Control Systems
5/2/2019(A)

Exercise 1 We have in Laplace domain

$$Y(s) = \frac{P(s)}{1+L(s)}d(s) + \frac{L(s)}{1+L(s)}v(s),\ L(s) = G(s)P(s),$$

so that $W_d(s) = \frac{P(s)}{1+L(s)}$ (disturbance-to-output transfer function) and $W_e(s) = \frac{1}{1+L(s)}$ (input-to-error transfer function).

(i) Since there is no integral action before the entering point of d, we set $G(s) = \frac{1}{s}\tilde{G}(s)$ so that the steady state response with constant disturbances is

$$y_0 = W_d(0) = 0$$

(ii),(iii) From (ii) we have the following constraint on $\tilde{G}(s) : |\tilde{G}(j\omega)| \leq 36$ dB for all ω. The Bode plots of $\tilde{P}(s) = \frac{1}{s}P(s)$ are drawn in Fig. 1.

We have form the Bode plots in Fig. 1

$$|\tilde{P}(j5)|_{dB} \approx -27.8dB,\ Arg(\tilde{P}(j5)) \approx -191^\circ$$
$$|\tilde{P}(j10)|_{dB} \approx 40dB,\ Arg(\tilde{P}(j10)) \approx -180^\circ$$

Let us place the new crossover frequency ω_t^* at 5 rad/sec with the desired phase margin $\geq 30^\circ$, using $\tilde{G}(s)$ and recalling that we must satisfy $vert\tilde{G}(j\omega)| \leq 36$ dB for all ω. For doing this, $\tilde{G}(s)$ must be such that

$$|\tilde{G}(j5)|_{dB} \approx 27.8dB,\ Arg(\tilde{G}(j5)) \approx 42^\circ$$

Let

$$\tilde{G}(s) = KR_a(s) = K\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$$

and choose (from the compensating functions Bode plots) $m_a = 6$, $\omega_N = 2$ rad/sec with $\omega_t^* = 5$. At $\omega_N = 2$ rad/sec we have magnitude increase equal to 6 dB and phase increase equal to 45°. For $\omega_t^* = 5$ we obtain $2 = \omega_N = \omega_t^*\tau_a = 5\tau_a \Rightarrow \tau_a = 2/5$.

We have $|R_a(j\omega_t^*)P(j\omega_t^*)| = -27.8 + 6dB = -21.8dB$ and $Arg() \approx -191^\circ + 45^\circ = 146^\circ$ which would imply a phase margin $\approx 34^\circ \geq 30^\circ$(as required by (iii)). For having an overall magnitude increase of $27.8dB$ at $\omega_t^* = 5$ rad/sec we choose a proportional action $K = 21.8dB$ so that to have $\omega_t^* \approx 5$ rad/sec. Our controller $G(s)$ is finally

$$\tilde{G}(s) = 12.28\frac{1+\frac{2}{5}s}{1+\frac{1}{15}s}$$

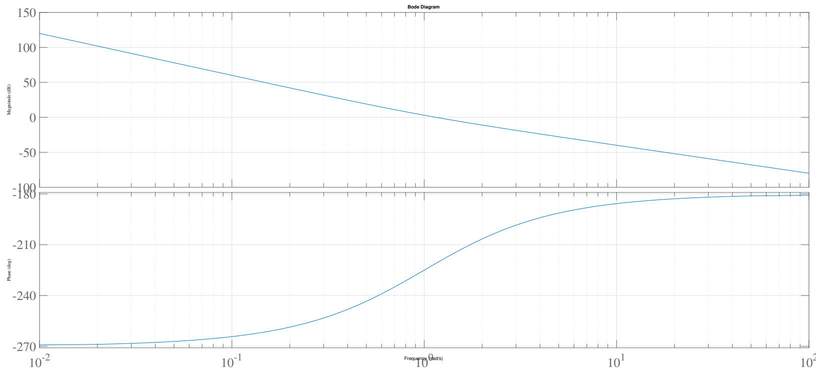

Figure 1: Bode plots of $\frac{1}{s}P(s)$

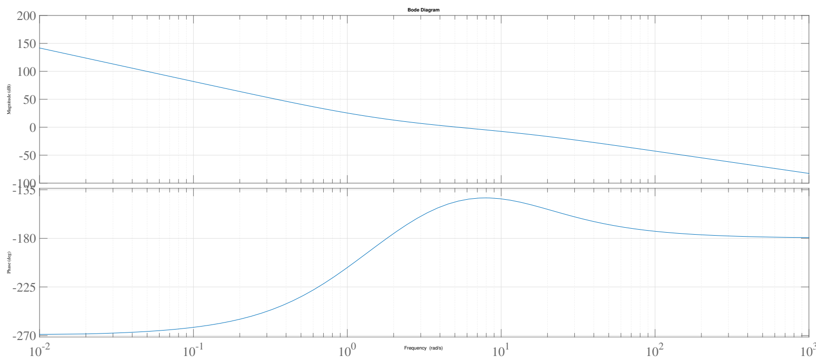

Figure 2: Bode plots of $G(s)P(s)$

The Bode plots of $G(s)P(s)$ and its Nyquist plot are drawn in Fig. 2 and 3. The Nyquist plot shows that the closed-loop system is asymptotically stable (we have $-1+1=0$ counter-clockwise tours around the point $-1+0j$).

