# Notes on Linear Control Systems: Module III

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***Abstract*—Laplace (and inverse Laplace) transformation and its properties. Differential models in Laplace domain. Modes, eigenvalues and poles. Transfer functions. Residuals theorem and decomposition of rational functions in simple fractional terms. Calculus of forced and unforced state and output responses in Laplace domain.**

## I. The mathematical model in Laplace domain

It is useful to consider a state space model, described by differential equations, from a different perspective using the so called Laplace transform (appendix A). The reasons for this change of perspective is both for the technical advantages the Laplace transform has over the time analysis and for its relevance and direct application in an experimental setup, which leads to the so called armonic or frequency analysis. To this aim, we want to obtain the Laplace transform of the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t),\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t),\end{aligned}$$

In particular, on account of proposition A.2 and the linearity property (see the appendix) of the Laplace transform

$$\begin{aligned}&\mathfrak{L}[\frac{d}{dt}\mathbf{x}(t)](s) = s\mathfrak{L}[\mathbf{x}(t)](s) - x_0 = \mathfrak{L}[A\mathbf{x}(t) + B\mathbf{u}(t)](s)\\ &= A\mathfrak{L}[\mathbf{x}(t)](s) + B\mathfrak{L}[\mathbf{u}(t)](s)\end{aligned}$$

Therefore

$$(sI - A)\mathfrak{L}[\mathbf{x}(t)](s) = x_0 + B\mathfrak{L}[\mathbf{u}(t)](s)$$

so that

$$\mathfrak{L}[\mathbf{x}(t)](s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\mathfrak{L}[\mathbf{u}(t)](s) \quad (1)$$

Recalling the form of the solution $\mathbf{x}(t)$, on account of proposition A.3 and the linearity property of the Laplace transform

$$\begin{aligned}&\mathfrak{L}[\mathbf{x}(t)](s) = \mathfrak{L}[e^{At}x_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}_\tau d\tau](s)\\ &= \mathfrak{L}[e^{At}](s)x_0 + \mathfrak{L}[e^{At}B](s)\mathfrak{L}[\mathbf{u}(t)](s)\\ &= \mathfrak{L}[\mathbf{\Phi}(t)](s)x_0 + \mathfrak{L}[\mathbf{H}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s) \quad (2)\end{aligned}$$

By comparison of (1) and (2)

$$\begin{aligned}&(sI - A)^{-1}x_0 + (sI - A)^{-1}B\mathfrak{L}[\mathbf{u}(t)](s)\\ &= \mathfrak{L}[\mathbf{\Phi}(t)](s)x_0 + \mathfrak{L}[\mathbf{H}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s)\end{aligned}$$

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for all input functions $\mathbf{u}$ and $x_0 \in \mathbb{R}^n$. It follows that

$$\begin{aligned}\mathfrak{L}[\mathbf{\Phi}(t)](s) &= (sI - A)^{-1}\\ \mathfrak{L}[\mathbf{H}(t)](s) &= (sI - A)^{-1}B \quad (3)\end{aligned}$$

The matrix $\mathfrak{L}[\mathbf{H}(t)](s)$ is known as *input-to-state (I/S) transfer function* matrix. Similarly, by linearity of the Laplace transform and (2)

$$\begin{aligned}&\mathfrak{L}[\mathbf{y}(t)](s) = \mathfrak{L}[C\mathbf{x}(t) + D\mathbf{u}(t)](s)\\ &= C\mathfrak{L}[\mathbf{x}(t)](s) + D\mathfrak{L}[\mathbf{u}(t)](s) = C(\mathfrak{L}[\mathbf{\Phi}(t)](s)x_0\\ &+\mathfrak{L}[\mathbf{H}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s)) + D\mathfrak{L}[\mathbf{u}(t)](s)\\ &= C\mathfrak{L}[\mathbf{\Phi}(t)](s)x_0 + [C\mathfrak{L}[\mathbf{H}(t)](s) + D]\mathfrak{L}[\mathbf{u}(t)](s)\\ &= C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]\mathfrak{L}[\mathbf{u}(t)](s) \quad (4)\end{aligned}$$

and recalling the form of the output $\mathbf{y}(t)$

$$\begin{aligned}&\mathfrak{L}[\mathbf{y}(t)](s) = \mathfrak{L}[Ce^{At}x_0\\ &+ \int_0^t (Ce^{A(t-\tau)}B + D\boldsymbol{\delta}^{(0)}_{t-\tau})\mathbf{u}_\tau d\tau](s)\\ &= \mathfrak{L}[Ce^{At}](s)x_0 + \mathfrak{L}[Ce^{At}B + D\boldsymbol{\delta}^{(0)}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s)\\ &= \mathfrak{L}[\mathbf{\Psi}(t)](s)x_0 + \mathfrak{L}[\mathbf{W}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s) \quad (5)\end{aligned}$$

By comparison of (4) and (5)

$$\begin{aligned}&C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]\mathfrak{L}[\mathbf{u}(t)](s)\\ &= \mathfrak{L}[\mathbf{\Psi}(t)](s)x_0 + \mathfrak{L}[\mathbf{W}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s)\end{aligned}$$

for all all input functions $\mathbf{u}$ and $x_0 \in \mathbb{R}^n$. It follows that

$$\begin{aligned}\mathfrak{L}[\mathbf{\Psi}(t)](s) &= C(sI - A)^{-1}\\ \mathfrak{L}[\mathbf{W}(t)](s) &= C(sI - A)^{-1}B + D \quad (6)\end{aligned}$$

The matrix $\mathfrak{L}[\mathbf{W}(t)](s)$ is known as *input-to-output (I/O) transfer function* matrix. Since

$$\mathfrak{L}[\mathbf{y}(t)](s) = \mathfrak{L}[\mathbf{\Psi}(t)](s)x_0 + \mathfrak{L}[\mathbf{W}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s)$$

clearly

$$\begin{aligned}\mathfrak{L}[\mathbf{y}^{(0)}(t, x_0)](s) &= \mathfrak{L}[\mathbf{\Psi}(t)](s)x_0\\ \mathfrak{L}[\mathbf{y}^{(u)}(t, \mathbf{u})](s) &= \mathfrak{L}[\mathbf{W}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s) \quad (7)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathfrak{L}[\mathbf{x}^{(0)}(t, x_0)](s) &= \mathfrak{L}[\mathbf{\Phi}(t)](s)x_0\\ \mathfrak{L}[\mathbf{x}^{(u)}(t, \mathbf{u})](s) &= \mathfrak{L}[\mathbf{H}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s) \quad (8)\end{aligned}$$

## II. Residuals method for decomposition of rational functions

As it results from (7) and (8) it is possible to calculate the state and output responses in the Laplace domain by means of the Laplace transforms $\mathfrak{L}[\mathbf{\Phi}(t)](s)$, $\mathfrak{L}[\mathbf{H}(t)](s)$, $\mathfrak{L}[\mathbf{\Psi}(t)](s)$ and $\mathfrak{L}[\mathbf{W}(t)](s)$ together with the Laplace transform of the input function. The state and output responses in the time domain are obtained as inverse Laplace transforms. For exercise, with an I/O transfer function $\mathfrak{L}[\mathbf{W}(t)](s) = \frac{1}{s+1}$ the forced output response to the unit step input $\mathbf{u}(t) = \boldsymbol{\delta}^{(-1)}(t)$ (appendix A1) is in Laplace domain

$$\mathfrak{L}[\mathbf{y}^{(u)}(t,\mathbf{u})](s) = \mathfrak{L}[\mathbf{W}(t)](s)\mathfrak{L}[\mathbf{u}(t)](s) = \frac{1}{s(s+1)}$$

and in the time domain is

$$\mathfrak{L}^{-1}[\frac{1}{s(s+1)}](t) = (\mathbf{y}^{(u)}(t,\mathbf{u}))_+ \tag{9}$$

where $(\mathbf{y}^{(u)}(t,\mathbf{u}))_+$ denotes the signal associated to $\mathbf{y}^{(u)}(t,\mathbf{u})$ (appendix A7). The inverse Laplace transform of $\frac{1}{s(s+1)}$ cannot be obtained from known transforms: we know the inverse Laplace transform of $\frac{1}{s}$ and $\frac{1}{s+1}$ separately but not of their product. We want to find a method for calculating the inverse Laplace transform of rational functions, which are the most common functions we have to deal with in the calculation of an inverse Laplace transform.

