Control Systems
9/01/2018

Exercise 1 By defining $\bar{P}(s) = \frac{1}{s}(P(s) - 2) = -2\frac{s-5}{s(s+10)}$ and $L(s) = G(s)\bar{P}(s)$, one has

$$y(s) = W(s)v(s) + W_d(s)d(s) \quad \text{with} \quad W(s) = \frac{L(s)}{1+L(s)}, \quad W_d(s) = \frac{1}{1+L(s)}.$$

Let us write $G(s) = G_2(s)G_1(s)$ so that $G_1(s)$ is designed for fulfilling steady-state specifications (i.e., (ii)) whereas $G_2(s)$ will be later set for stability and transient performances (i.e., (i) and (iii)).

(ii) Since the input-to-error transfer function $W_e(s) = \frac{1}{1+L(s)}$ and recalling that the steady state response to $v(t) = t$ is given $e_1(t) = W_e(0)t + \frac{dW_e}{ds}(0)$, for the requirement to be satisfied one needs $W_e(0) = 0$ and $\left|\frac{dW_e}{ds}\right|_{s=0} \leq 0.2$.
In this case, because an integrator is already located before the entering point of the disturbance, one has $W_e(0) = 0$ so that for (ii) to be solved one sets $G_1(s) = k_1$ with $k_1 \in \mathbb{R}$ such that

$$\left|\frac{W_e(s)}{s}\right|_{s=0} \leq 0,2 \implies k_1 \geq 5.$$

Thus, one can fix $k_1 = 5$ while guaranteeing, for (ii) to be fulfilled by the closed-loop system, that $G_2(0) \geq 1$.

(iii) For assigning $\omega_t^* = 2rad/sec$ and $m_\phi^* \geq 50^o$ let us first draw the Bode plots of

$$L_1(s) = G_1(s)\bar{P}(s) = -10\frac{s-5}{s(s+10)} = 5\frac{1-\frac{s}{5}}{s(1+\frac{s}{10})} \tag{1}$$

which are reported in Figure 1. As $\omega_t^* = 2rad/sec$ is the desired crossover frequency, we notice that

$$|L_1(j\omega_t^*)|_{dB} = 8.433 \quad \angle L_1(j\omega_t^*) = -123.1113.$$

Accordingly, $G_2(s)$ needs to be chosen in such a way that

$$|G_2(j\omega_t^*)|_{dB} + |L_1(j\omega_t^*)|_{dB} = 0 \tag{2}$$
$$180^o + \angle L_1(j\omega_t^*) + \angle G_2(j\omega_t^*) \geq 50^o \tag{3}$$

with the further requirement $|G_2(0)| \geq 1$ to preserve (ii).
It is a matter of computations to verify that, with no need of further actions, (3) is already satisfied as $180^o + \angle L_1(j\omega_t^*) = 56.8^o$. Accordingly, one can satisfy the specification by assigning the cross-over frequency to $\omega_t^* = 2rad/sec$. Thus, $G_2(s)$ needs to be designed so to decrease the magnitude at $\omega_t^* = 2rad/sec$ without possibly affecting the phase in the corresponding neighborhood. By noticing that a simple proportional action is not

