Notes on Linear Control Systems: Module VI

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***Abstract*—Bode Plots of the armonic I/O response $\mathbf{W}(j\omega)$. Bode form and monomial, binomial, trinomial terms.**

I. The Bode plots

We have seen that $\mathbf{W}(j\omega)$ can be experimentally determined by forcing an asymptotically stable system with a sinusoidal input $\mathbf{u}_t = \sin \omega t$ and by observing the steady-state output response $\mathbf{y}^{(ss)}(\mathbf{u}) = |\mathbf{W}(j\omega)| \sin(\omega t + \mathrm{Arg}(\mathbf{W}(j\omega))$. The values of the magnitude $|\mathbf{W}(j\omega)|$ and phase $\mathrm{Arg}\{\mathbf{W}(j\omega)\}$ are usually plotted versus ω (rad/sec) on a suitable semilogarithmic chart (Figure 1). The values of $|\mathbf{W}(j\omega)|$ are measured in dB

$$|\mathbf{W}(j\omega)|_{dB} := 20log_{10}|\mathbf{W}(j\omega)|$$

and the values of $\mathrm{Arg}\{\mathbf{W}(j\omega)\}$ in degrees or radiant, while ω is plotted as $\log_{10}\omega$. These plots are commonly known as *Bode plots* of $\mathbf{W}(j\omega)$. In this section we will study the structure of these plots and how to obtain an approximate and shorthand representation.

Recall that $\mathbf{W}(s)$ is a proper rational function of s with a certain number of real and complex conjugate zeroes and poles. By denoting as

- $m_0^{(N)}$ (resp. $m_0^{(D)}$) the multiplicity of the zeroes $s=0$ (resp. poles $s=0$),
- $\lambda_1^{(N)}, \ldots, \lambda_{\mathfrak{r}^{(N)}}^{(N)}$ the distinct real zeroes (resp. $\lambda_1^{(D)}, \ldots, \lambda_{\mathfrak{r}^{(D)}}^{(D)}$ the distinct real poles) with multiplicity $\gamma_1^{(N)}, \ldots, \gamma_{\mathfrak{r}^{(N)}}^{(N)}$ (resp. $\gamma_1^{(D)}, \ldots, \gamma_{\mathfrak{r}^{(D)}}^{(D)}$),
- $\mu_1^{(N)}, {\mu_1^{(N)}}^*, \ldots, \mu_{\mathfrak{s}^{(N)}}^{(N)}, {\mu_{\mathfrak{s}^{(N)}}^{(N)}}^*$ the distinct complex conjugate zeroes (resp. $\mu_1^{(D)}, {\mu_1^{(D)}}^*, \ldots, \mu_{\mathfrak{s}^{(D)}}^{(D)}, {\mu_{\mathfrak{s}^{(D)}}^{(D)}}^*$ the distinct complex conjugate poles) with multiplicity $\delta_1^{(N)}, \ldots, \delta_{\mathfrak{s}^{(N)}}^{(N)}$ (resp. $\delta_1^{(D)}, \ldots, \delta_{\mathfrak{s}^{(D)}}^{(D)}$),

we can represent $\mathbf{W}(s)$ as:

$$\mathbf{W}(s) = K s^{m_0^{(N)} - m_0^{(D)}} \frac{\prod_{j=1}^{\mathfrak{r}^{(N)}} (s - \lambda_j^{(N)})^{\gamma_j^{(N)}}}{\prod_{j=1}^{\mathfrak{r}^{(D)}} (s - \lambda_j^{(D)})^{\gamma_j^{(D)}}} \times$$
$$\frac{\prod_{j=1}^{\mathfrak{s}^{(N)}} (s - \mu_j^{(N)})^{\delta_j^{(N)}} (s - {\mu_j^{(N)}}^*)^{\delta_j^{(N)}}}{\prod_{j=1}^{\mathfrak{s}^{(D)}} (s - \mu_j^{(D)})^{\delta_j^{(D)}} (s - {\mu_j^{(D)}}^*)^{\delta_j^{(D)}}} \tag{1}$$

where $K \in \mathbb{R}$. In terms of time constants τ_j, damping ζ_j and natural frequencies ω_n, we can re-write (1) in the so-called *Bode form*

$$\mathbf{W}(s) = K' s^{m_0^{(N)} - m_0^{(D)}} \frac{\prod_{j=1}^{\mathfrak{r}^{(N)}} (1 + s\tau_j^{(N)})^{\gamma_j^{(N)}}}{\prod_{j=1}^{\mathfrak{r}^{(D)}} (1 + s\tau_j^{(D)})^{\gamma_j^{(D)}}} \times$$
$$= \times \frac{\prod_{j=1}^{\mathfrak{s}^{(N)}} (1 + \frac{2s\zeta_j^{(N)}}{\omega_{n,j}^{(N)}} + \frac{s^2}{(\omega_{n,j}^{(N)})^2})^{\delta_j^{(N)}}}{\prod_{j=1}^{\mathfrak{s}^{(D)}} (1 + \frac{2s\zeta_j^{(N)}}{\omega_{n,j}^{(D)}} + \frac{s^2}{(\omega_{n,j}^{(D)})^2})^{\delta_j^{(D)}}} \tag{2}$$

for some $K' \in \mathbb{R}$. Note that in the Bode form we have *constant* terms (K'), *monomial* terms (s^h), *binomial* terms ($1+s\tau_j$) and *trinomial* terms ($1 + \frac{2s\zeta_j}{\omega_{nj}} + \frac{s^2}{\omega_{nj}^2}$).

As an example, consider

$$\mathbf{W}(s) = 2\frac{(s-1)^3(1+s+s^2)}{(1+2s)(s-3)} \tag{3}$$

The Bode form of $\mathbf{W}(s)$ is

$$\mathbf{W}(s) = \frac{2}{3}\frac{(1-s)^3(1+s+s^2)}{(1+2s)(1-\frac{s}{3})} \tag{4}$$

Therefore, $K' = \frac{2}{3}$. We have a numerator binomial term $(1-s)$ with multiplicity 3 and time constant $\tau = -1$, a numerator trinomial term $(1+s+s^2)$ with multiplicity 1, damping $\zeta = \frac{1}{2}$ and natural frequency $\omega_n = 1$, a denominator binomial term $(1+2s)$ with multiplicity 1 and time constant $\tau = 2$ and a denominator binomial term $(1 - \frac{s}{3})$ with multiplicity 1 and time constant $\tau = -\frac{1}{3}$.