Let $\mathbf{G}(s)$ be a $(n \times m)$ matrix of proper rational functions. We recall that a rational function $\mathbf{g}(s)$ has the form

$$\mathbf{g}(s) = \frac{\mathbf{a}(s)}{\mathbf{b}(s)}$$

where $\mathbf{a}(s)$ and $\mathbf{b}(s)$ are polynomials and deg$[\mathbf{a}]$ and deg $[\mathbf{b}]$ denote the degree of $\mathbf{a}(s)$ and, respectively, $\mathbf{b}(s)$. The rational function $\mathbf{G}(s)$ is *strictly proper* if deg$[\mathbf{a}] <$ deg$[\mathbf{a}]$, *proper* if deg$(a) \leqslant$ deg$[\mathbf{b}]$ and *improper* if deg$[\mathbf{a}] >$ deg$[\mathbf{b}]$. We want to study a method for finding a $(n \times m)$ matrix $\mathbf{Q}(t)$ such that

$$\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)](t). \tag{10}$$

First of all, we can always find a constant $(n \times m)$ matrix $G_0$ such that $\mathbf{G}(s) = G_0 + \mathbf{G}_1(s)$ and $\mathbf{G}_1(s)$ is a $(n \times m)$ matrix of strictly proper rational functions (either use the comparison method or divide the numerator of each element of $\mathbf{G}(s)$ by its denominator). For instance, the function

$$\frac{s+2}{s+1} \tag{11}$$

can be decomposed as

$$\frac{s+2}{s+1} = 1 + \frac{1}{s+1} \tag{12}$$

This can be done by using a comparison method or, alternatively, Ruffini's method. With a comparison method, for some reals $\alpha, \beta$ to be determined we have

$$\frac{s+2}{s+1} = \alpha + \frac{\beta}{s+1} \tag{13}$$

Therefore,

$$\frac{s+2}{s+1} = \alpha + \frac{\beta}{s+1} = \frac{\alpha s + \alpha + \beta}{s+1} \tag{14}$$

and by comparing the coefficients of the corresponding powers of $s$ in the right and left-hand polynomials of (14) we obtain the equations in the unknowns $\alpha, \beta$

$$\begin{aligned} 1 &= \alpha \\ 2 &= \alpha + \beta \end{aligned} \tag{15}$$

i.e. $\alpha = 1$ and $\beta = 1$.

It is easy to see that any strictly proper $\mathbf{G}(s)$ can be always written as

$$\mathbf{G}(s) = \frac{\overline{\mathbf{G}}(s)}{\mathbf{m}(s)} \tag{16}$$

where $\overline{\mathbf{G}}(s)$ is a $(n \times m)$ matrix of polynomial functions and $\mathbf{m}(s)$ is the m.c.m. of all the polynomials $\mathbf{b}_{i,j}(s)$, where $\mathbf{b}_{i,j}(s)$ is the denominator of $[\mathbf{G}(s)]_{i,j} := \frac{\mathbf{a}_{i,j}(s)}{\mathbf{b}_{i,j}(s)}$, the $(i,j)$ element of the matrix $\mathbf{G}(s)$. We can assume that $\mathbf{m}(s)$ has the form

$$\mathbf{m}(s) = \prod_{j=1}^{r}(s-\lambda_j)^{\mu_j} \tag{17}$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct roots of $\mathbf{m}(s)$ and $\mu_j$ denotes the multiplicity of $\lambda_j$. The numbers $\lambda_1, \ldots, \lambda_r$ are the *poles* of $\mathbf{G}(s)$. We want to prove the following result.

**Proposition** *2.1: (Residuals theorem). Let $\mathbf{G}(s)$ be a $(n\times m)$ matrix of strictly proper rational functions and let $\lambda_1, \ldots, \lambda_r$ be its distinct poles with multiplicity $\mu_1, \ldots, \mu_r$. Then*

$$\mathbf{G}(s) = \sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\frac{R_{i,j}}{(s-\lambda_i)^j} \tag{18}$$

*where $R_{i,j}$ is the so-called residual of order $j$ associated to the pole $\lambda_i$ of $\mathbf{G}(s)$ defined as*

$$R_{i,j} := \frac{1}{(\mu_i - j)!}\lim_{s\to\lambda_i}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}[\mathbf{G}(s)(s-\lambda_i)^{\mu_i}] \tag{19}$$

*Proof (can be omitted).* The decomposition (18) for some matrices $R_{i,j}$ to be determined follows from the fact that $\lambda_i$ is a pole of $\mathbf{G}(s)$ with multiplicity $\mu_i$. We have

$$\mathbf{G}(s)(s-\lambda_i)^{\mu_i} = \sum_{h=1}^{r}\sum_{k=1}^{\mu_h}\frac{R_{h,k}}{(s-\lambda_h)^k}(s-\lambda_i)^{\mu_i} \tag{20}$$

Moreover, since

$$\lim_{s\to\lambda_i}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}\frac{(s-\lambda_i)^{\mu_i}}{(s-\lambda_h)^k} = 0$$

for all $h \neq i$, $h, i = 1, \ldots, r$ and $j = 1, \ldots, \mu_i$

$$\begin{aligned} &\lim_{s\to\lambda_i}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}[\mathbf{G}(s)(s-\lambda_i)^{\mu_i}] \\ &= \lim_{s\to\lambda_i}\sum_{h=1}^{r}\sum_{k=1}^{\mu_h}R_{h,k}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}\frac{(s-\lambda_i)^{\mu_i}}{(s-\lambda_h)^k} \\ &= \lim_{s\to\lambda_i}\sum_{k=1}^{\mu_i}R_{i,k}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}(s-\lambda_i)^{\mu_i-k} \end{aligned}$$

and

$$\lim_{s\to\lambda_i}\frac{d^{\mu_i-j}}{ds^{\mu_i-j}}(s-\lambda_i)^{\mu_i-k}$$

for all $k \neq j$ and $j, k = 1, \ldots, \mu_i$,

$$\lim_{s\to\lambda_i} \sum_{k=1}^{\mu_i} R_{i,k} \frac{d^{\mu_i - j}}{ds^{\mu_i - j}} (s-\lambda_i)^{\mu_i - k}$$
$$= \lim_{s\to\lambda_i} \sum_{k=1}^{\mu_i} R_{i,k} (\mu_i - k)(\mu_i - k - 1)\cdots$$
$$\cdots(-k+j+1)(s-\lambda_i)^{-k+j}$$
$$= R_{i,j}(\mu_i - j)(\mu_i - j - 1)\cdots 2\cdot 1 = R_{i,j}(\mu_i - j)!$$

which proves formula (19). □

Recall that (appendix A4)

$$\mathfrak{L}[\frac{t_+^{j-1}}{(j-1)!}](s) = \frac{1}{s^j} \tag{21}$$

and (frequency translation property in appendix A7)

$$\mathfrak{L}[\frac{e^{\lambda_i t} t_+^{j-1}}{(j-1)!}](s) = \frac{1}{(s-\lambda_i)^j} \tag{22}$$

Therefore, in view of (18)

$$\mathbf{Q}(t) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} R_{i,j} e_+^{\lambda_i t} \frac{t_+^{j-1}}{(j-1)!} \tag{23}$$

is such that $\mathbf{G}(s) = \mathfrak{L}[\mathbf{Q}(t)](s)$ and therefore $\mathbf{Q}(t)$ solves our problem (10).

**Exercize** *2.1: Find* $\mathbf{Q}(t)$ *such that* $\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)](t)$ *where* $\mathbf{G}(s) := \frac{10}{s(s+1)}$.

We have from (18)

$$\mathbf{G}(s) = \frac{10}{s(s+1)} = \frac{R_1}{s} + \frac{R_2}{s+1} \tag{24}$$

and

$$R_1 = \lim_{s\to 0} \mathbf{G}(s)s = 10$$
$$R_2 = \lim_{s\to -1} \mathbf{G}(s)(s+1) = -10.$$

Therefore,

$$\mathbf{G}(s) = \frac{s}{(s-1)^2} = \frac{10}{s} - \frac{10}{s+1} \tag{25}$$

and using the linearity property of the Laplace transform and the frequency translation property (appendix A7)

$$\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s-1)](t) + \mathfrak{L}^{-1}[\mathfrak{L}[t_+](s+1)](t)$$
$$= 10\boldsymbol{\delta}^{(-1)}(t) - 10e_+^t = 10\boldsymbol{\delta}^{(-1)}(t)[1+e_+^t].\lhd \tag{26}$$

**Exercize** *2.2: Find* $\mathbf{Q}(t)$ *such that* $\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)](t)$ *where* $\mathbf{G}(s) := \frac{s}{(s-1)^2}$.