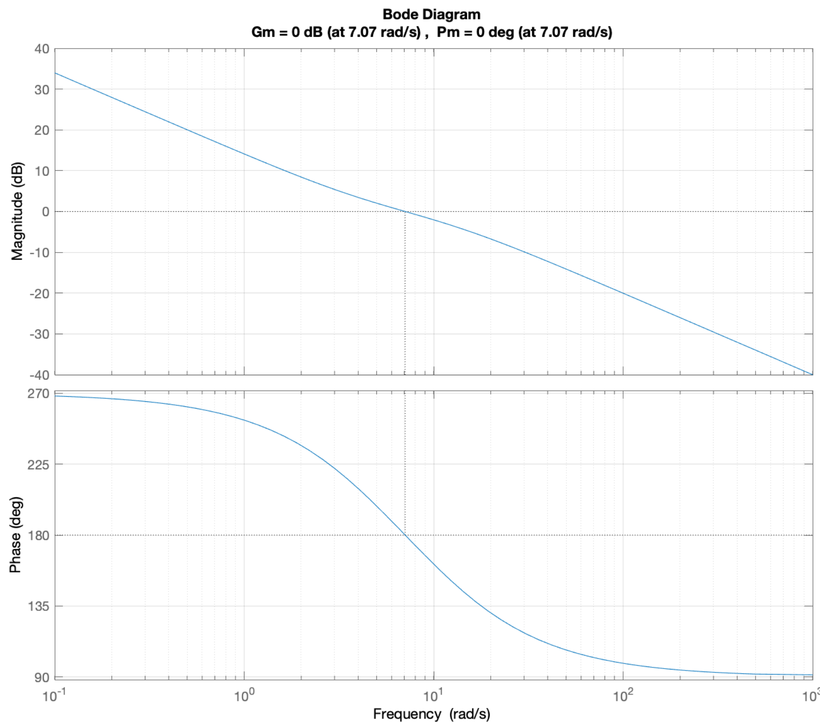

Figure 1: Bode plots of (1)

compatible with the requirement $|G_2(0)| \geq 1$, then an attenuating action of the form

$$G_2(s) = k_i \Big(\frac{1 + \frac{\tau_i}{m_1} s}{1 + \tau_i s} \Big)^{n_i}$$

is needed in such a way to guarantee

$$|G_2(j\omega_t^*)|_{dB} = -8.433, \quad \angle G_2(j\omega_t^*) \geq -6.8^o, \quad |k_i|_{dB} > 0.$$

As at ω_t^* must be affected as least as possible by the feedback action one has $k_i > 0$. Denoting $\omega_n = \omega_t^* \tau_i$, it is evident that for decreasing the phase contribution at $\omega_t^* = 2\ rad/s$ as much as possible, one needs ω_n to act at high frequency. Thus, we set $m_i = 3$, $\omega_n = 100$ and $n_i = 1$ so that

$$|G_2(j\omega_t^*)|_{dB} = |k_i|_{dB} - 9.5390, \quad \angle G_2(j\omega_t^*) = -1.1454^o.$$

Finally, the gain is set in such a way that $|k_i|_{dB} - 9.5390 = -8.433$ so resulting in $|k_i|_{dB} = 1.106 > 0$ which is indeed compatible with specification (ii) as $k_i = 1.1358 > 1$. Figure 2 depicts the Bode plots of the open loop transfer function

$$L(s) = G_2(s)G_1(s)\bar{P}(s) = 11.358 \frac{1 + \frac{50}{3}s}{1 + 50s} \frac{s - 5}{s(s + 10)}. \tag{4}$$