Since for any $a, b \in \mathbb{C}$

$$|ab|_{dB} = |a|_{dB} + |b|_{dB}$$

and in particular

$$|a^h|_{dB} = h|a|_{dB}$$

it follows that

$$\begin{aligned}
|\mathbf{W}(j\omega)|_{dB} &= |K'|_{dB} + (m_0^{(N)} - m_0^{(D)})|j\omega|_{dB} \\
&+ \sum_{j=1}^{\mathfrak{r}^{(N)}} \gamma_j^{(N)} |1 + j\omega\tau_j^{(N)}|_{dB} \\
&+ \sum_{j=1}^{\mathfrak{s}^{(N)}} \delta_j^{(N)} |1 + \frac{2j\omega\zeta_j^{(N)}}{\omega_{n,j}^{(N)}} - \frac{\omega^2}{(\omega_{n,j}^{(N)})^2}|_{dB} \\
&- \sum_{j=1}^{\mathfrak{r}^{(D)}} \gamma_j^{(D)} |1 + j\omega\tau_j^{(D)}|_{dB} \\
&- \sum_{j=1}^{\mathfrak{s}^{(D)}} \delta_j^{(D)} |1 + \frac{2j\omega\zeta_j''}{\omega_{n,j}^{(D)}} - \frac{\omega^2}{(\omega_{n,j}^{(D)})^2}|_{dB}
\end{aligned}$$

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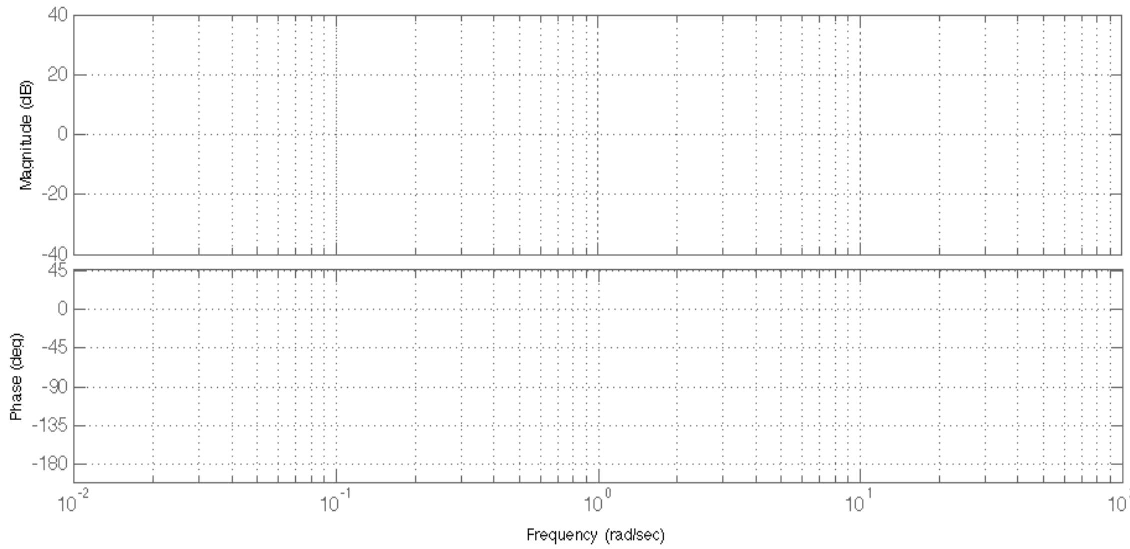

Figure 1. Semilogarithmic chart or Bode chart.

Moreover, since for any pair of complex numbers a and b

$$\text{Arg}\{ab\} = \text{Arg}\{a\} + \text{Arg}\{b\}$$

and in particular

$$\text{Arg}\{a^h\} = h\text{Arg}\{a\}$$

it follows that

$$\begin{aligned}
&\text{Arg}\{\mathbf{W}(j\omega)\} = \text{Arg}\{K'\} + (m_0^{(N)} - m_0^{(D)})\text{Arg}\{j\omega\} \\
&+ \sum_{j=1}^{\mathfrak{r}^{(N)}} \gamma_j^{(N)} \text{Arg}\{1 + j\omega\tau_j^{(N)}\} \\
&+ \sum_{j=1}^{\mathfrak{s}^{(N)}} \delta_j^{(N)} \text{Arg}\{1 + \frac{2j\omega\zeta_j^{(N)}}{\omega_{n,j}^{(N)}} - \frac{\omega^2}{(\omega_{n,j}^{(N)})^2}\} \\
&- \sum_{j=1}^{r_1''} \gamma_j^{(D)} \text{Arg}\{1 + j\omega\tau_j^{(D)}\} \\
&- \sum_{j=1}^{r_2''} \delta_j^{(D)} \text{Arg}\{1 + \frac{2j\omega\zeta_j^{(D)}}{\omega_{n,j}^{(D)}} - \frac{\omega^2}{(\omega_{n,j}^{(D)})^2}\}
\end{aligned}$$

This means that in order to obtain the Bode plot of $\mathbf{W}(j\omega)$ it is sufficient to draw the Bode plot of four kind of terms (constant K, monomial $j\omega$, binomial $1 + j\omega\tau$ and trinomial $1 + \frac{2j\omega\zeta}{\omega_n} - \frac{\omega^2}{\omega_n^2}$) and, finally, sum them all together at each ω.

It should be noted that the Bode plot of the binomial $1+j\omega\tau$ with $\tau < 0$ is the same as the one of the binomial $1 + j\omega|\tau|$, except that the phase plot has opposite sign: indeed, $1 + j\omega\tau$ with $\tau < 0$ can be written as $1 - j\omega|\tau|$ which has the same magnitude of $1 + j\omega|\tau|$ and opposite phase (since $1 - j\omega|\tau|$ and $1 + j\omega|\tau|$ are complex conjugate). Likewise, the Bode plot of the trinomial $1 + \frac{2j\omega\zeta}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ with $\zeta < 0$ is the same as the one of the trinomial $1 + \frac{2j\omega|\zeta|}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$, except that the phase plot has opposite sign: indeed, $1 + \frac{2j\omega\zeta}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ with $\zeta < 0$ can be written as $1 - \frac{2j\omega|\zeta|}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ which has the same magnitude of $1 + \frac{2j\omega|\zeta|}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ and opposite phase (since $1 - \frac{2j\omega|\zeta|}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ and $1 + \frac{2j\omega|\zeta|}{\omega_n} - \frac{\omega^2}{(\omega_n)^2}$ are complex conjugate).

All this means that it is sufficient to draw the Bode plot of four kind of terms (constants K, monomials $j\omega$, binomials $1 + j\omega\tau$ and trinomials $1 + \frac{2j\omega\zeta}{\omega_n} - \frac{\omega^2}{\omega_n^2}$) with $\tau > 0$ and $\zeta > 0$.