We have from (18)

$$\mathbf{G}(s) = \frac{s}{(s-1)^2} = \frac{R_1}{s-1} + \frac{R_2}{(s-1)^2} \tag{27}$$

and

$$R_2 = \lim_{s\to 1} \mathbf{G}(s)(s-1)^2 = 1$$
$$R_1 = \lim_{s\to 1} \frac{d}{ds}(\mathbf{G}(s)(s-\lambda_i)^{\mu_i}) = 1$$

Therefore,

$$\mathbf{G}(s) = \frac{s}{(s-1)^2} = \frac{1}{s-1} + \frac{1}{(s-1)^2} \tag{28}$$

and using the linearity property of the Laplace transform and the frequency translation property (appendix A7)

$$\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s-1)](t) + \mathfrak{L}^{-1}[\mathfrak{L}[t_+](s-1)](t)$$
$$= e_+^t + e_+^t t_+ = e_+^t[1+t_+].\lhd \tag{29}$$

**Exercize** *2.3: Find the function* $\mathbf{Q}(t)$ *such that* $\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)](t)$ *where* $\mathbf{G}(s) := \frac{s+1}{s^2+s+1}$.

We have from (18)

$$\mathbf{G}(s) = \frac{s+1}{s^2+s+1} = \frac{R_1}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}} + \frac{R_2}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

and

$$R_1 = \lim_{s\to -\frac{1}{2}-j\frac{\sqrt{3}}{2}} [\mathbf{G}(s)(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})]$$
$$= \frac{\frac{1}{2}-j\frac{\sqrt{3}}{2}}{-j\sqrt{3}} = \frac{1}{2}+j\frac{\sqrt{3}}{6}$$
$$R_2 = \lim_{s\to -\frac{1}{2}+j\frac{\sqrt{3}}{2}} [\mathbf{G}(s)(s+\frac{1}{2}-j\frac{\sqrt{3}}{2})]$$
$$= \frac{1}{2}-j\frac{\sqrt{3}}{6} = R_1^*$$

Therefore,

$$\mathbf{G}(s) = \frac{s+1}{s^2+s+1} = \frac{\frac{1}{2}+j\frac{\sqrt{3}}{6}}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}} + \frac{\frac{1}{2}-j\frac{\sqrt{3}}{6}}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

and using the linearity and frequency translation properties of the Laplace transform

$$\mathbf{Q}(t) = \mathfrak{L}^{-1}[\frac{\frac{1}{2}+j\frac{\sqrt{3}}{6}}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}}](t) + \mathfrak{L}^{-1}[\frac{\frac{1}{2}-j\frac{\sqrt{3}}{6}}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}](t)$$
$$= (\frac{1}{2}+j\frac{\sqrt{3}}{6})\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s+\frac{1}{2}+j\frac{\sqrt{3}}{2})](t)$$
$$+(\frac{1}{2}-j\frac{\sqrt{3}}{6})\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s+\frac{1}{2}-j\frac{\sqrt{3}}{2})](t)$$
$$= (\frac{1}{2}+j\frac{\sqrt{3}}{6})e_+^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t} + (\frac{1}{2}-j\frac{\sqrt{3}}{6})e_+^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}$$
$$= e^{-\frac{t}{2}}[(\frac{1}{2}+j\frac{\sqrt{3}}{6})e_+^{-j\frac{\sqrt{3}}{2}t} + (\frac{1}{2}-j\frac{\sqrt{3}}{6})e_+^{j\frac{\sqrt{3}}{2}t}]$$
$$= e^{-\frac{t}{2}}\Big((\frac{1}{2}+j\frac{\sqrt{3}}{6})(\cos_+(\frac{\sqrt{3}}{2}t) - j\sin_+(\frac{\sqrt{3}}{2}t))$$
$$+(\frac{1}{2}-j\frac{\sqrt{3}}{6})(\cos_+(\frac{\sqrt{3}}{2}t) + j\sin_+(\frac{\sqrt{3}}{2}t))\Big)$$
$$= e^{-\frac{t}{2}}\Big(\cos_+(\frac{\sqrt{3}}{2}t) + \frac{\sqrt{3}}{3}\sin_+(\frac{\sqrt{3}}{2}t)\Big) \tag{30}$$

Note that for obtaining the inverse transform of second order functions $\mathbf{G}(s)$ with complex conjugate poles, alternatively it is possible to proceed as follows. Rewrite

$$\begin{aligned}
\mathbf{G}(s) &= \frac{s+1}{s^2+s+1} = \frac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} \\
&= \mathfrak{L}[\cos_+(\frac{\sqrt{3}}{2}t)](s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\mathfrak{L}[\sin_+(\frac{\sqrt{3}}{2}t)](s+\frac{1}{2})
\end{aligned}$$

Therefore, using the inverse transforms of sinus and cosinus and the frequency translation property of the Laplace transform

$$\begin{aligned}
\mathbf{Q}(t) &= \mathfrak{L}^{-1}[\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}](t) + \mathfrak{L}^{-1}[\frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}](t) \\
&= \mathfrak{L}^{-1}[\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}](t) + \frac{1}{\sqrt{3}}\mathfrak{L}^{-1}[\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}](t) \\
&= \mathfrak{L}^{-1}[\mathfrak{L}[\cos_+(\frac{\sqrt{3}}{2}t)](s+\frac{1}{2})](t) \\
&+ \frac{1}{\sqrt{3}}\mathfrak{L}^{-1}[\mathfrak{L}[\sin_+(\frac{\sqrt{3}}{2}t)](s+\frac{1}{2})](t) \\
&= e_+^{-\frac{t}{2}}[\cos(\frac{\sqrt{3}}{2}t) + \frac{\sqrt{3}}{3}\sin(\frac{\sqrt{3}}{2}t)].\lhd \qquad (31)
\end{aligned}$$

**Exercize** *2.4: Find a function* $\mathbf{Q}(t)$ *such that* $\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)](t)$ *where*

$$\mathbf{G}(s) := \frac{1}{(s-3)(s+2)}\begin{pmatrix} s-2 & 4 \\ 1 & s+1 \end{pmatrix}$$

We have from (18)

$$\begin{aligned}
\mathbf{G}(s) &= \frac{1}{(s-3)(s+2)}\begin{pmatrix} s-2 & 4 \\ 1 & s+1 \end{pmatrix} \\
&= \frac{R_1}{s-3} + \frac{R_2}{s+2} \qquad (32)
\end{aligned}$$

and

$$\begin{aligned}
R_1 &= \lim_{s\to 3}(\mathbf{G}(s)(s-3)) = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \\
R_2 &= \lim_{s\to -2}(\mathbf{G}(s)(s+2)) = \begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \qquad (33)
\end{aligned}$$

Therefore,

$$\mathbf{G}(s) = \frac{\begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}}{s-3} + \frac{\begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}}{s+2} \qquad (34)$$

and the linearity and frequency translation properties of the transform

$$\begin{aligned}
\mathbf{Q}(t) &= \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}}{s-3}](t) + \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}}{s+2}](t) \\
&= \begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s-3)](t) \\
&+ \begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s+2)](t) \\
&= \begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} e_+^{3t} + \begin{pmatrix} \frac{4}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} e_+^{-2t}.\lhd \qquad (35)
\end{aligned}$$

**Exercize** *2.5: Calculate* $\mathbf{x}^{(0)}(t, x_0)$ *with* $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ *and* $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We have

$$\begin{aligned}
\mathfrak{L}[\mathbf{\Phi}(t)](s) &= (sI-A)^{-1} = \begin{pmatrix} s-2 & 1 \\ 2 & s-3 \end{pmatrix}^{-1} \\
&= \frac{1}{s^2-5s+4}\begin{pmatrix} s-3 & -1 \\ -2 & s-2 \end{pmatrix} \qquad (36)
\end{aligned}$$