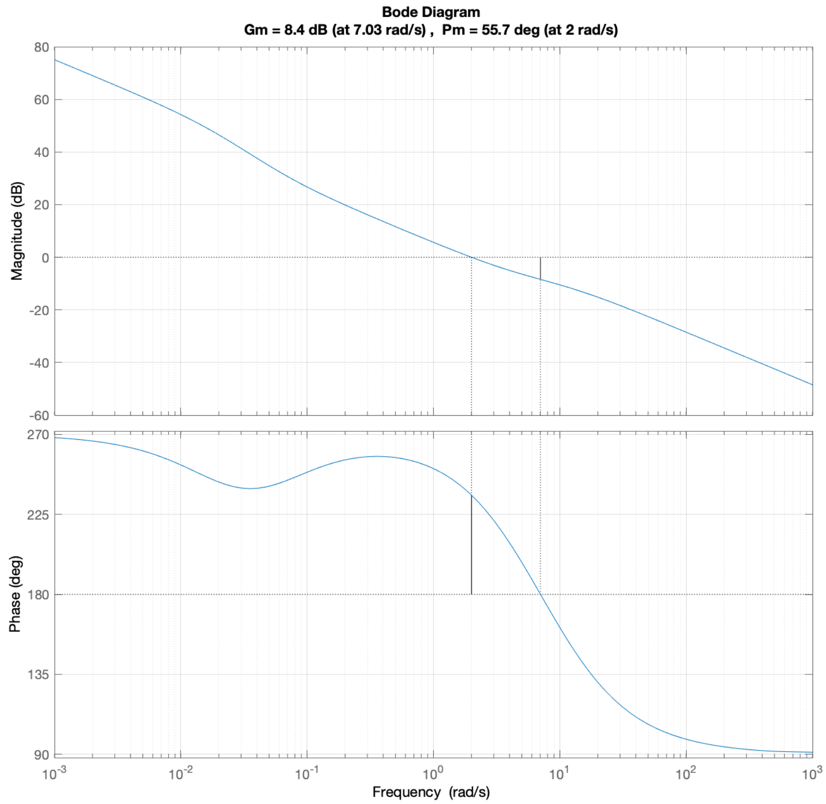

Figure 2: Bode plots of (4)

(ii) The open loop system $L(s) = G_2(s)G_1(s)\bar{P}(s)$ possesses no poles with positive real part and one pole in zero with multiplicity one. Thus, the feedback system is asymptotically stable if and only if the number of counter-clockwise tours around $-1 + j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0. As Figure 3 suggests, the feedback system is asymptotically stable.

Exercise 2 **(a)** As the root locus of $P(s)$ describes the location of the poles of the feedback system under static feedback $G(s) = k$ (i.e., of the transfer function $W(s) = \frac{kP(s)}{1+kP(s)}$) the root locus of $P(s)$ is equivalent to the one of

$$\bar{P}(s) = \frac{1}{(s-1)(s+2)}$$

deduced when neglecting the proportional term 3 and relabeling $\tilde{k} = 3k$.

Denoting by n and m respectively the number of poles and zeros, the relative degree is $r = n - m = 2$, the positive and negative locus possess respectively two branches. Moreover, the positive locus exhibits two vertical asymptotes centered at $s_0 = -1$.

Defining $p(s, \tilde{k}) = (s-1)(s+2) + \tilde{k}$ as the polynomial of the closed-loop poles, singularities