A. *Constant term*

For the constant K (Figure 2) we have

$$|K|_{dB} = 20log_{10}|K|$$
$$\text{Arg}\{K\} = \begin{cases} 2h\pi,\ h = 0, 1, \dots & \text{if } K > 0 \\ -\pi + 2h\pi,\ h = 0, 1, \dots & \text{otherwise} \end{cases} \tag{5}$$

Note that the phase is determined up to multiples of 2π. From now on we choose always the principal value of the phase (i.e. for $h = 0$).

B. *Monomial term*

For the monomial $j\omega$ (Figure 3) we have

$$|j\omega|_{dB} = 20log_{10}|j\omega| = 20log_{10}\omega$$
$$\text{Arg}\{j\omega\} = \frac{\pi}{2} + 2h\pi,\ h = 0, 1, \dots \tag{6}$$

C. *Binomial term*

For the binomial $1 + j\omega\tau$ we have

$$|1 + j\omega\tau|_{dB} = 20log_{10}|1 + j\omega\tau| = 20log_{10}\sqrt{1 + \omega^2\tau^2}$$
$$\text{Arg}\{1 + j\omega\tau\} = \arctan\{\omega\tau\} + 2h\pi,\ h = 0, 1, \dots \tag{7}$$

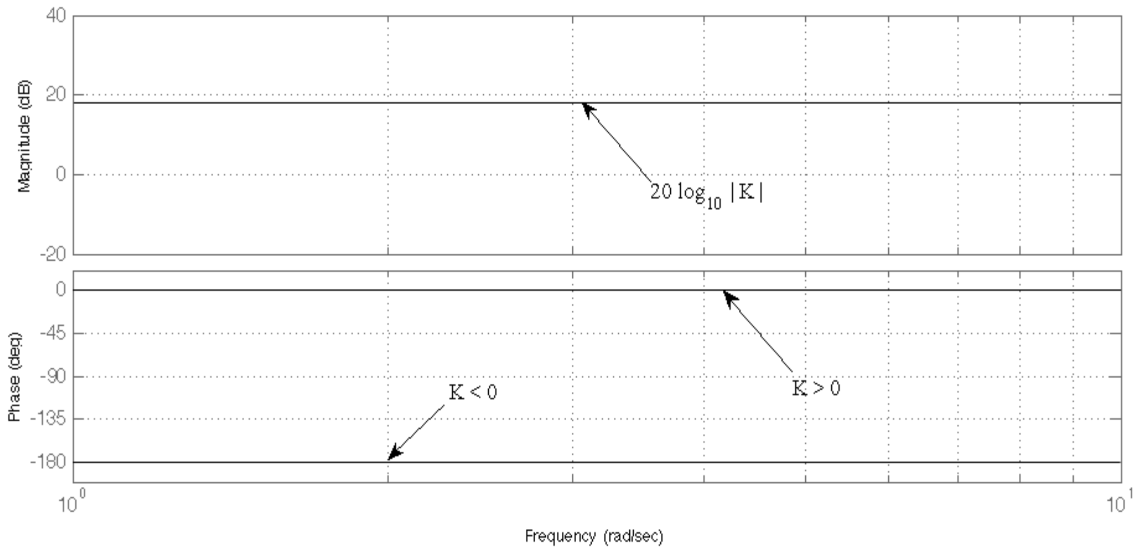

Figure 2. The Bode plot of a constant term.

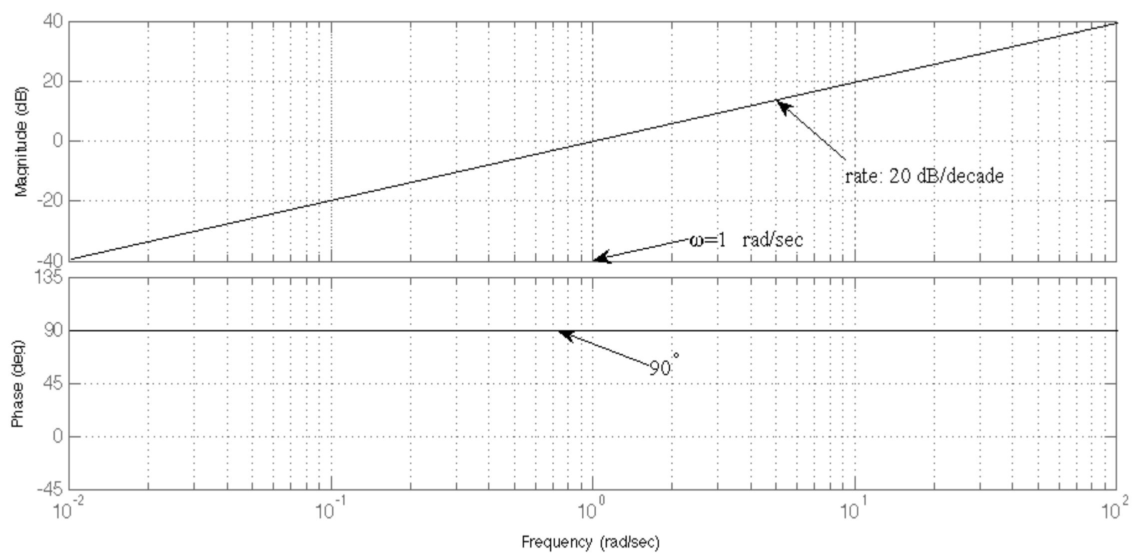

Figure 3. The Bode plot of a monomial term.

We can draw the Bode plot of the binomial by approximating it in the following way. If $\omega << \frac{1}{\tau}$

$$|1+j\omega\tau|_{dB} = 20log_{10}\sqrt{1+\omega^2\tau^2} \approx 0dB$$

$$\text{Arg}\{1+j\omega\tau\} = \arctan\{\omega\tau\} + 2h\pi \approx 2h\pi,\ h = 0, 1, \dots$$

If $\omega >> \frac{1}{\tau}$

$$\begin{aligned} &|1+j\omega\tau|_{dB} = 20log_{10}\sqrt{1+\omega^2\tau^2} \\ &\approx 20log_{10}\omega\tau = 20log_{10}\omega + 20log_{10}\tau \\ &\text{Arg}\{1+j\omega\tau\} = \arctan\{\omega\tau\} + 2h\pi \approx \frac{\pi}{2} + 2h\pi, \\ &\ h = 0, 1, \dots \end{aligned}$$