Since $s^2-5s+4 = (s-4)(s-1)$ from (18)

$$\begin{aligned}
(sI-A)^{-1} &= \frac{1}{(s-4)(s-1)}\begin{pmatrix} s-3 & -1 \\ -2 & s-2 \end{pmatrix} \\
&= \frac{R_1}{s-4} + \frac{R_2}{s-1} \qquad (37)
\end{aligned}$$

and

$$\begin{aligned}
R_1 &= \lim_{s\to 4}((sI-A)^{-1}(s-4)) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \\
R_2 &= \lim_{s\to 1}((sI-A)^{-1}(s-1)) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad (38)
\end{aligned}$$

Therefore,

$$(sI-A)^{-1} = \frac{\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}}{s-4} + \frac{\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}}{s-1} \qquad (39)$$

and the linearity and frequency translation properties of the transform

$$\begin{aligned}
e_+^{At} &= \mathfrak{L}^{-1}[(sI-A)^{-1}](t) \\
&= \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}}{s-4}](t) + \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}}{s-1}](t) \\
&= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s-4)](t) \\
&+ \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s-1)](t) \\
&= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix} e_+^{4t} + \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} e_+^{t} \qquad (40)
\end{aligned}$$

The unforced state response ensuing from $x_0$ is

$$\begin{aligned}
\mathbf{x}^{(0)}(t, x_0) &= e_+^{At}x_0 \\
&= \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e_+^{4t} + \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e_+^{t} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e_+^{t}.\lhd \qquad (41)
\end{aligned}$$

**Exercize** *2.6: Calculate the forced output response with*

$$W(s) = \frac{s^2+7s+1}{s^2+5s+6}$$

*and input function*$\mathbf{u}(t) := (1+2\sin(2t))\boldsymbol{\delta}^{(-1)}(t)$. Using the linearity and time translation properties of the transform

$$\mathfrak{L}[\mathbf{u}(t)](s) = \frac{1}{s} + \frac{4}{s^2+4} = \frac{s^2+4s+4}{s(s^2+4)} \tag{42}$$

Therefore,

$$y(s) = W(s)u(s) = \frac{s^2+7s+1}{s^2+5s+6}\frac{s^2+4s+4}{s(s^2+4)} \tag{43}$$

Using the residuals theorem

$$\begin{aligned}\mathbf{G}(s) &= \frac{s^2+7s+1}{s^2+5s+6}\frac{s^2+4s+4}{s(s^2+4)} \\ &= \frac{R_1}{s+3} + \frac{R_2}{s+2} + \frac{R_3}{s} + \frac{Ds+E}{s^2+4} \end{aligned} \tag{44}$$

where $R_1, R_2$ and $R_3$ can be computed with the residuals' formula and $D$ and $E$ can be computed with comparison method (see exercize 2.3). By the linearity and frequency translation properties of the transform

$$\mathfrak{L}^{-1}[\mathbf{G}(s)](t) = R_1 e_+^{-3t} + R_2 e_+^{-2t} + R_3\boldsymbol{\delta}^{(-1)}(t)$$
$$+D\cos_+(2t) + E\sin_+(2t).\lhd$$

The method of residuals for calculating the inverse Laplace transform of rational function can be extended to products of rational functions with exponentials $e^{-as}$, which correspond to time translations in the time domain. Let $\mathbf{G}(s)$, $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_r$ be as above. In what follows, we point out the method for finding a $(n \times m)$ matrix $\mathbf{Q}(t)$ such that

$$\mathbf{Q}(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)\sum_{j=1}^{N} e^{-a_j s}](t) \tag{45}$$

where $N \in \mathbb{N}$ and $a_1, \ldots, a_N$ are positive reals. In this case the above residual method cannot be applied since $e^{-a_j s}$ is not a rational function. We can proceed as follows. In view of (18)

$$\mathbf{f}(t) := \sum_{i=1}^{r}\sum_{j=1}^{\mu_i} R_{i,j} e_+^{\lambda_i t}\frac{t_+^{j-1}}{(j-1)!} \tag{46}$$

where $R_{i,j}$ are determined as in (19), has Laplace transform $\mathbf{G}(s)$. On account of the time translation property (appendix A7),

$$\mathbf{Q}(t) := \sum_{j=1}^{N}\mathbf{f}(t-a_j) \tag{47}$$

has Laplace transform $\mathbf{G}(s)\sum_{j=1}^{N} e^{-a_j s}$ and therefore $\mathbf{Q}(t)$ solves our problem (45).

**Exercize** *2.7: Calculate the forced output response with*

$$W(s) = \frac{1}{s+2}$$

*and input function*

$$\mathbf{u}(t) := \begin{cases} t & \text{for } 0 \leqslant t < 1 \\ -t+2 & \text{for } 1 \leqslant t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

The input function can be written as

$$\begin{aligned}\mathbf{u}(t) &= t\boldsymbol{\delta}^{(-1)}(t) - 2(t-1)\boldsymbol{\delta}^{(-1)}(t-1) \\ &+ (t-3)\boldsymbol{\delta}^{(-1)}(t-3)\end{aligned} \tag{49}$$

Using the linearity and time translation properties of the transform

$$\mathfrak{L}[\mathbf{u}(t)](s) = \mathfrak{L}[t_+](s) - 2\mathfrak{L}[(t-1)_+](s) + \mathfrak{L}[(t-2)_+](s)$$
$$= \frac{1}{s^2(s+2)}(1-2e^{-s}+e^{-2s}) \tag{50}$$

We cannot apply the residual method to $\mathfrak{L}[\mathbf{y}(t)^{(\text{forced})}](s)$ for the presence of the irrational functions $e^{-s}$ and $e^{-2s}$. We use the residual method to get the inverse transform of the rational component $\frac{1}{s^2(s+2)}$ of $\mathfrak{L}[\mathbf{y}(t)^{(\text{forced})}](s)$. If

$$\mathbf{G}(s) := \frac{1}{s^2(s+2)} = \frac{R_{1,1}}{s} + \frac{R_{1,2}}{s^2} + \frac{R_2}{s+2}$$

and

$$R_{1,2} = \lim_{s\to 0}(\mathbf{G}(s)s^2) = \frac{1}{2}$$
$$R_{1,1} = \lim_{s\to 0}\frac{d}{ds}(\mathbf{G}(s)s^2) = -\frac{1}{4}$$
$$R_2 = \lim_{s\to -2}(\mathbf{G}(s)(s+2)) = \frac{1}{4}$$

Therefore,

$$\mathbf{G}(s) = -\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} \tag{51}$$

and by the linearity and frequency translation properties of the transform

$$\mathfrak{L}^{-1}[\mathbf{G}(s)](t) = -\frac{1}{4}\mathfrak{L}^{-1}[\frac{1}{s}](t) + \frac{1}{2}\mathfrak{L}^{-1}[\frac{1}{s^2}](t)$$
$$+\frac{1}{4}\mathfrak{L}^{-1}[\frac{1}{s+2}](t) = -\frac{1}{4}\boldsymbol{\delta}^{(-1)}(t) + \frac{1}{2}t_+ + \frac{1}{4}e_+^{-2t} \tag{52}$$

It follows that

$$\mathbf{y}^{(u)}(t,\mathbf{u}) = \mathfrak{L}^{-1}[\mathbf{G}(s)(1-2e^{-s}+e^{-2s})](t)$$
$$= \mathfrak{L}^{-1}[\mathbf{G}(s)](t) - 2\mathfrak{L}^{-1}[\mathbf{G}(s)](t-1)$$
$$+\mathfrak{L}^{-1}[\mathbf{G}(s)](t-2).\lhd$$

**Exercize** *2.8: Calculate the forced output response with*

$$A = \begin{pmatrix} -1 & -1 & 0 \\ -2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

*and input function*

$$\mathbf{u}(t) := \begin{cases} 1 & \text{for } 1 \leqslant t < 3 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

The input function can be written as

$$\mathbf{u}(t) = \boldsymbol{\delta}^{(-1)}(t-1) - \boldsymbol{\delta}^{(-1)}(t-3) \tag{54}$$

Using the linearity and time translation properties of the transform

$$\begin{aligned} &\mathfrak{L}[\mathbf{u}(t)](s) = \mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t-1)](s) - \mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t-3)](s) \\ &= \frac{1}{s}(e^{-s} - e^{-3s}) \end{aligned} \tag{55}$$