Exercise 2 We have in Laplace domain

$$Y(s) = \frac{L_1(s)}{1+L_2(s)}d_1(s) + \frac{1}{P(s)}\frac{L_1(s)}{1+L_2(s)}d_2(s) + \frac{L_2(s)}{1+L_2(s)}v(s)$$

where $L_1(s) = \frac{P(s)}{1+P(s)}$ and $L_2(s) = G(s)L_1(s)$.

(ii) Since the d_1 to y transfer function is $W_{d_1}(s) = \frac{L_1(s)}{1+L_2(s)}$ we must have for unit ramp disturbance d_1

$$\Big|\frac{1}{s}\frac{L_1(s)}{1+L_2(s)}\Big|_{s=0} \leq 0.1 \Rightarrow \Big|\frac{NUM(G(s))|_{s=0}}{DEN(sG(s))|_{s=0}} \leq 0.1$$

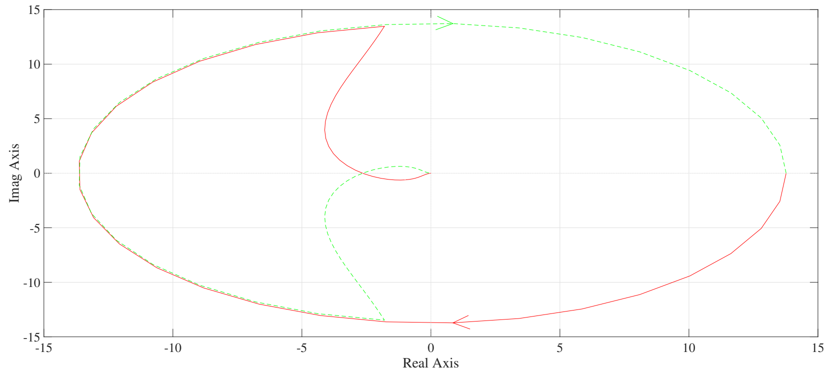

Figure 3: Nyquist plot of $G(s)P(s)$

which implies that $G(s) = \frac{K_{G,1}}{s} G_2(s)$ with

$$|K_{G,1}| \geq 10.$$

Choose $|K_{G,1}| = 10$.

(iii) Since the d_1 to y transfer function is $W_{d_2}(s) = \frac{1}{P(s)} \frac{L_1(s)}{1+L_2(s)}$ we must have for constant disturbance d_2

$$\left. \frac{1}{P(s)} \frac{L_1(s)}{1 + L_2(s)} \right|_{s=0} = 0$$

which is true thanks to the pole at $s = 0$ in $G(s)$.

(i) Recall that $G(s)$ is required to be two-dimensional. Therefore, $\tilde{G}(s)$ may have the form $\frac{K_{G_2}(s+z)^2}{s+p}$ so that $G(s) = \frac{1}{s}\tilde{G}(s)$ is indeed two dimensional and realizable (two pole-zero actions plus a proportional action). The direct path transfer function is

$$L_2(s) = G(s)L_1(s) = 10 \frac{s+2}{s^2(s-1)^2} \tilde{G}(s)$$

the first zero of $\tilde{G}(s)$ will decrease the zero-pole excess from 3 to 2 and the zero-pole action will move the asymptote center to the left: the new asymptote center will be required to satisfy

$$s_0' = \frac{4 - p + 2z}{2} < -1$$

Moreover, notice that the zeroes of $L_2(s)$ must be all with real part < -1 (in such a way that by increasing the gain the closed-loop poles will move to the left of $Re(s) = -1$). We choose $z = 3$ and $p = 20$. Next, we choose K_{G_2} from the Routh table of $NUM(1 + G(s)P(s))|_{s-1} = s^5 + 13s^4 + (K - 10)s^3 + (5K + 235)s^2(8K - 224)s + 4K + 76$. We obtain as first column of