Note that for $\omega >> \frac{1}{\tau}$ the Bode plot of the magnitude of $1 + j\omega\tau$ is a straight line with rate 20 dB/decade, passing through the point 0 dB at $\omega = 1$rad/sec (recall that the values of ω are plotted as $\log_{10}\omega$). A reasonable approximation for the Bode plot of $1+j\omega\tau$ is shown in Figure 4 while the exact Bode plot is shown in Figure 4 for values of $\tau = \frac{1}{1}, \tau = \frac{1}{3}, \tau = \frac{1}{9}$. In few words:

- the approximate plot of $|1+j\omega\tau|_{dB}$ is 0 dB for all $\omega \leqslant \frac{1}{\tau}$ and linearly increasing 20 dB per decade for $\omega \geqslant \frac{1}{\tau}$,
- the approximate plot of $\text{Arg}\{1 + j\omega\tau\}$ is 0 rad for all $\omega \leqslant \frac{0.1}{\tau}$, $\frac{\pi}{2}$ rad for all $\omega \geqslant \frac{10}{\tau}$ and linearly increasing 45° per decade for $\omega \in [\frac{0.1}{\tau}, \frac{10}{\tau}]$.

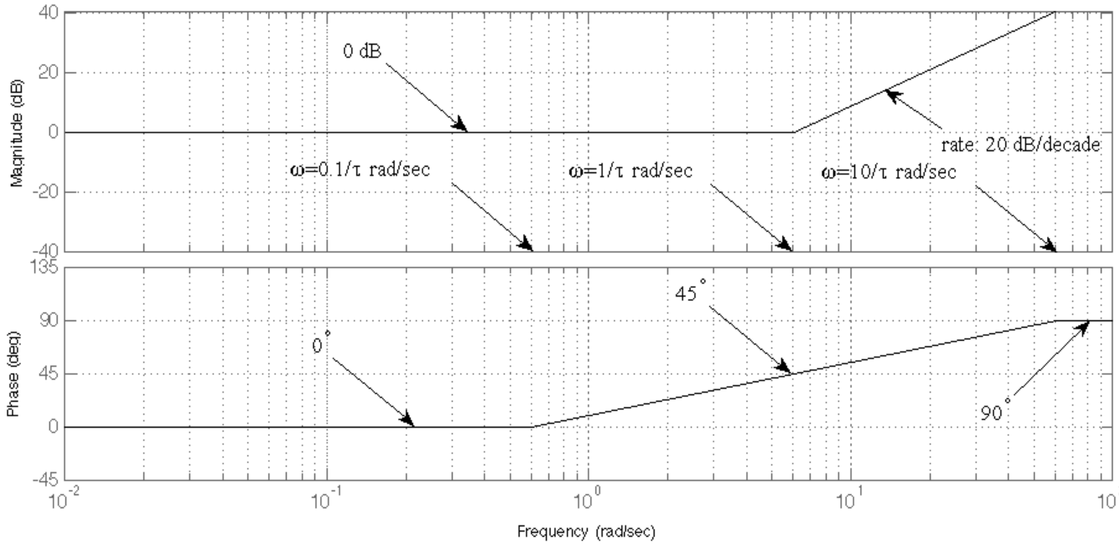

Figure 4. The approximate Bode plot of a binomial term.

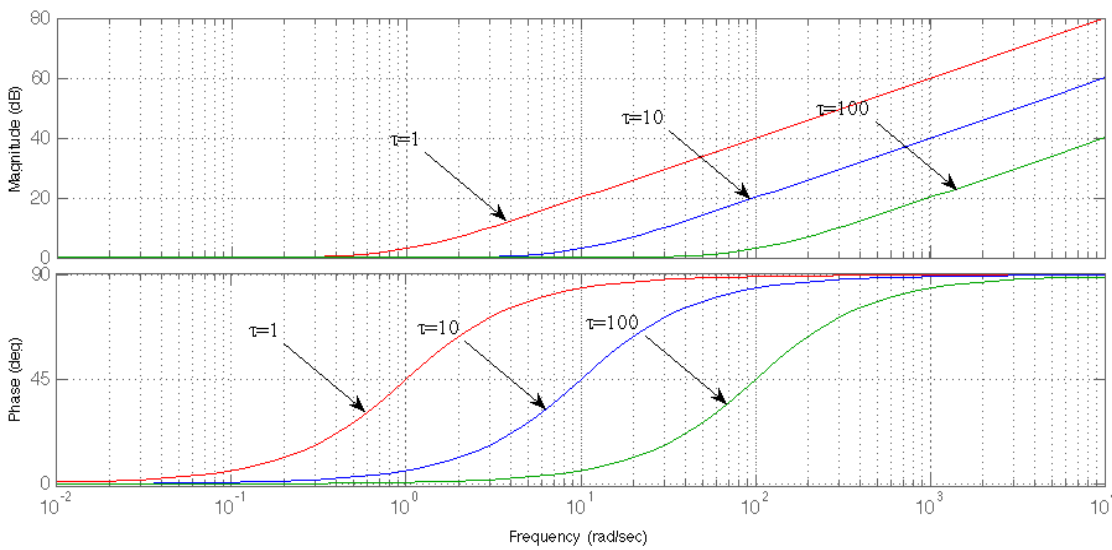

Figure 5. The exact Bode plot of a binomial term with $\tau = 1, 0.1, 0.01$ (*Errata Corrige*: in the above figures the labels $\tau = 10$ and $\tau = 100$ must be corrected as $\tau = 0.1$ and $\tau = 0.01$).

It is possible to evaluate the error between the approximate Bode plot of $1 + j\omega\tau$ and the exact Bode plot. Since the magnitude error is

$$\begin{aligned} &20log_{10}\sqrt{1+\omega^2\tau^2}, \text{ if } \omega\tau \leqslant 1, \\ &20log_{10}\sqrt{\frac{1+\omega^2\tau^2}{\omega^2\tau^2}}, \text{ if } \omega\tau \geqslant 1, \end{aligned} \tag{8}$$

we find out that the maximal magnitude error is achieved in $\omega = \frac{1}{\tau}$ and it is 3dB.

On the other hand, since the phase error is

$$\begin{aligned} &\arctan\{\omega\tau\}, \text{ if } \omega\tau \leqslant 0.1, \\ &\arctan\{\omega\tau\} - \frac{\pi}{4}(\log_{10}\omega - \log_{10}\frac{0.1}{\tau}), \text{ if } 0.1 \leqslant \omega\tau \leqslant 10, \\ &\arctan\{\omega\tau\} - \frac{\pi}{2}, \text{ if } \omega\tau \geqslant 10, \end{aligned}$$

we find out that the maximal phase error is achieved in $\omega = \frac{0.1}{\tau}$ and $\omega = \frac{10}{\tau}$ and it is $\arctan\{0.1\} = |\arctan\{10\} - \frac{\pi}{2}| = 0.0997\text{rad}$.

