \{-2\} \notin \mathfrak{F}_O \end{aligned}$$

and

$$\begin{aligned} \lambda = -1 &\Rightarrow \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} \right\} \\ &= \text{rank}_{\mathbb{R}}\left\{ \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & \alpha \end{pmatrix} \right\} = n = 2 \Rightarrow \{-1\} \notin \mathfrak{F}_O . \triangleleft \end{aligned}$$

G. Eigenvalues assignment by output feedback: the separation principle

The control laws used for eigenvalue assignment in section III-A are not practically implementable since the state information is required. Consider the class of control laws

$$\begin{aligned} \mathbf{u}(t) &= F\hat{\mathbf{x}}(t) + \mathbf{v}(t) \\ \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K(\mathbf{y}(t) - C\hat{\mathbf{x}}(t)) \end{aligned} \tag{114}$$

with matrices $F(1 \times n)$, $K(n \times 1)$ and with v the new control input. These control laws are commonly referred as *output feedback* laws, in the sense that only the output and input information is used to implement the control law. With this control law the resulting system is represented by the new equations

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{pmatrix} = \begin{pmatrix} A & BF \\ KC & A + BF - KC \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} \mathbf{v}(t) \tag{115}$$

which is a system with states (x, ξ) and inputs v. If we are able to find matrices F and K in such a way that the eigenvalues of

$$\begin{pmatrix} A & BF \\ KC & A + BF - KC \end{pmatrix} \tag{116}$$

are equal to a given set

$$\{\lambda_1^*, \dots, \lambda_n^*\} \cup \{\mu_1^*, \dots, \mu_n^*\}$$

with negative real part, then the natural modes are all convergent with rate convergence corresponding to the given negative real parts. The problem can be formulated as follows.

Definition *3.3: (Spectrum assignment by output feedback). Given numbers $\{\lambda_1^*, \dots, \lambda_n^*\} \cup \{\mu_1^*, \dots, \mu_n^*\} \subset \mathbb{C}^-$, find matrices $F(1 \times n)$, $K(n \times 1)$ such that*

$$\sigma \begin{pmatrix} A & BF \\ KC & A + BF - KC \end{pmatrix} = \{\lambda_1^*, \dots, \lambda_n^*\} \cup \{\mu_1^*, \dots, \mu_n^*\}$$

A necessary and sufficient condition for the existence of F, H, G and K is the following.

Proposition *3.9: The Spectrum assignment problem by output feedback is solvable if and only if the system is controllable and observable, i.e. the controllability and observability matrices R and O are nonsingular. In particular, if R and O are nonsingular then for any given numbers $\{\lambda_1^*, \dots, \lambda_n^*\} \cup$*

$\{\mu_1^*, \dots, \mu_n^*\} \subset \mathbb{C}^-$, *the Spectrum assignment problem with output feedback is solvable with*

$$F = -\gamma_R p_R^*(A),$$
$$K = p_O^*(A)\gamma_O,$$

where γ_R *is the last row of* R^{-1}, γ_O *is the last column of* O^{-1}, $p_R^*(\lambda) := \prod_{j=1}^n (\lambda - \lambda_j^*)$ *and* $p_O^*(\lambda) := \prod_{j=n+1}^{2n} (\lambda - \mu_j^*)$.

Assume that the system is controllable and observable. By changing state coordinates as

$$\begin{pmatrix} x \\ e \end{pmatrix} := \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \tag{117}$$

the system (115) in the new coordinates is

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t) - BF\mathbf{e}(t) + B\mathbf{v}(t)$$
$$\dot{\mathbf{e}}(t) = (A - KC)\mathbf{x}(t)$$

or in compact form

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A - KC \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \mathbf{v}(t)$$

Therefore, the eigenvalues of (116) are equal to the eigenvalues of

$$\begin{pmatrix} A + BF & -BF \\ 0 & A - KC \end{pmatrix}$$

since these two matrices differ by a coordinate transformation, and clearly

$$\sigma \begin{pmatrix} A + BF & -BF \\ 0 & A - KC \end{pmatrix} = \sigma(A + BF) \cup \sigma(A - KC)$$