Moreover

$$\begin{aligned} &\mathfrak{L}[\mathbf{W}(t)](s) = C(sI-A)^{-1}B \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} s+1 & 1 & 0 \\ 2 & s+2 & 0 \\ -1 & -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{s(s+1)(s+3)} \times \\ &\times \begin{pmatrix} \frac{2}{3}(s+1)(s+3) + \frac{1}{3}(s+1)s \\ -\frac{2}{3}(s+1)(s+3) + \frac{3}{2}s(s+3) + \frac{1}{6}(s+1)s \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{L}[\mathbf{y}(t)^{(\text{forced})}](s) = C(sI-A)^{-1}B\mathfrak{L}[\mathbf{u}(t)](s) \\ &= \frac{e^{-s} - e^{-3s}}{s^2(s+1)(s+3)} \times \\ &\times \begin{pmatrix} \frac{2}{3}(s+1)(s+3) + \frac{1}{3}(s+1)s \\ -\frac{2}{3}(s+1)(s+3) + \frac{3}{2}s(s+3) + \frac{1}{6}(s+1)s \end{pmatrix} \end{aligned} \tag{56}$$

We cannot apply the residual method to $\mathfrak{L}[\mathbf{y}(t)^{(\text{forced})}](s)$ for the presence of the irrational functions $e^{-s}$ and $e^{-3s}$. We use the residual method to get the inverse transform of the rational component of $\mathfrak{L}[\mathbf{y}(t)^{(\text{forced})}](s)$. If

$$\begin{aligned} &\mathbf{G}(s) := \frac{1}{s^2(s+1)(s+3)} \times \\ &\times \begin{pmatrix} \frac{2}{3}(s+1)(s+3) + \frac{1}{3}(s+1)s \\ -\frac{2}{3}(s+1)(s+3) + \frac{3}{2}s(s+3) + \frac{1}{6}(s+1)s \end{pmatrix} \end{aligned}$$

from (18)

$$\mathbf{G}(s) = \frac{R_{1,1}}{s} + \frac{R_{1,2}}{s^2} + \frac{R_2}{s+3} + \frac{R_3}{s+1}$$

and

$$\begin{aligned} R_{1,2} &= \lim_{s\to 0}(\mathbf{G}(s)s^2) = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \\ R_{1,1} &= \lim_{s\to 0}\frac{d}{ds}(\mathbf{G}(s)s^2) = \begin{pmatrix} \frac{1}{9} \\ -\frac{14}{9} \end{pmatrix} \\ R_2 &= \lim_{s\to -3}(\mathbf{G}(s)(s+3)) = \begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{18} \end{pmatrix} \\ R_3 &= \lim_{s\to -1}(\mathbf{G}(s)(s+1)) = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix} \end{aligned} \tag{57}$$

Therefore,

$$\mathbf{G}(s) = \frac{\begin{pmatrix} \frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}}{s} + \frac{\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}}{s^2} + \frac{\begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{18} \end{pmatrix}}{s+3} + \frac{\begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}}{s+1} \tag{58}$$

and by the linearity and frequency translation properties of the transform

$$\begin{aligned} &\mathfrak{L}^{-1}[\mathbf{G}(s)](t) = \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{1}{9} \\ -\frac{14}{9} \end{pmatrix}}{s}](t) + \mathfrak{L}^{-1}[\frac{\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}}{s^2}](t) \\ &+\mathfrak{L}^{-1}[\frac{\begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{18} \end{pmatrix}}{s+3}](t) + \mathfrak{L}^{-1}[\frac{\begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}}{s+1}](t) \\ &= \begin{pmatrix} \frac{1}{9} \\ -\frac{14}{9} \end{pmatrix} \mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s)](t) + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \mathfrak{L}^{-1}[\mathfrak{L}[t_+](s)](t) \\ &+ \begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{18} \end{pmatrix} \mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s+3)](t) \\ &+ \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix} \mathfrak{L}^{-1}[\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s+1)](t) \\ &= \begin{pmatrix} \frac{1}{9} \\ -\frac{14}{9} \end{pmatrix} \boldsymbol{\delta}^{(-1)}(t) + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} t_+ + \begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{18} \end{pmatrix} e_+^{-3t} + \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix} e_+^{-t} \end{aligned} \tag{59}$$

It follows from (47) that

$$\begin{aligned} &\mathbf{y}^{(u)}(t,\mathbf{u}) = \mathfrak{L}^{-1}[\mathbf{G}(s)(e^{-s} - e^{-3s})](t) \\ &= \mathfrak{L}^{-1}[\mathbf{G}(s)](t-1) - \mathfrak{L}^{-1}[\mathbf{G}(s)](t-3). \lhd \end{aligned}$$

## Appendix

### A. Laplace transform

Let $I$ be an interval containing $[0, +\infty)$ and let $\mathbf{f} : I \to \mathbb{C}$ be a real or complex-valued function.

**Definition** *A.1: The function $\mathbf{f} : I \to \mathbb{C}$ has a $\mathfrak{L}$-transform if $\exists s \in \mathbb{C}$ such that $\boldsymbol{\phi} : t \to \boldsymbol{\phi}(t) := e^{-st}\mathbf{f}(t)$ is absolutely integrable over $[0, +\infty)$, i.e.*

$$\int_0^{+\infty} |e^{-st}\mathbf{f}(t)| dt < +\infty \tag{60}$$

If the integral on the left of (60) exists for $s = s_0$ then it exists for all $s \in \mathbb{C}$ such that $\text{Re}(s) > \text{Re}(s_0)$. Indeed, for all $s \in \mathbb{C}$ such that $\text{Re}(s) > \text{Re}(s_0)$

$$\begin{aligned} &|e^{-st}\mathbf{f}(t)| = e^{-\text{Re}(s)t}|\mathbf{f}(t)| \leqslant e^{-\text{Re}(s_0)t}|\mathbf{f}(t)| \\ &= |e^{-s_0 t}\mathbf{f}(t)| \end{aligned} \tag{61}$$

since $|e^{-j\text{Im}(s_0)t}| = |e^{-j\text{Im}(s)t}| = 1$. Therefore, $|e^{-st}\mathbf{f}(t)|$ is majorized by an integrable function $|e^{-s_0 t}\mathbf{f}(t)|$ for all $s \in \mathbb{C}$ such that $\text{Re}(s) > \text{Re}(s_0)$ and it is integrable for all such $s$.

We conclude that, if the set of $s \in \mathbb{C}$ for which the integral on the left of (60) exists is not empty, it is an open half-plane in the complex plane, in particular the set $\{s \in \mathbb{C} : \text{Re}(s) > \sigma[\mathbf{f}]\}$ where $\sigma[\mathbf{f}]$ is the infimum of the real parts of the points $s \in \mathbb{C}$ for which the integral on the left of (60) exists.

**Definition** *A.2: Let $\mathbf{f} : I \to \mathbb{C}$ have a $\mathfrak{L}$-transform. If*

$$\sigma[\mathbf{f}] := \inf[\text{Re}(s) : \int_0^{+\infty} |e^{-st}\mathbf{f}(t)| dt < +\infty], \tag{62}$$

*for each $s \in \mathbb{C} : \text{Re}(s) > \sigma[\mathbf{f}]$ we define the Laplace transform of $\mathbf{f}$ as*

$$\mathfrak{L}[\mathbf{f}](s) := \int_0^{+\infty} |e^{-st}\mathbf{f}(t)| dt \tag{63}$$

and $\sigma[\mathbf{f}]$ is called the *convergence abscissa* of $\mathbf{f}$. Note that if

$$|\mathbf{f}(t)| \leqslant Me^{\alpha t} \tag{64}$$

for some $M > 0$, real $\alpha$ and for all $t \geqslant 0$, then $\mathbf{f}$ has a $\mathfrak{L}$-transform and $\sigma[\mathbf{f}] \leqslant \alpha$. Indeed, for all $s \in \mathbb{C} : \mathrm{Re}(s) > \alpha$

$$\begin{aligned} |e^{-st}\mathbf{f}(t)| &= e^{-\mathrm{Re}(s)t}|\mathbf{f}(t)| \leqslant e^{-\mathrm{Re}(s)t}Me^{\alpha t} \\ &= Me^{(\alpha-\mathrm{Re}(s))t} \end{aligned} \tag{65}$$

and since $Me^{(\alpha-\mathrm{Re}(s))t}$ is integrable over $[0,+\infty)$, also $e^{-st}\mathbf{f}(t)$ is.