D. Trinomial term

For the trinomial $1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}$ we have

$$|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB} = 20log_{10}|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|$$
$$= 20log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\frac{\omega^2\zeta^2}{\omega_n^2}}$$
$$\text{Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\} = \arctan\left\{\frac{\frac{2\omega\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right\}+2h\pi,$$
$$h=0,1,\dots \tag{9}$$

We can draw the Bode plot of the trinomial term by approximating it in the following way. If $\omega << \omega_n$

$$|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}$$
$$= 20log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\frac{\omega^2\zeta^2}{\omega_n^2}} \approx 0dB$$
$$\text{Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}$$
$$= \arctan\left\{\frac{\frac{2\omega\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right\}+2h\pi \approx 2h\pi,\ h=0,1,\dots$$

If $\omega >> \omega_n$

$$|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB} = 20log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\frac{\omega^2\zeta^2}{\omega_n^2}}$$
$$\approx 20log_{10}\frac{\omega^2}{\omega_n^2} = 40log_{10}\omega - 40log_{10}\omega_n$$
$$\text{Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}$$
$$= \arctan\left\{\frac{\frac{2\omega\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right\}+2h\pi \approx \pi+2h\pi,\ h=0,1,\dots$$

Note that for $\omega >> \omega_n$ the Bode plot of the magnitude of $1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}$ is a straight line with rate 40 dB/decade, passing through the point 0 dB at $\omega=\omega_n$rad/sec (recall that the values of ω are plotted as $\log_{10}\omega$). A tentative approximation for the Bode plot of the magnitude of $1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}$ is a double binomial term with $\tau=\frac{1}{\omega_n}$. Unfortunately, this approximation may differ significantly from the real Bode plot according to the values of ζ. Indeed when $\zeta=0$

$$\text{if } \omega\to\omega_n^+ \text{ then } |1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}\to-\infty$$
$$\text{if } \omega\to\omega_n^- \text{ then } |1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}\to-\infty$$

and

$$\text{if } \omega\to\omega_n^+$$
$$\text{then Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}\to\pi+2h\pi,\ h=0,1,\dots$$
$$\text{if } \omega\to\omega_n^-$$
$$\text{then Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}\to 2h\pi,\ h=0,1,\dots$$

Therefore, the magnitude has a second kind discontinuity at $\omega=\omega_n$ and the phase has a first kind discontinuity with jump $+\pi$ at $\omega=\omega_n$.

For all values of $\zeta\in(0,\frac{1}{\sqrt{2}})$, the magnitude $|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}$ has a minimum at $\omega=\omega_R:=\omega_n\sqrt{1-2\zeta^2}$ (ω_R is called the *resonance frequency*). In particular, the minimum value of the magnitude is

$$M_R := |1+\frac{2j\omega_R\zeta}{\omega_n}-\frac{\omega_R^2}{\omega_n^2}|_{dB} = |2\zeta\sqrt{1-\zeta^2}|_{dB}$$

which tends to $-\infty$ as $\zeta\to 0$. The value M_R is called the *resonance peak* of the trinomial. It can be seen from the exact plot of the trinomial that

- For all values of $\zeta\in(0,\frac{1}{2})$ the Bode plot of the magnitude crosses 0 dB for some $\omega>\omega_n$
- For $\zeta=\frac{1}{2}$ the Bode plot of the magnitude crosses 0 dB at $\omega=\omega_n$
- For all values of $\zeta\in(\frac{1}{2},\frac{1}{\sqrt{2}}]$ the Bode plot of the magnitude crosses 0 dB for some $\omega<\omega_n$
- For all values of $\zeta\in[\frac{1}{\sqrt{2}},1)$ the Bode plot of the magnitude never crosses 0 dB and stays for all frequency below 0 dB

A reasonable approximation for the Bode plots of $1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}$ is shown in Figure 6 for $\zeta\in(0,\frac{1}{\sqrt{2}})$ and in Figure 7 for $\zeta\in[\frac{1}{\sqrt{2}},1)$. In few words, for all values of $\zeta\in[\frac{1}{\sqrt{2}},1)$:

- the approximate plot of $|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}$ is 0 dB for all $\omega\leqslant\omega_n$ and linearly increasing 40 dB per decade for $\omega\geqslant\omega_n$ from 0 dB,
- the approximate plot of $\text{Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}$ is 0 rad for all $\omega\leqslant 10^{-\zeta}\omega_n$, linearly increasing 90° per decade for $\omega\in[10^{-\zeta}\omega_n,10^{\zeta}\omega_n]$ from 0° and, finally, π rad for all $\omega\geqslant 10^{\zeta}\omega_n$ and

while for all values of $\zeta\in(0,\frac{1}{\sqrt{2}})$:

- the plot of $|1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}|_{dB}$ is 0 dB for all $\omega\leqslant 10^{-\zeta}\omega_n$, linearly decreasing for $\omega\in[10^{-\zeta}\omega_n,\omega_n]$ from 0 dB to M_R, linearly increasing for $\omega\in[\omega_n,10^{\zeta}\omega_n,]$ from M_R to 0 dB and, finally, linearly increasing 40 dB per decade for $\omega\geqslant 10^{\zeta}\omega_n$ from 0 dB,
- the plot of $\text{Arg}\{1+\frac{2j\omega\zeta}{\omega_n}-\frac{\omega^2}{\omega_n^2}\}$ is 0 rad for all $\omega\leqslant 10^{-\zeta}\omega_n$, π rad for all $\omega\geqslant 10^{\zeta}\omega_n$ and linearly increasing 90° per decade for $\omega\in[0.1\omega_n,10\omega_n]$.

Exercize *1.1: Draw the Bode plot of*

$$\mathbf{W}(s) = \frac{s+4}{s(s-1)} \tag{10}$$

First of all, rewrite $\mathbf{W}(s)$ in the Bode form

$$\mathbf{W}(s) = -4\frac{1+\frac{s}{4}}{s(1-s)} \tag{11}$$

Therefore, we have a constant term $K=-4$, one monomial term, one binomial term with $\tau=\frac{1}{4}$ and one binomial term with $\tau=-1$. The Bode plot of $K=-4$ is given in Figure 8.

The Bode plot of $\frac{1}{j\omega}$ is given in Figure 9.