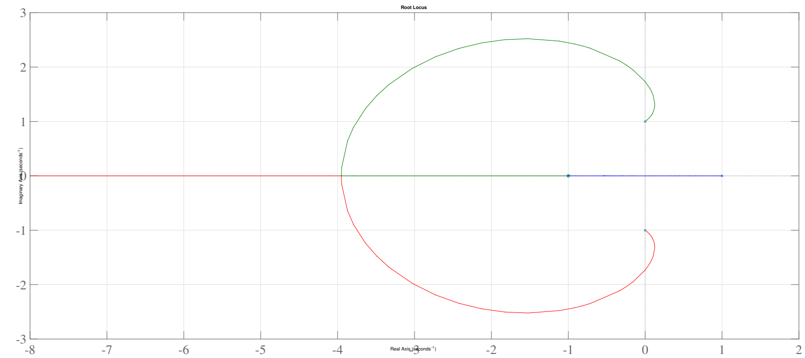

Figure 4: Positive root locus of $P(s) = \frac{(s+1)^2}{(s-1)(s^2+1)}$

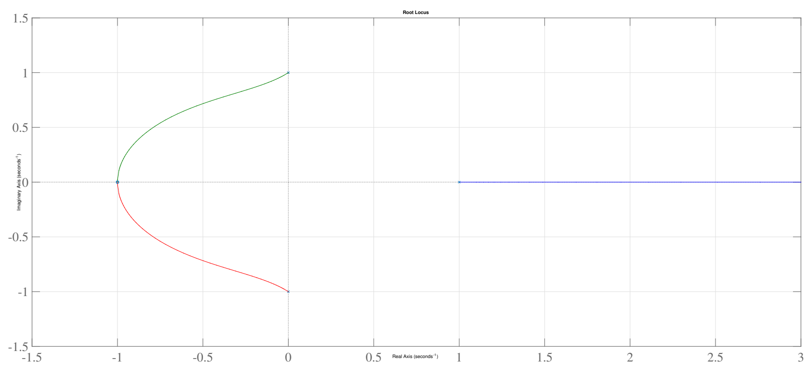

Figure 5: Negative root locus of $P(s) = \frac{(s+1)^2}{(s-1)(s^2+1)}$

the Routh table

$$1$$

$$13$$

$$2K_{G_2} - 62$$

$$\frac{10K_{G_2}^2 - 165K_{G_2} - 4859}{2(K_{G_2} - 31)}$$

$$\frac{18K_{G_2}^3 - 839K_{G_2}^2 + 404K_{G,2} + 256733}{10K_{G_2}^2 - 165K_{G_2} - 4859}$$

$$4K_{G_2} + 76$$

which gives $K_{G_2} > \max\{31, 19, 31.78\} = 31.78$ for having no sign variations.

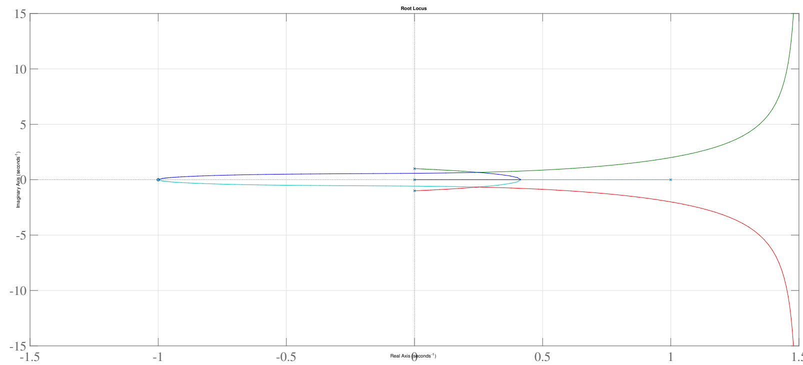

Figure 6: Positive root locus of $P(s) = \frac{(s+1)^2}{s(s-1)(s^2+1)}$

Exercise 3. (i) The zero-pole excess is $n - m = 1$, the asymptote center $s_0 = 1 + 21 = 3$ (it is not useful for $n - m = 1$) and the singular points are determined by the equations:

$$p(s,k) = (s^2+1)(s-1) + K(s+1)^2 = 0$$
$$\frac{d}{ds}p(s,k) = 2s^2 - 2s + s^2 + 1 + 2K(s+1) = 0$$

We obtain as solution $s \approx -3.95$. From the Routh table of $NUM(1+KP(s)) = s^3 + (K-1)s^2 + (1+2K)s + K - 1$ we obtain as first column

$$\begin{array}{r} 1 \\ K-1 \\ 2K \\ K-1 \end{array}$$

which implies $K > 1$ for having no sign variations. Therefore, the closed-loop system with any $G(s) = K > 1$ is asymptotically stable. The root locuses of $P(s) = \frac{(s+1)^2}{(s-1)(s^2+1)}$ have been drawn in Figg. 4 and 5.