*1) Laplace transform of the Heaviside (or unit step) function:* The Heaviside (or unit step function is

$$\boldsymbol{\delta}^{(-1)}(t) := \begin{cases} 1 & \text{for } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{66}$$

The function $t \to \boldsymbol{\delta}^{(-1)}(t)e^{-st}$ is integrable over $[0,+\infty)$ for all $s \in \mathbb{C} : \mathrm{Re}(s) > 0$. Indeed,

$$|e^{-st}\boldsymbol{\delta}^{(-1)}(t)| = e^{-\mathrm{Re}(s)t} \tag{67}$$

which is integrable over $[0,+\infty)$ if and only if $\mathrm{Re}(s) > 0$. Therefore, $\sigma[\mathbf{f}] = 0$ and the Laplace transform of $f$ is

$$\begin{aligned} \mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s) &:= \int_0^{+\infty} |e^{-st}\boldsymbol{\delta}^{(-1)}(t)|dt \\ &= \int_0^{+\infty} |e^{-st}|dt = \frac{1}{s} \end{aligned} \tag{68}$$

*2) Laplace transform of exponential functions:* As further exercise, consider the function

$$\mathbf{f}(t) := e^{at} \tag{69}$$

with $a = \alpha + j\beta$. The function $t \to \mathbf{f}(t)e^{-st}$ is integrable over $[0,+\infty)$ for all $s \in \mathbb{C} : \mathrm{Re}(s) > \alpha$. Indeed,

$$|e^{-st}\mathbf{f}(t)| = e^{-(\mathrm{Re}(s)-\alpha)t} \tag{70}$$

which is integrable over $[0,+\infty)$ if and only if $\mathrm{Re}(s) > \alpha$. Therefore, $\sigma[\mathbf{f}] = \alpha$ and the Laplace transform of $f$ is

$$\begin{aligned} \mathfrak{L}[\mathbf{f}(t)](s) &:= \int_0^{+\infty} |e^{-st}\mathbf{f}(t)|dt \\ &= \int_0^{+\infty} |e^{-(s-a)t}|dt = \frac{1}{s-a} \end{aligned} \tag{71}$$

*3) Laplace transform of impulsive functions:* Consider the impulse function with duration $T > 0$

$$\begin{aligned} \mathbf{f}(t) &:= \begin{cases} 1 & \text{for } 0 \leqslant t < T \\ 0 & \text{otherwise} \end{cases} \\ &= \boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t-T) \end{aligned} \tag{72}$$

The function $t \to \mathbf{f}(t)e^{-st}$ is absolutely integrable over $[0,+\infty)$ for all $s \in \mathbb{C}$. Indeed, for all $s \neq 0$

$$\int_0^{+\infty} |e^{-st}\mathbf{f}(t)|dt = \int_0^{T} |e^{-st}|dt = \frac{1-e^{-sT}}{s} \tag{73}$$

This function has a singularity at $s = 0$ but

$$\lim_{s\to 0} \frac{1-e^{-sT}}{s} = \lim_{s\to 0} \frac{1-e^{-sT}}{Ts}T = T \tag{74}$$

Therefore, for all $s$

$$\mathfrak{L}[\mathbf{f}(t)](s) := \int_0^{+\infty} |e^{-st}\mathbf{f}(t)|dt = \int_0^{T} |e^{-st}|dt = \frac{1-e^{-sT}}{s}$$

and $\sigma[\mathbf{f}] = -\infty$. This is a general fact: if $f$ is null outside a compact set of $\mathbb{R}$, then $\sigma[\mathbf{f}] = -\infty$.

Next, consider the impulse function with duration $T > 0$

$$\begin{aligned} \mathbf{f}(t) &:= \begin{cases} \frac{1}{T} & \text{for } 0 \leqslant t < T \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{T}[\boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t-T)] \end{aligned} \tag{75}$$

This impulse is called *normalized* since

$$\int_{-\infty}^{+\infty} \mathbf{f}(t)dt = \int_0^{+\infty} \mathbf{f}(t)dt = 1 \tag{76}$$

Reasoning as in the former exercise, we find out that the function $f$ has a Laplace transform

$$\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1-e^{-sT}}{Ts} \tag{77}$$

with $\sigma[\mathbf{f}] = -\infty$. Moreover, for each $s$

$$\lim_{T\to 0} \mathfrak{L}[\mathbf{f}(t)](s) = \lim_{T\to 0} \frac{1-e^{-sT}}{Ts} = 1 \tag{78}$$

If we choose $T = \frac{1}{n}$, $n \in \mathbb{N}$, we obtain the family of functions $\{\mathbf{f}_n(t)\}$ defined as $t \to \mathbf{f}_n(t) := N[\boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t)(t-\frac{1}{n})]$. Note that

$$\lim_{n\to+\infty} \mathbf{f}_n(t) = \boldsymbol{\delta}^{(0)}(t) := \begin{cases} 0 & \text{for } t \neq 0 \\ +\infty & t = 0 \end{cases} \tag{79}$$

where $\boldsymbol{\delta}^{(0)}$ is the *Dirac impulse* function and, on account of (78) we define the Laplace transform of $\boldsymbol{\delta}^{(0)}$ as

$$\mathfrak{L}[\boldsymbol{\delta}^{(0)}(t)](s) = \lim_{n\to+\infty} \mathfrak{L}[\mathbf{f}_n(t)](s) = 1 \tag{80}$$

*4) Laplace transform of polynomial functions:* As a final exercise, consider the function

$$\mathbf{f}(t) := \frac{t^k}{k!} \tag{81}$$

with $k \in \mathbb{N}$. Select $k = 0$ and note that in this case $\mathbf{f}(t) = \boldsymbol{\delta}^{(-1)}(t)$ over $[0,+\infty)$. As we have already seen, the function $\mathbf{f}(t)$ is integrable over $[0,+\infty)$ for all $s \in \mathbb{C} : \mathrm{Re}(s) > 0$ and

$$\mathfrak{L}[\mathbf{f}(t)](s) := \frac{1}{s} \tag{82}$$

with $\sigma[\mathbf{f}] = 0$. On the other hand, for all $k \in \mathbb{N}\backslash[0]$ and for all $s \in \mathbb{C} : \mathrm{Re}(s) > 0$

$$\begin{aligned} \mathfrak{L}[\frac{t^k}{k!}](s) &:= \int_0^{+\infty} |e^{-st}\frac{t^k}{k!}|dt \\ &= \Big[-\frac{e^{-st}}{s}\frac{t^k}{k!}\Big]_{t=0}^{t=+\infty} + \frac{1}{s}\int_0^{+\infty} |e^{-st}\frac{t^{k-1}}{(k-1)!}|dt \\ &= \frac{1}{s}\mathfrak{L}[\frac{t^{k-1}}{(k-1)!}](s) \end{aligned} \tag{83}$$

Therefore

$$\mathfrak{L}[\frac{t^k}{k!}](s) = \frac{1}{s^{k+1}} \tag{84}$$

with $\sigma[\frac{t^k}{k!}] = 0$.

*5) Properties of Laplace transform:* The Laplace transform is a*linear* operator, i.e. for all pairs of $c_1, c_2 \in \mathbb{C}$ and $\mathbf{f}_1, \mathbf{f}_2$ with Laplace transforms and convergence abscissa $\sigma[\mathbf{f}_1]$ and, respectively, $\sigma[\mathbf{f}_2]$

- (LINEARITY): $\mathfrak{L}[c_1\mathbf{f}_1(t)+c_2\mathbf{f}_2(t)](s) = c_1\mathfrak{L}[\mathbf{f}_1(t)](s)+c_2\mathfrak{L}[\mathbf{f}_2(t)](s)$

with $\sigma[c_1\mathbf{f}_1 + c_2\mathbf{f}_2] := \max[\sigma[\mathbf{f}_1], \sigma[\mathbf{f}_2]]$.