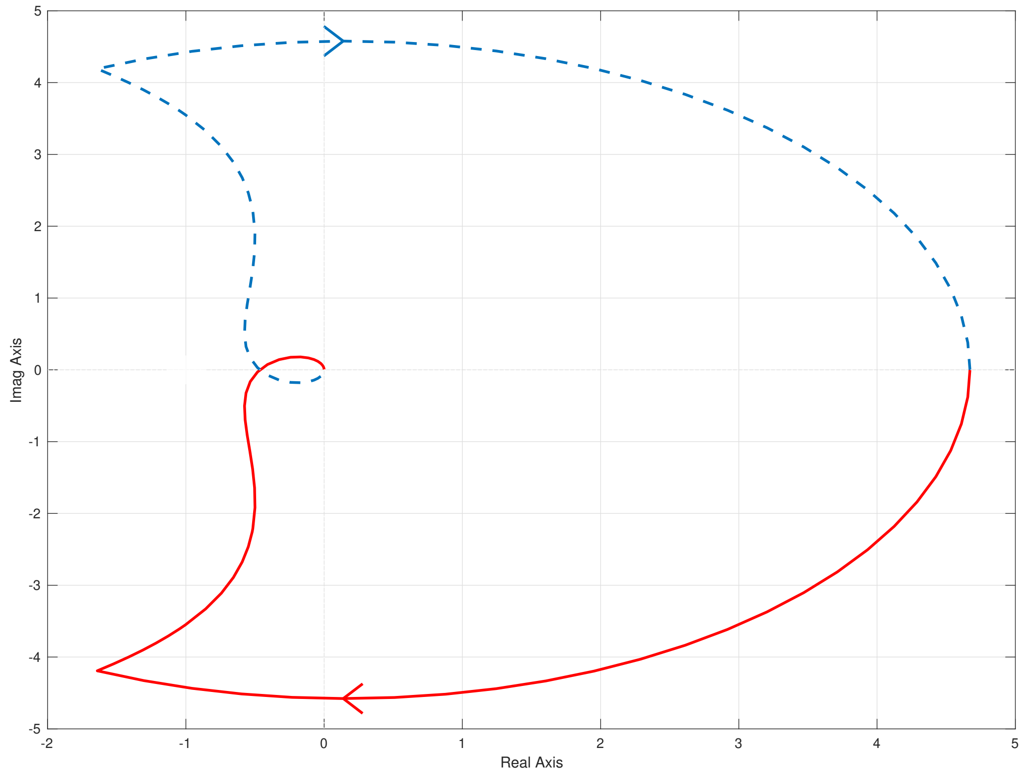

Figure 3: Nyquist plot of (4)

$(s^*, \tilde{k}^*) \in \mathbb{C} \times \mathbb{R}$ are given by the solution of the coupled equations

$$p(s, \tilde{k}) = 0, \quad \frac{\partial}{\partial s} p(s, \tilde{k}) = 0$$

given by $s^* = -\frac{1}{2}$ and $\tilde{k}^* = 2$ (and thus $k = \frac{2}{3}$). Thus, the positive locus possesses a singularity of order two at $s^* = -\frac{1}{2}$ corresponding to $k = \frac{2}{3}$.
The positive and negative locus of $P(s)$ are thus reported in Figures 4 and 5

(b) For assigning all poles with damping $\geq .7$, it is enough to assign them real. Moreover, as the root locus suggests, a static feedback is not enough for assigning all poles with real part ≥ 1 as the center of the asymptotes is at $s_0 = -\frac{1}{2}$. Thus, a feedback $G(s)$ with dimension at least one is needed so to move the center of the asymptotes beyond $s = -1$. To preserve the relative degree and simplify the design, let us design a feedback of the form

$$G(s) = k_1 \frac{s+2}{s+p}$$

with $k_1, p \in \mathbb{R}$ also generating uncontrollability of the mode associated to the eigenvalue -2.

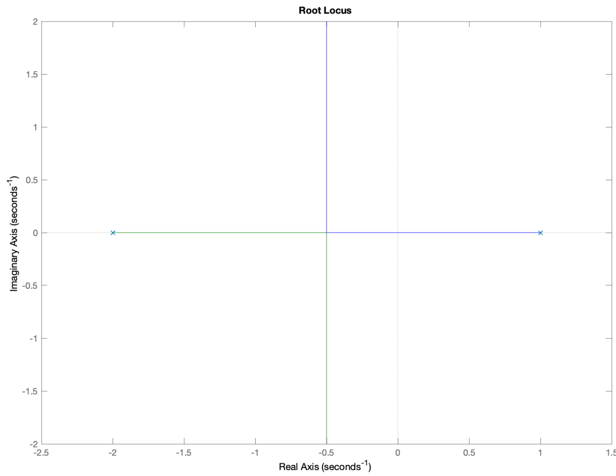

Figure 4: Positive root locus of $P(s) = \frac{3}{(s-1)(s+2)}$

At this point $p > 0$ needs to be set in such a way that

$$s_0' = \frac{1-p}{2} \leq -1 \implies p > 3.$$

For completing the design, it is enough to assign p so to generate a singularity of order 2 at some $s^* \in \mathbb{R}$ and $s^* \leq -1$. It is a matter of computations to verify that such a singularity is unavoidably located in correspondence of the center of the asymptotes s_0'. Hence, one can set $p = 5$ in such a way that the closed-loop poles are located at -2 corresponding to $k_1 = 3$. Thus, the closed-loop transfer function is given by

$$W(s) = \frac{9}{(s+2)^2}. \tag{5}$$

(c) As the disturbance is affecting the output, the output-disturbance transfer function is given by

$$W_d(s) = \frac{1}{1+G(s)P(s)} = \frac{(s+5)(s-1)}{(s+2)^2}.$$

Accordingly, as $d(t) = t$, the steady state response is given by

$$y_{d,ss}(t) = \nabla_s W_d(0) + W_d(0)t$$

with

$$W_d(0) = -\frac{5}{4}, \quad \nabla_s W_d(0) = -\frac{1}{4}.$$

Exercise 3.