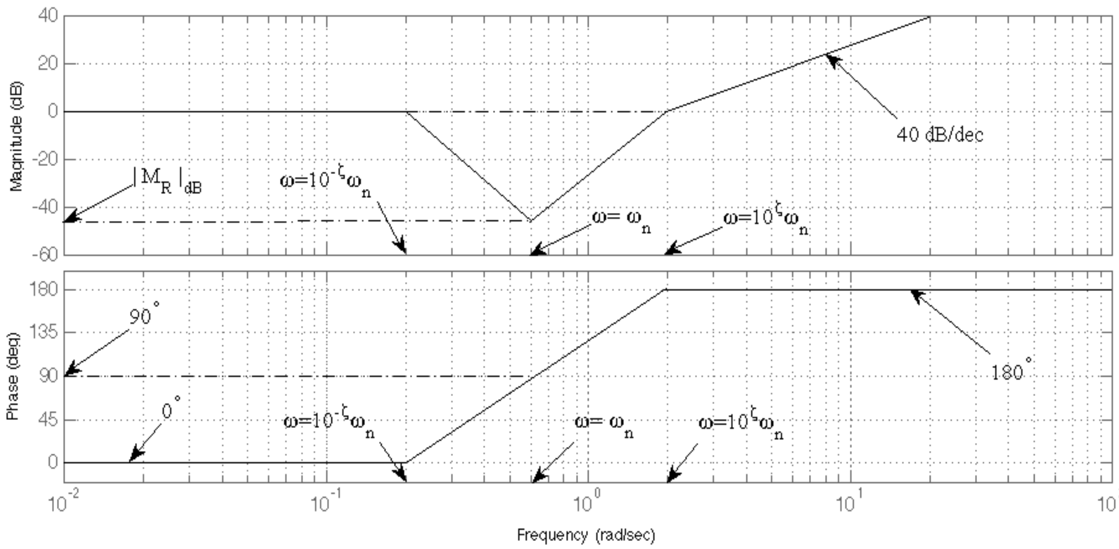

Figure 6. The approximate Bode plot of a trinomial term for $\zeta \in (0, \frac{1}{\sqrt{2}})$.

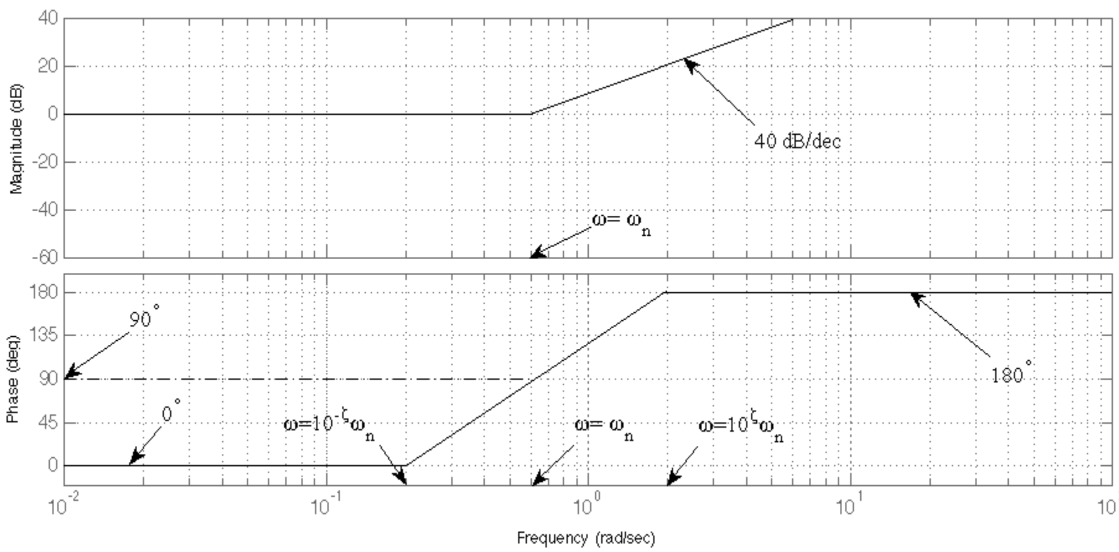

Figure 7. The approximate Bode plot of a trinomial term for $\zeta \in [\frac{1}{\sqrt{2}}, 1)$.

By summing up these plot at each ω we obtain the Bode plot of the product $-4\frac{1}{j\omega}$ (Figure 10). The Bode plot of $\frac{1}{1-j\omega}$ is given in Figure 11. By summing up the plot of $-4\frac{1}{j\omega}$ and of $\frac{1}{1-j\omega}$ at each ω we obtain the Bode plot of the product $-4\frac{1}{j\omega(1-j\omega)}$ (Figure 12).

Finally, the Bode plot of $1 + \frac{j\omega}{4}$ is given in Figure 13. By summing up the plot of $-4\frac{1}{j\omega(1-j\omega)}$ and of $1 + \frac{j\omega}{4}$ at each ω we obtain the Bode plot of the product $\mathbf{W}(j\omega)$ (Figure 14). $\lhd$

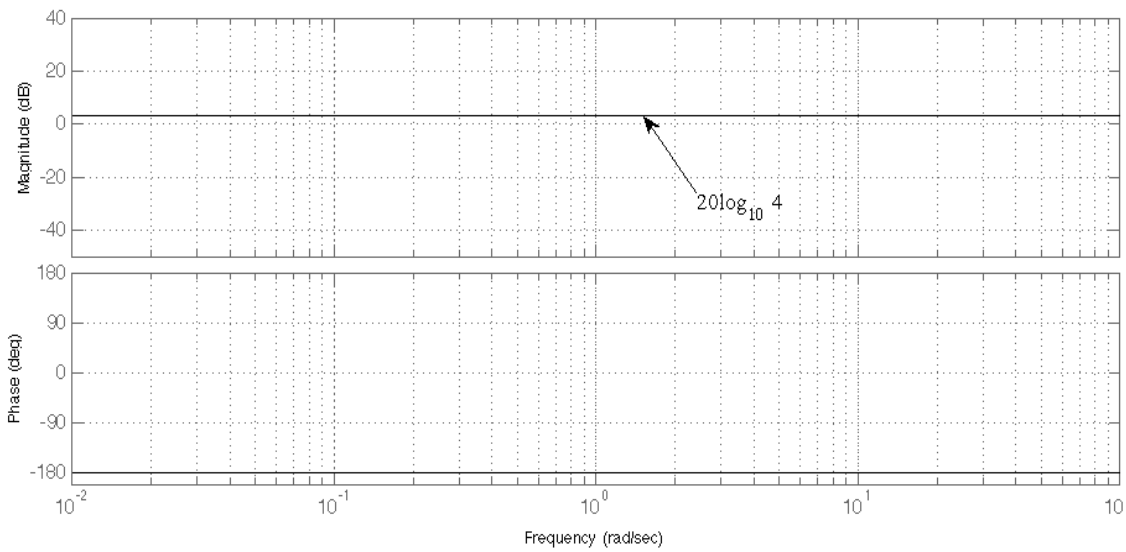

Figure 8. The Bode plot of $K = -4$.