*6) Laplace transform of sinusoidal and cosinusoidal functions:* As we have already seen, for each real $\omega$

$$\mathfrak{L}[e^{j\omega t}](s) = c_1\mathfrak{L}[\mathbf{f}_1(t)](s) + c_2\mathfrak{L}[\mathbf{f}_2(t)](s) \tag{85}$$

for all $s$ such that Re $(s) > 0$. By the linearity of the Laplace transform

$$\begin{aligned}\mathfrak{L}[\sin(\omega t)](s) &= \mathfrak{L}[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}](s) \\ &= \frac{1}{2j}[\mathfrak{L}[e^{j\omega t}](s) - \mathfrak{L}[e^{-j\omega t}](s)] \\ &= \frac{1}{2j}[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}] = \frac{\omega}{s^2+\omega^2}\end{aligned} \tag{86}$$

for all $s$ such that Re $(s) > 0$. Likewise,

$$\begin{aligned}\mathfrak{L}[\cos(\omega t)](s) &= \mathfrak{L}[\frac{e^{j\omega t} + e^{-j\omega t}}{2}](s) \\ &= \frac{1}{2}[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}] = \frac{s}{s^2+\omega^2}\end{aligned} \tag{87}$$

for all $s$ such that Re $(s) > 0$.

*7) Signals:* The definition of Laplace transform depends only on the values of the function $f$ over $[0, +\infty)$. Therefore if

$$\mathbf{f}_+(t) := \begin{bmatrix}\mathbf{f}(t) & \text{for } t \geqslant 0 \\ 0 & \text{otherwise}\end{bmatrix} \tag{88}$$

then $\mathbf{f}_+(t) = \boldsymbol{\delta}^{(-1)}(t)\mathbf{f}(t)$ and $\mathfrak{L}[\mathbf{f}_+(t)](s) = \mathfrak{L}[\mathbf{f}(t)](s)$. Some remarkable properties of $\mathbf{f}_+$ are:

- (TIME TRANSLATION): $\mathfrak{L}[\mathbf{f}_+(t - T)](s) = e^{-sT}\mathfrak{L}[\mathbf{f}_+(t)](s), \quad \forall T > 0, \ \forall s : \text{Re}(s) > \sigma[\mathbf{f}]$
- (FREQUENCY TRANSLATION): $\mathfrak{L}[e^{at}\mathbf{f}_+(t)](s) = \mathfrak{L}[\mathbf{f}_+(t)](s-a), \quad \forall a \in \mathbb{C}, \ \forall s : \text{Re} > \sigma[\mathbf{f}] + \text{Re}(a)$.

Consider $\mathbf{f}(t) = \sin t$. Clearly

$$\mathfrak{L}[\mathbf{f}_+(t)](s) = \mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{s^2+1} \tag{89}$$

and from the time-translation property for all $T > 0$

$$\mathfrak{L}[\mathbf{f}_+(t-T)](s) = e^{-sT}\mathfrak{L}[\mathbf{f}_+(t)](s) = \frac{e^{-sT}}{s^2+1} \tag{90}$$

Therefore,

$$\mathfrak{L}[\mathbf{f}_+(t) + \mathbf{f}_+(t-\pi)](s) = \frac{1+e^{-s\pi}}{s^2+1} \tag{91}$$

and

$$\mathbf{f}_+(t) + \mathbf{f}_+(t-\pi) = \begin{cases}\sin t & \text{for } 0 \leqslant t < 2\pi \\ 0 & \text{otherwise}\end{cases} \tag{92}$$

Note that since $s = \pm j$ are at the same time poles and zeroes of $\frac{1+e^{-s\pi}}{s^2+1}$ then $\sigma[\mathbf{f}_+(t) + \mathbf{f}_+(t-T)] = -\infty$.

Likewise, consider $\mathbf{f}(t) = \boldsymbol{\delta}^{(-1)}(t)$. Clearly,

$$\mathfrak{L}[\mathbf{f}_+(t)](s) = \mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{s} \tag{93}$$

and from the time-translation property for all $T > 0$

$$\mathfrak{L}[\mathbf{f}_+(t-T)](s) = e^{-sT}\mathfrak{L}[\mathbf{f}_+(t)](s) = \frac{e^{-sT}}{s} \tag{94}$$

Therefore,

$$\mathfrak{L}[\mathbf{f}_+(t) - \mathbf{f}_+(t-T)](s) = \mathfrak{L}[\mathbf{f}(t)](s) = \frac{1-e^{-sT}}{s} \tag{95}$$

and

$$\mathbf{f}_+(t) - \mathbf{f}_+(t-T) = \begin{cases}1 & \text{for } 0 \leqslant t < T \\ 0 & \text{otherwise}\end{cases} \tag{96}$$

Note that also in this exercise since $\frac{1-e^{-sT}}{s}$ has no singularity at $s = 0$, since $s = 0$ is at the same time pole and zero, then $\sigma[\mathbf{f}_+(t) - \mathbf{f}_+(t-T)] = -\infty$.

Finally, consider $\mathbf{f}(t) = t$. Clearly,

$$\mathfrak{L}[\mathbf{f}_+(t)](s) = \mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{s^2} \tag{97}$$

and from the time-translation property

$$\mathfrak{L}[\mathbf{f}_+(t) - 2\mathbf{f}_+(t-1) + \mathbf{f}_+(t-2)](s) = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$

with

$$\begin{aligned}&\mathbf{f}_+(t) - 2\mathbf{f}_+(t-1) + \mathbf{f}_+(t-2) \\ &= \begin{cases}t & \text{for } 0 \leqslant t < 1 \\ 2-t & \text{for } 1 \leqslant t < 2 \\ 0 & \text{for } t \geqslant 2\end{cases}\end{aligned}$$

*8) Laplace transform of periodic functions:* We can prove the following result for the Laplace transform of periodic functions $\mathbf{f}(t)$.

**Proposition** *A.1: Let* $\mathbf{f}$ *be periodic with period* $T > 0$, *i.e.* $\mathbf{f}(t+T) = \mathbf{f}(t)$ *for all* $t \geqslant 0$. *If* $\mathbf{f}$ *is integrable over* $[0, T]$, *then*

$$\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\mathbf{f}(t)dt \tag{98}$$

*with* $\sigma[\mathbf{f}] = 0$.

Consider the square wave

$$\begin{aligned}\mathbf{f}(t) &:= \begin{cases}1 & \text{for } 2n \leqslant t < 2n+1, \ n \in \mathbb{N} \\ 0 & \text{otherwise}\end{cases} \\ &= \sum_{j=0}^{+\infty}[\boldsymbol{\delta}^{(-1)}(t-j) - \boldsymbol{\delta}^{(-1)}(t-j-1)]\end{aligned} \tag{99}$$

The function $\mathbf{f}(t)$ is periodic with period $T = 2$. By proposition A.1

$$\begin{aligned}\mathfrak{L}[\mathbf{f}(t)](s) &= \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\mathbf{f}(t)dt \\ &= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}dt = \frac{1}{1-e^{-2s}}\frac{1-e^{-s}}{s} \\ &= \frac{1}{s(1+e^{-s})}\end{aligned}$$

with $\sigma[\mathbf{f}] = 0$.

As a further exercise, consider the function

$$\mathbf{f}(t) := \begin{cases} \sin t & \text{for } 2n\pi \leqslant t < (2n+1)\pi,\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$
$$= \sum_{j=0}^{+\infty} [\sin_+(t - j\pi) + \sin_+(t-(j+1)\pi)] \quad (100)$$

The function $\mathbf{f}$ is periodic with period $T = 2\pi$. By proposition A.1

$$\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}\mathbf{f}(t)dt$$
$$= \frac{1}{1-e^{-2s}} \int_0^{\pi} e^{-st}[\sin_+(t) + \sin_+(t-\pi)]dt$$
$$= \frac{1}{1-e^{-2s}} \frac{1+e^{-s}}{s^2+1} \quad (101)$$

with $\sigma[\mathbf{f}] = 0$, where we used the fact that

$$\int_0^{\pi} e^{-st}(\sin_+(t) + \sin_+(t-T))dt$$
$$= \int_0^{+\infty} e^{-st}(\sin_+(t) + \sin_+(t-T))dt$$
$$= \mathfrak{L}(\sin_+(t) + \sin_+(t-T))(s) = \frac{1+e^{-s}}{s^2+1}$$

Finally, consider $\mathbf{f}(t) := |\sin(t)|$. The function $\mathbf{f}(t)$ is periodic with period $T = \pi$. By proposition A.1

$$\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st}\mathbf{f}(t)dt$$
$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st}\sin(t)dt$$
$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st}(\sin_+(t) + \sin_+(t-T))dt$$
$$= \frac{1}{1-e^{-\pi s}} \int_0^{+\infty} e^{-st}(\sin_+(t) + \sin_+(t-T))dt$$
$$= \frac{1}{1-e^{-\pi s}} \mathfrak{L}(\sin_+(t) + \sin_+(t-T))(s)$$
$$= \frac{1}{1-e^{-\pi s}} \frac{1+e^{-s}}{s^2+1}$$

with $\sigma[\mathbf{f}] = 0$.