(ii) The root locuses of $P(s) = \frac{(s+1)^2}{s(s-1)(s^2+1)}$ have been drawn in Figg. 4 and 5. Notice the singular points $s \approx 0.2 \pm 0.6j$ for $K \approx 0.2$ and $s \approx 0.4$ for $K \approx 0.1$ for the positive locus (Fig. 6) and $s \approx -2.4$ for $K \approx -28$ for the negative locus (Fig. 7). From the Routh table of $NUM(1+KP(s)) = s^4 - s^3 + (K+1)s^2 + (2K-1)s + K$ we obtain as first column

$$\begin{array}{r} 1 \\ -1 \\ 3K \\ 2K(3K-1) \\ K \end{array}$$

which implies there is no $G(s) = K$ for which the closed-loop system is asymptotically stable.

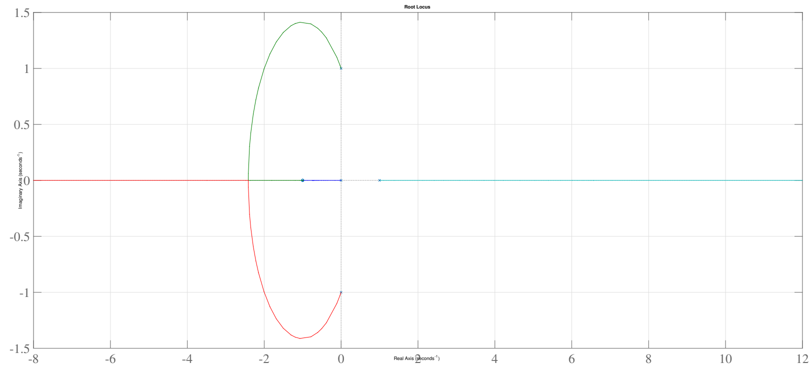

Figure 7: Negative root locus of $\frac{(s+1)^2}{s(s-1)(s^2+1)}$

Exercise 4. Our process

$$\dot{x} = Ax + Bu + \tilde{P}d, \; y = Cx \tag{1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \tilde{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

We first check that (A, B) is stabilizable. Indeed, it is even controllable ($R = \begin{pmatrix} B & 1AB \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).

We solve the problem with the output regulation procedure. Since $d = D \sin t$ we choose an exosystem for d of the form

$$\dot{w}_d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w_d = S_d w_d$$

and its solutions have the form

$$w_d(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} w_d(0)$$

so that the disturbance is generated as $d(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} w_d(t) = Q_d d(t)$ corresponding to the initial conditions $w_d(0) = \begin{pmatrix} 0 \\ D \end{pmatrix}$.

Since $v = \delta_{-1}(t)$ we choose an exosystem for v of the form

$$\dot{w}_v = 0 = S_v w_v$$

whose solutions have the form

$$\dot{w}_v = w_v(0)$$

so that the reference input v is generated as $v(t) = w_v(t) = Q_v v(t)$ corresponding to the initial conditions $w_v(0) = 1$. In the overall, we have the exosystem

$$\dot{w} = \begin{pmatrix} S_d & 0 \\ 0 & S_v \end{pmatrix} w = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} w$$

and $w = \begin{pmatrix} w_d \\ w_v \end{pmatrix}$. The output of the exosystem $q = Qw$ for generating the vector $\begin{pmatrix} d \\ v \end{pmatrix}$ (distrubances and reference inputs) will be

$$q = \begin{pmatrix} d \\ v \end{pmatrix} = Qw = \begin{pmatrix} Q_d & 0 \\ 0 & Q_v \end{pmatrix} w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} w$$

Finally the tracking error is defined as

$$e = y - v = Cx - Q_v w = \begin{pmatrix} 1 & 0 \end{pmatrix} x - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} w$$

The process (1) together with the exosystem becomes

$$\begin{aligned} \dot{x} &= Ax + Bu + Pd, \\ \dot{w} &= Sw, \\ e &= Cx + Qw, \end{aligned} \tag{2}$$

$$\tag{3}$$

with

$$P = \tilde{P} Q_w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q = -Q_v = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}.$$

The regulator equations to be solved foursome $\Pi \in \mathbb{R}^{2\times 3}$ and $\Gamma \in \mathbb{R}^{1\times 3}$ are

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P \\ C\Pi &= Q \end{aligned}$$

From the second equation

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \Rightarrow \pi_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

and using this in the first equation we get

$$\pi_2 = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \; \Gamma = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}.$$

Therefore,

$$\Pi = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \; \Gamma = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$$

and the (state feedback) regulator is given by

$$u = F(x - \Pi w) + \Gamma w$$

where $F \in \mathbb{R}^{1\times 2}$ is any matrix for which $\sigma(A + BF) \in \mathbb{C}^-$ (use Ackermann's formula for finding F: $F = -\gamma p^*(A)$). For example, with $F = \begin{pmatrix} -1 & -2 \end{pmatrix}$ we assign the eigenvalues of $A + BF$ both in -1.