Therefore, the spectrum $\{\lambda_1^*, \dots, \lambda_n^*\} \cup \{\mu_1^*, \dots, \mu_n^*\}$ can be assigned to (116) if and only if the spectrums $\{\lambda_1^*, \dots, \lambda_n^*\}$ and $\{\mu_1^*, \dots, \mu_n^*\}$ can be assigned to $A+BF$ and, respectively, $A-KC$. This proves the proposition by means of propositions 3.1 and 3.5.

H. Design of output feedback stabilizers

If the system is not controllable (resp. not observable) it is clear that there is a subset of the eigenvalues of $A + BF$ (resp. $A - KC$) that is invariant under any choice of F and K (*invariant spectrum*). Even if the system is either not controllable or not observable, it is possible to find a F and K such that $\sigma(A + BF) = \{\lambda_1^*, \dots, \lambda_n^*\}$ and $\sigma(A - KC) = \{\mu_1^*, \dots, \mu_n^*\}$, where $\{\mu_1^*, \dots, \mu_n^*\}$ and $\{\lambda_1^*, \dots, \lambda_n^*\}$ are given numbers, as long as the invariant spectrum of $A + BF$ (resp. $A - KC$) is a subset of the given set $\{\lambda_1^*, \dots, \lambda_n^*\}$ (resp. a subset of $\{\mu_1^*, \dots, \mu_n^*\}$). Denote by $\mathfrak{F}_R$ this invariant spectrum of $A + BF$ and $\mathfrak{F}_O$ the invariant spectrum of $A - KC$. The following proposition can be proved as a combination of propositions 3.2 and 3.6.

Proposition *3.10: Given two sets of numbers* $\mathfrak{L}_R := \{\lambda_1^*, \dots, \lambda_n^*\} \subset \mathbb{C}^-$ *and* $\mathfrak{L}_O := \{\mu_1^*, \dots, \mu_n^*\} \subset \mathbb{C}^-$, *there exist matrices* $F(1 \times n)$, $K(n \times 1)$ *such that*

$$\sigma \begin{pmatrix} A & BF \\ KC & A - KC \end{pmatrix} = \mathfrak{L}_R \cup \mathfrak{L}_O$$

if and only if $\mathfrak{F}_R \subset \mathfrak{L}_R$ *and* $\mathfrak{F}_O \subset \mathfrak{L}_O$.

The construction of the matrices F and K can be summed up as follows:

Step procedure for the design of output feedback stabilizers:

(i) Let $r := \mathrm{rank}_{\mathbb{R}}\{R\}$. Find $w_1, \dots, w_{n-r} \in \mathbb{R}^n$ such that $B, AB, \cdots, A^{r-1}B, w_1, \cdots, w_{n-r}$ are a basis of $\mathbb{R}^n$ and define T as

$$T := \begin{pmatrix} B & AB & \cdots & A^{r-1}B & w_1 & \cdots & w_{n-r} \end{pmatrix}^{-1} \tag{118}$$

(ii) Find the matrices $\tilde{A}_{11}(r \times r), \tilde{A}_{12}(r \times (n-r)), \tilde{A}_{22}((n-r) \times (n-r))$ and $\tilde{B}_1(r \times 1)$ for which

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \; \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \tag{119}$$