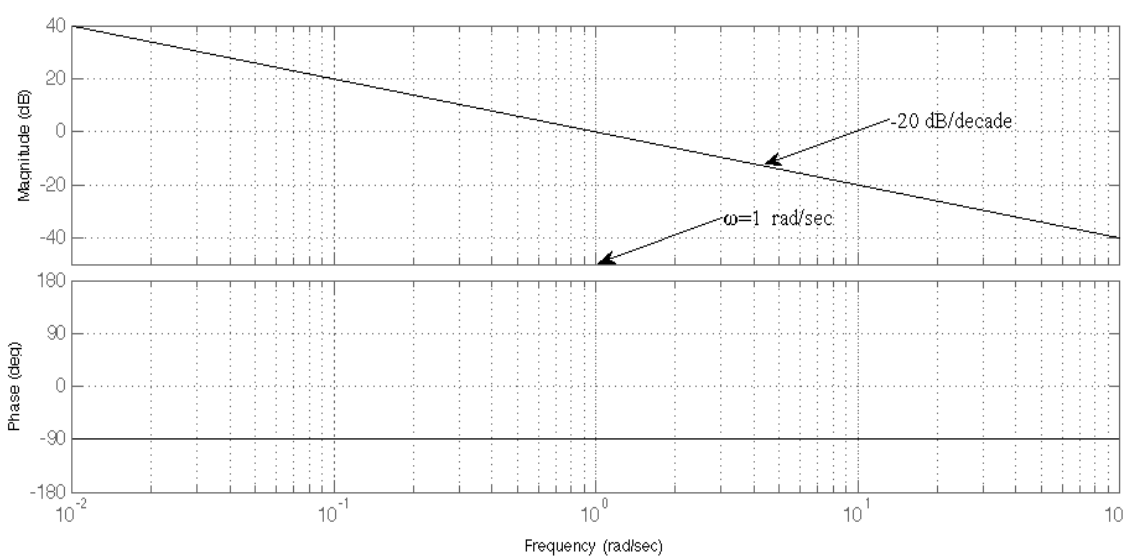

Figure 9. The Bode plot of $\frac{1}{j\omega}$.

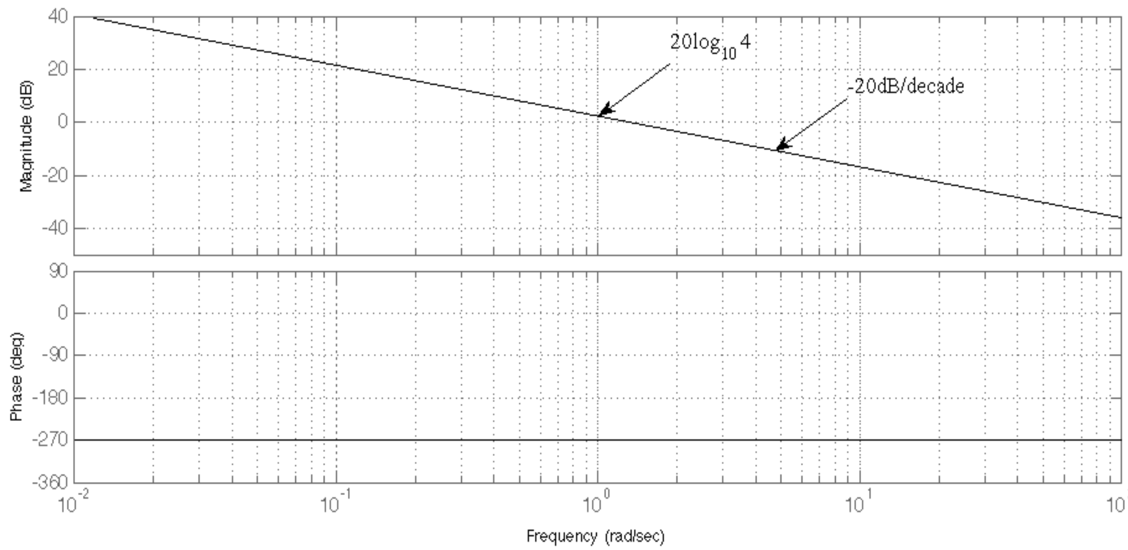

Figure 10. The Bode plot of $-4\frac{1}{j\omega}$.

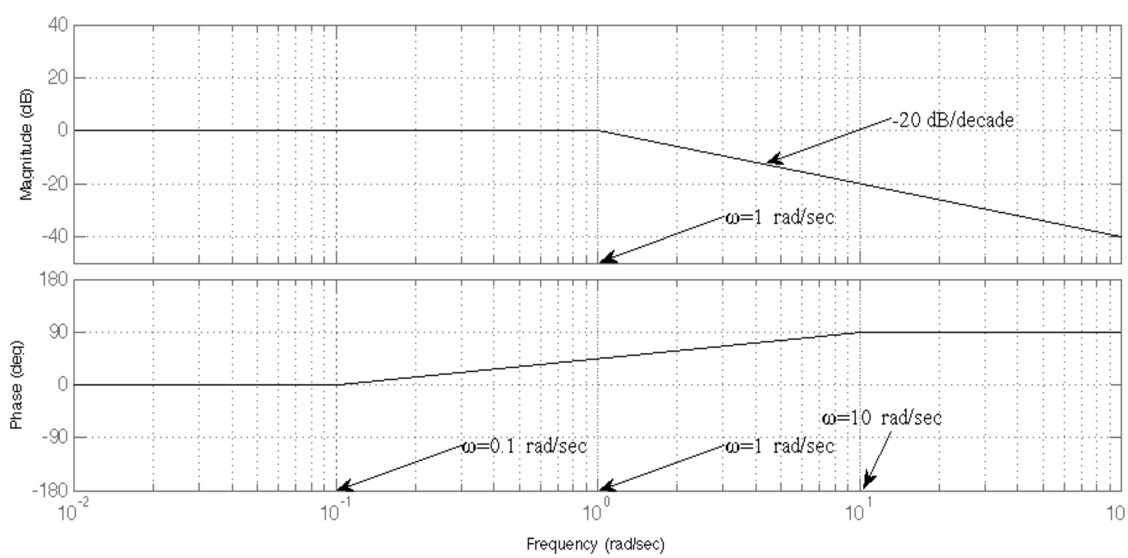

Figure 11. The Bode plot of $\frac{1}{1-j\omega}$.