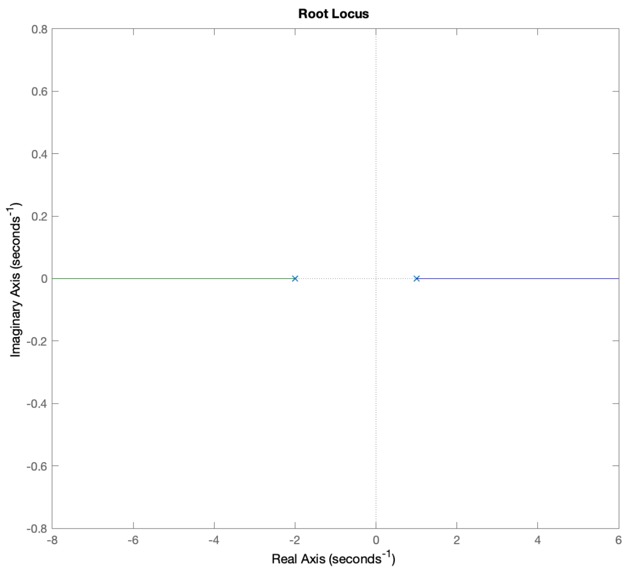

Figure 5: Negative root locus of $P(s) = \frac{3}{(s-1)(s+2)}$

(i) The forced response to the input $u(t) = cost_+$ can be computed as

$$y_f(t) = \mathcal{L}^{-1}(Y(s))[t], \qquad Y(s) = P(s)U(s) \qquad U(s) = \mathcal{L}(u(t))[s]$$

with $\mathcal{L}$ and $\mathcal{L}^{-1}$ being the Laplace and inverse Laplace transforms.

As the system is in canonical controllable form, the transfer function $P(s) = C(sI - A)^{-1}B$ is given by

$$P(s) = \frac{1}{(s+a)^2}.$$

Accordingly, as $U(s) = \mathcal{L}(cost)[s] = \frac{s}{s^2+1}$ one has

$$Y(s) = \frac{s}{(s+a)^2(s^2+1)} = \frac{R_{11}}{s+a} + \frac{R_{12}}{(s+a)^2} + \frac{As+B}{s^2+1}$$

with

$$\begin{aligned}
R_{11} &= \lim_{s\to -a} \nabla_s Y(s)(s+a)^2 = \frac{1-a^2}{(1+a^2)^2} \\
R_{12} &= \lim_{s\to -a} Y(s)(s+a)^2 = -\frac{a}{1+a^2} \\
A &= \frac{a^2-1}{a^4+2a^2+1} \\
B &= \frac{2a}{a^4+2a^2+1}.
\end{aligned}$$

Accordingly, by exploiting linearity of the Laplace operator one gets

$$\begin{aligned} y_f(t) =& R_{11}\mathcal{L}^{-1}(\frac{1}{s+a})[t] + R_{12}\mathcal{L}^{-1}(\frac{1}{(s+a)^2})[t] + A\mathcal{L}^{-1}(\frac{s}{s^2+1})[t] + B\mathcal{L}^{-1}(\frac{1}{s^2+1})[t] \\ =& (R_{11} + tR_{12}t)e_+^{-at} + A\cos t_+ + B\sin t_+. \end{aligned}$$

(ii) As the system only possesses one eigenvalue at $s = -a$ with multiplicity 2, the output steady-state response only if the system is asymptotically stable that is if $a > 0$. Accordingly, it can be easily deduced from the forced response by neglecting the terms whose effect vanish in time so getting

$$y_{ss}(t) = A\cos t + B\sin t$$

which can be rewritten as

$$y_{ss}(t) = M\cos(t + \varphi)$$

with $M = |P(j)| = \frac{1}{a^2-1}$, $\varphi = -\angle(j+a)^2$.