(iii) Find $\tilde{F}_1(1 \times r)$ such that $\sigma(\tilde{A}_{11} + \tilde{F}_1\tilde{B}_1) = \mathfrak{L}\backslash\mathfrak{F}_R$. In particular,

$$\tilde{F}_1 = -\gamma_s p_s^*(\tilde{A}_{11}) \tag{120}$$

where $p_s^*(\lambda) := \alpha_0^* + \alpha_1^*\lambda + \cdots + \alpha_{r-1}^*\lambda^{r-1} + \lambda^r$ is the polynomial which has the roots in $\mathfrak{L}\backslash\mathfrak{F}_R$ and

$$p_s^*(\tilde{A}_{11}) := \alpha_0^* I + \alpha_1^*\tilde{A}_{11} + \cdots + \alpha_{r-1}^*\tilde{A}_{11}^{r-1} + \tilde{A}_{11}^r \tag{121}$$

and γ_s is the last row of the inverse of

$$R_s := \begin{pmatrix} \tilde{B}_1 & \tilde{A}_1\tilde{B}_1 & \cdots & \tilde{A}_1^{r-1}\tilde{B}_1 \end{pmatrix} \tag{122}$$

(iv) Define

$$\tilde{F} := \begin{pmatrix} \tilde{F}_1 & 0 \end{pmatrix} \tag{123}$$

and, finally,

$$F := \tilde{F}T \tag{124}$$

(v) Let $s := \mathrm{rank}\{O\}$. Find $w_1, \dots, w_s \in \mathbb{R}^n$ such that $C^\top, (CA)^\top, \cdots, (CA^{s-1})^\top, w_1, \cdots, w_{n-s}$ are a basis of $\mathbb{R}^n$ and define S as

$$S = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \\ w_1^\top \\ \vdots \\ w_{n-s}^\top \end{pmatrix} \tag{125}$$

(vi) Find the matrices $\tilde{A}_{11}(s \times s), \tilde{A}_{12}((n-s) \times s), \tilde{A}_{22}((n-s) \times (n-s))$ and $\tilde{C}_1(1 \times s)$ for which

$$\tilde{A} = SAS^{-1} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix}, \; \tilde{C} = CS^{-1}$$
$$= \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} \tag{126}$$

(vii) Find with $\tilde{K}_1(1 \times (n-s))$ such that $\sigma(\tilde{A}_{11} - \tilde{C}_1\tilde{K}_1) = \mathfrak{L}\backslash\mathfrak{F}_O$. In particular,

$$\tilde{K}_1 = p_o^*(\tilde{A}_{11})\gamma_o \tag{127}$$

where $p_o^*(\lambda) := \beta_0^* + \beta_1^*\lambda + \cdots + \beta_{n-s-1}^*\lambda^{n-s-1} + \lambda^{n-s}$ is the polynomial which has the roots in $\mathfrak{L}\backslash\mathfrak{F}_O$ and

$$p_o^*(\tilde{A}_{11}) := \beta_0^* I + \beta_1^*\tilde{A}_{11} + \cdots + \beta_{n-s-1}^*\tilde{A}_{11}^{n-s-1} + \tilde{A}_{11}^{n-s}$$

and γ_o is the last column of the inverse of

$$O_o := \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_1\tilde{A}_{11} \\ \vdots \\ \tilde{C}_1\tilde{A}_{11}^{n-s-1} \end{pmatrix}$$

(viii) Define

$$\tilde{K} := \begin{pmatrix} \tilde{K}_1 \\ 0 \end{pmatrix} \tag{128}$$

and eventually

$$K := S^{-1}\tilde{K} \tag{129}$$

IV. Tracking of reference inputs with spectrum assignment

In this section we want to show, given a n-times continuously differentiable function $y_{ref}(t)$ together with a system $\dot{x} = Ax + Bu$, $y = Cx$, in the (controllable canonical) form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots &= \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + u, \\ y &= x_1, \end{aligned} \tag{130}$$

with $a_0, a_1, \cdots, a_{n-1} \in \mathbb{R}$, how it is possible to design a state-feedback control $u = Fx + \Phi(y_{ref}(t), y_{ref}^{(1)}(t), \cdots y_{ref}^{(n)}(t)) + v$ with $F \in \mathbb{R}^{1\times n}$ and $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots &= \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + Fx + \Phi(t), \\ y &= x_1, \end{aligned} \tag{131}$$

has the property that $\lim_{t\to+\infty} \|y(t) - y_{ref}(t)\| = 0$ for all $x_0 \in \mathbb{R}^n$. In this problem we want to design our control input $u(t)$ in such a way that the steady-state output response $y_{ss}(t)$ of the closed-loop system is identically equal to $y_{ref}(t)$.