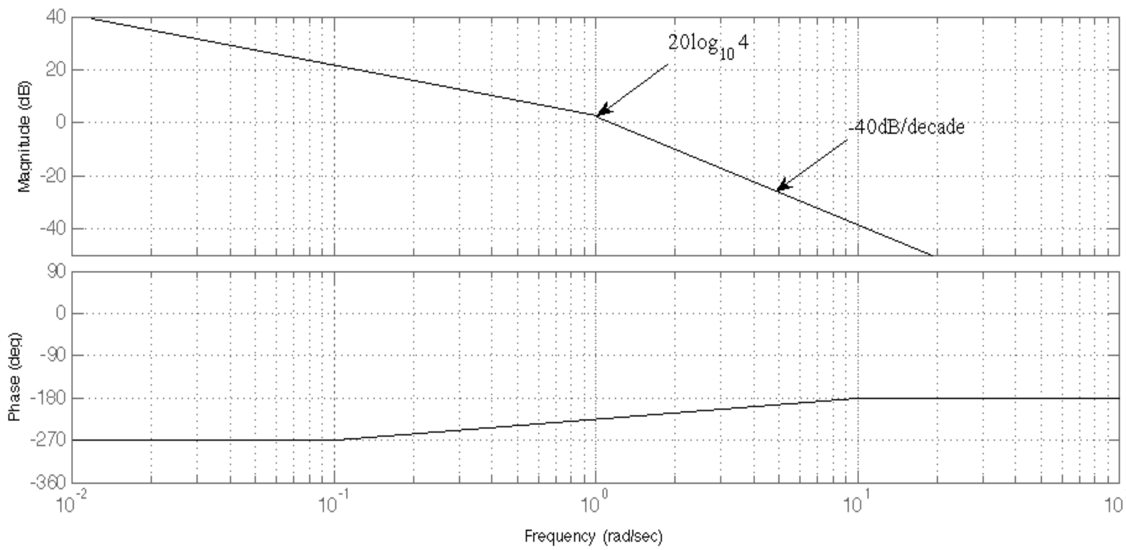

Figure 12. The Bode plot of $-4\frac{1}{j\omega(1-j\omega)}$.

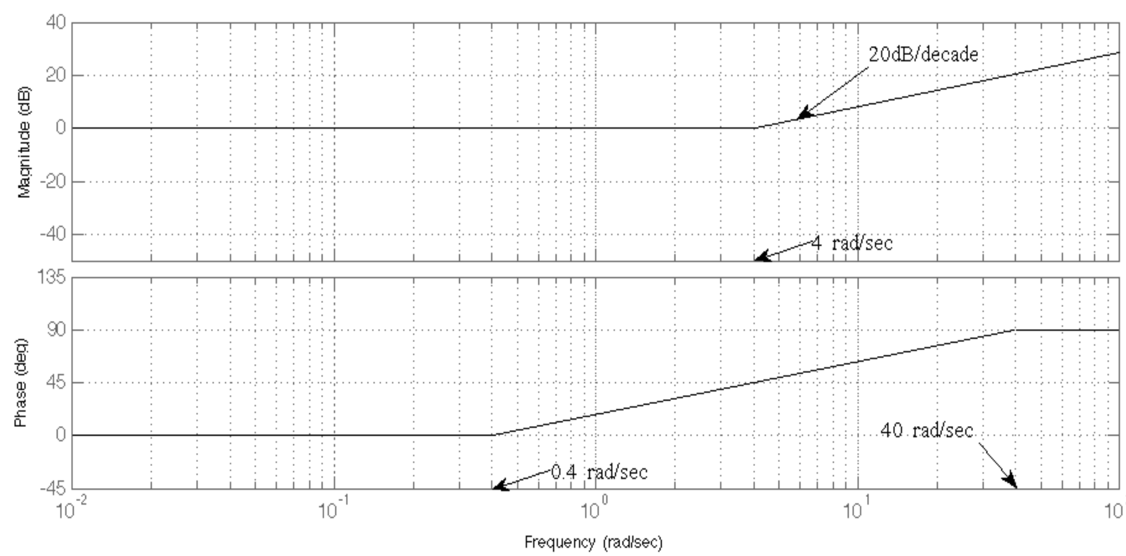

Figure 13. The Bode plot of $1+\frac{j\omega}{4}$.

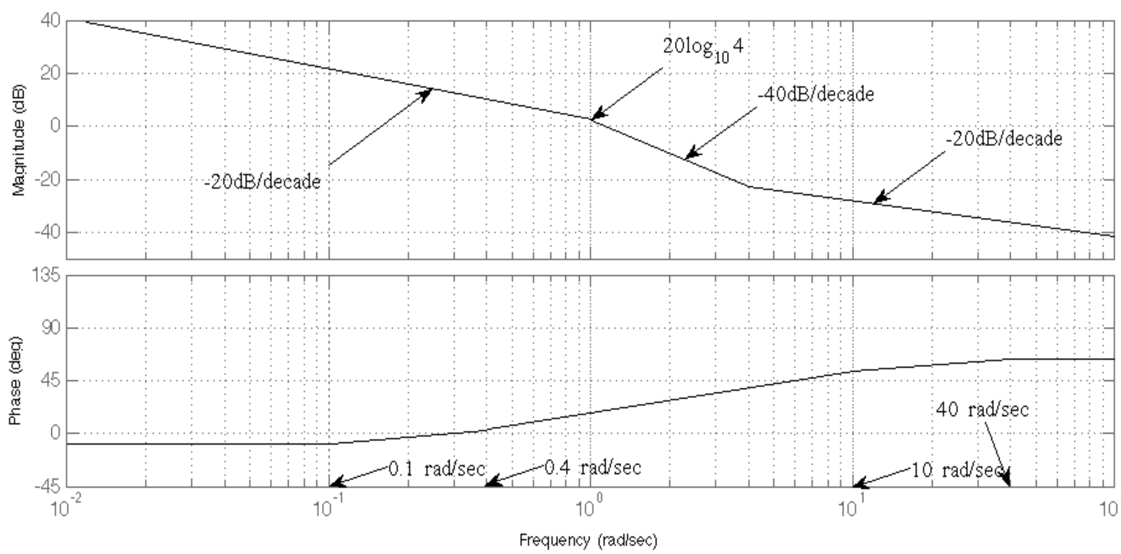

Figure 14. The Bode plot of $W(j\omega) = -4\frac{1+\frac{j\omega}{4}}{j\omega(1-j\omega)}$ (*Errata Corrige*: in the above figure the phase plot is wrong while the phase value at $\omega = 10^{-2}$ is $-270°$ and the phase value at $\omega = 10^2$ is $-90°$).