As it can be seen from the above examples, the term $\int_0^T e^{-st}\mathbf{f}(t)dt$ in (98) coincides with $\mathfrak{L}[\mathbf{f}_0(t)](s)$ where $\mathbf{f}_0(t)$ is a function which is zero outside $[0, T]$ and such that $\mathbf{f}(t) = \sum_{j=0}^{\infty}(\mathbf{f}_0)_+(t - jT)$.

*9) Further results on Laplace transforms:* For the derivative of a function $\mathbf{f}(t)$ we have the following result.

**Proposition** *A.2:* (TIME DERIVATIVE):*Let $\mathbf{f}$ be a signal, continuous for all $t > 0$ and continuous from the right in $t = 0$, with piecewise continuous derivative with a Laplace transform. Then for all $s$ such that Re$(s) > \max[\sigma[\mathbf{f}], \sigma[\frac{d\mathbf{f}}{dt}]$*

$$\mathfrak{L}[\frac{d\mathbf{f}}{dt}(t)](s) = s\mathfrak{L}[\mathbf{f}(t)](s) - \mathbf{f}(0^+) \quad (102)$$

*where $\mathbf{f}(0^+)$ is the limit of $\mathbf{f}$ at $t = 0$ from the right, i.e.* $\lim_{t\to 0^+}\mathbf{f}(t)$.

If $\mathbf{f}$ is once continuously differentiable and $\frac{d\mathbf{f}}{dt}$ is continuous for all $t > 0$ and continuous from the right in $t = 0$, with piecewise continuous derivative with a Laplace transform, by applying twice proposition A.2 we get

$$\mathfrak{L}[\frac{d^2\mathbf{f}}{dt^2}(t)](s) = \mathfrak{L}[\frac{d}{dt}\frac{d\mathbf{f}}{dt}(t)](s) = s\mathfrak{L}[\frac{d\mathbf{f}}{dt}(t)](s) - \frac{d\mathbf{f}}{dt}\Big|_{t=0}$$
$$= s[s\mathfrak{L}[\mathbf{f}(t)](s) - f(0)] - \frac{d\mathbf{f}}{dt}\Big|_{t=0}$$
$$= s^2\mathfrak{L}[\mathbf{f}(t)](s) - sf(0) - \frac{d\mathbf{f}}{dt}\Big|_{t=0} \quad (103)$$

In general, if $\mathbf{f}$ is $(k-1)$-times continuously differentiable and $\frac{d^{k-1}\mathbf{f}}{dt^{k-1}}(t)$ is continuous for all $t > 0$ and continuous from the right in $t = 0$, with piecewise continuous derivative with a Laplace transform, by applying twice proposition A.2 we get

$$\mathfrak{L}[\frac{d^k\mathbf{f}}{dt^k}(t)](s) = s^k\mathfrak{L}[\mathbf{f}(t)](s) - \sum_{j=0}^{k-1} s^j \frac{d^{k-1-j}\mathbf{f}}{dt^{k-1-j}}\Big|_{t=0}$$

The Laplace transform of the time convolution of two signals is given by the following result. We recall that the *time convolution* of two time functions $\mathbf{f}$ and $\mathbf{g}$ integrable over $\mathbb{R}$ is defined as

$$(\mathbf{f} * \mathbf{g})(t) = \int_{-\infty}^{+\infty} \mathbf{f}(\tau)\mathbf{g}(t-\tau)d\tau \quad (104)$$

If $\mathbf{f}$ and $\mathbf{g}$ are signals

$$(\mathbf{f} * \mathbf{g})(t) = \begin{cases} \int_0^t \mathbf{f}(\tau)\mathbf{g}(t-\tau)d\tau & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (105)$$

**Proposition** *A.3:* (TIME CONVOLUTION): *Let $f$ and $g$ be two signals with Laplace transforms and convergence abscissa $\sigma[\mathbf{f}]$ and, respectively, $\sigma[\mathbf{g}]$. Then $\mathbf{f} * \mathbf{g}$ has a Laplace transform for all $s$ such that Re$(s) > \max[\sigma[\mathbf{f}], \sigma[\mathbf{g}]]$ and*

$$\mathfrak{L}[(\mathbf{f} * \mathbf{g})(t)](s) = \mathfrak{L}[\mathbf{f}(t)](s)\mathfrak{L}[\mathbf{g}(t)](s). \quad (106)$$

For exercise, the time convolution of a signal $\mathbf{f}$, with Laplace transforms and convergence abscissa $\sigma[\mathbf{f}]$, and $\boldsymbol{\delta}^{(-1)}(t)$ is

$$(\mathbf{f} * \boldsymbol{\delta}^{(-1)})(t) = \int_0^t \mathbf{f}(\tau)\boldsymbol{\delta}^{(-1)}(t-\tau)d\tau = \int_0^t \mathbf{f}(\tau)d\tau$$

Moreover, $\mathfrak{L}[\boldsymbol{\delta}^{(-1)}(t)](s) = \frac{1}{s}$ with $\sigma[\boldsymbol{\delta}^{(-1)}] = 0$. By application of proposition A.3

$$\mathfrak{L}[\int_0^t \mathbf{f}(\tau)d\tau](s) = \mathfrak{L}[(\mathbf{f} * \boldsymbol{\delta}^{(-1)})(t)](s) = \frac{1}{s}\mathfrak{L}[\mathbf{f}(t)](s) \quad (107)$$

with $\sigma[\int_0^t \mathbf{f}(\tau)d\tau] = \max[\sigma[\mathbf{f}], 0]$.

We have already seen that the impulse function with duration $T > 0$

$$\mathbf{f}(t) := \begin{cases} \frac{1}{T} & \text{for } 0 \leqslant t < T \\ 0 & \text{otherwise} \end{cases}$$
$$= \frac{1}{T}(\boldsymbol{\delta}^{(-1)}(t) - \boldsymbol{\delta}^{(-1)}(t-T)) \quad (108)$$

and

$$\int_{-\infty}^{+\infty} \mathbf{f}(t)dt = 1 \tag{109}$$

Also, its Laplace transform is

$$\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1 - e^{-sT}}{Ts} \tag{110}$$

with $\sigma[\mathbf{f}] = -\infty$. The primitive of $\mathbf{f}$ is

$$\begin{aligned} \int_0^t \mathbf{f}(\tau)d\tau \quad &:= \quad \begin{cases} 1 & \text{for } t > T \\ \frac{t}{T} & \text{for } 0 \leqslant t \leqslant T \end{cases} \\ &= \quad \frac{1}{T}(t_+ - (t-T)_+) \end{aligned}$$

By application of (107)

$$\mathfrak{L}[\int_0^t \mathbf{f}(\tau)d\tau](s) = \frac{1}{s}\mathfrak{L}[\mathbf{f}(t)](s) = \frac{1}{s}\frac{1 - e^{-sT}}{Ts}$$

with $\sigma[\int_0^t \mathbf{f}(\tau)d\tau] = -\infty$.

### *B. Inverse Laplace transform*

It is possible, under certain conditions, to reconstruct a signal $\mathbf{f}_+$ from its Laplace transform, defining in some sense an inverse Laplace transform. However, we are interested in this kind of issues, except for the following important remark. For any function $\mathbf{f}$ which has a Laplace transform $\mathfrak{L}[\mathbf{f}(t)](s)$, we define as its inverse Laplace transform $\mathbf{f}_+(t) := \mathfrak{L}^{-1}[\mathfrak{L}[\mathbf{f}(t)](s)](t)$, which is the signal of $\mathbf{f}(t)$. Indeed, we identify all the inverse transforms of $\mathfrak{L}[\mathbf{f}(t)](s)$ with the signal of $\mathbf{f}(t)$ (the inverse tranforms of $\mathfrak{L}[\mathbf{f}(t)](s)$ form an equivalence class).

On the other hand, any proper rational function $\mathbf{F}(s)$ can be first decomposed into simple fractional terms using the residuals theorem (theorem 2.1) and each fractional term can be inverse transformed by obtaining as a final result the inverse transform of $\mathbf{F}(s)$.