Let

$$z_j := x_j - y_{ref}^{(j-1)}(t),\ j = 1, \dots, n,$$

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_n \end{pmatrix},\ x_{ref}(t) = \begin{pmatrix} y_{ref}(t) \\ y_{ref}^{(1)}(t) \\ \cdots \\ y_{ref}^{(n-1)}(t) \end{pmatrix}, \tag{132}$$

where $y_{ref}^{(j)}(t)$ is the j-th order derivative of $y_{ref}(t)$. In these coordinates

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \vdots &= \vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= -a_0z_1 - a_1z_2 - \cdots - a_{n-1}z_n \\ &-a_0y_{ref}(t) - a_1y_{ref}^{(1)}(t) - \cdots - a_{n-1}y_{ref}^{(n-1)}(t) - y_{ref}^{(n)}(t) + u, \\ y &= z_1 + y_{ref}(t), \end{aligned} \tag{133}$$

Let $\{\lambda_1^*, \dots, \lambda_n^*\} \subset \mathbb{C}^-$ be a given set of eigenvalues we want to assign to (133) and $a_0^*, \dots, a_{n-1}^* \in \mathbb{R}$ be such that

$$p^*(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j^*) = a_0^* + a_1^*\lambda + \cdots + a_{n-1}^*\lambda^{n-1} + \lambda^n.$$

Define the state feedback control on (133)

$$u = Fz + \Phi(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t)) + v \tag{134}$$

where

$$F = \begin{pmatrix} a_0 - a_0^* & a_1 - a_1^* & \cdots & a_{n-1} - a_{n-1}^* \end{pmatrix}$$

i.e. it is such that $\sigma(A + BF) = \{\lambda_1^*, \dots, \lambda_n^*\} \subset \mathbb{C}^-$, and

$$\begin{aligned} &\Phi(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t)) \\ &= a_0y_{ref}(t) + a_1y_{ref}^{(1)}(t) + \cdots + a_{n-1}y_{ref}^{(n-1)}(t) + y_{ref}^{(n)}(t). \end{aligned}$$

The closed-loop system resulting from (133) is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \vdots &= \vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= -a_0^*z_1 - a_1^*z_2 - \cdots - a_{n-1}^*z_n \\ y &= z_1 + y_{ref}(t), \end{aligned} \tag{135}$$

and such that $\lim_{t\to+\infty} \|z(t)\| = 0$ for all $z_0 \in \mathbb{R}^n$ or, equivalently in x-coordinates, $\lim_{t\to+\infty} \|x(t) - x^{ref}(t)\| = 0$ for all $x_0 \in \mathbb{R}^n$. In particular, $\lim_{t\to+\infty} \|y(t) - y^{ref}(t)\| = 0$, which is our control objective.

In x-coordinates we have

$$\begin{aligned} u &= Fz + \Phi(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t) + v \\ &= Fx + \Phi^*(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t) + v \end{aligned} \tag{136}$$

where

$$\begin{aligned} &\Phi^*(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t)) \\ &= a_0^*y_{ref}(t) + a_1^*y_{ref}^{(1)}(t) + \cdots + a_{n-1}^*y_{ref}^{(n-1)}(t) + y_{ref}^{(n)}(t). \end{aligned}$$

Notice that if we take $y_{ref}(t) \equiv 0$ then

$$u = Fx + v \tag{137}$$

which is the state feedback controller which assigns the spectrum $\{\lambda_1^*, \dots, \lambda_n^*\} \subset \mathbb{C}^-$ to the system (130). Hence, the additional control term $\Phi^*(y_{ref}(t), y_{ref}^{(1)}(t), \cdots, y_{ref}^{(n)}(t)$ in (136) is exactly the one that enforces the system's output to track the reference signal $y_{ref}(t)$.