Control Systems
4/6/2019(A)

Exercise 1 The armonic response $P(j\omega)$ has constant phase equal to $-270°$. Therefore, in order to obtain stability of the closed-loop system we need to increase the phase of $P(j\omega)$ and obtain the largest phase margin as possible (under the constraint of at most two anticipative actions: see the form of $G(s)$). The controller has the form

$$G(j\omega) = K_{G,1}\Big(\frac{1+\tau_1 s}{1+\tau_2 s}\Big)^2 = K_{G,1}\Big(\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}\Big)^2 = K_{G,1}G_2(s)$$

with $tau_a > 0, m_a > 1$ to be chosen (from the compensating functions plots). Let us consider first the steady state requirement (i) on the response to the disturbance $d(t)$. In order to guarantee the required attenuation of the steady state response it is necessary to guarantee that

$$|W_d(j\omega)| = \Big|\frac{1}{1+G(j\omega)P(j\omega)}\Big| \leq 0.11, \forall\omega \in [0, 0.1]rad/sec$$

i.e.

$$\Big|1+G(j\omega)P(j\omega)\Big| \geq \frac{1}{0.11}, \forall\omega \in [0, 0.1]rad/sec.$$

Since for all $\omega > 0$

$$\Big|1+G(j\omega)P(j\omega)\Big| \geq |G(j\omega)P(j\omega)| - 1$$

our condition becomes

$$|G(j\omega)P(j\omega)| \geq \frac{1}{0.11} + 1 \approx 10 = 20dB, \forall\omega \in [0, 0.1]rad/sec. \quad (1)$$

Since this means that the Bode plot of the magnitude of $G(j\omega)P(j\omega)$ must not enter the rectangle delimited by the frequencies $\omega \in [0, 0.1]rad/sec$ and the magnitudes $[0, 20]dB$. This implies that the crossover frequency ω_t^* must be certainly greater than 0.1 rad/sec.

As to requirement (ii) the gain $K_{G,1}$ must be sufficiently less than 1 and positive in order counteract the magnitude increase due to the anticipative actions (this also introduce a further limitation on the maximum achievable phase margin). However, the value of $K_{g,1}$ cannot be arbitrarily small on account of the constraint (1). Inspection of the Bode plot of $P(j\omega)$ (Fig. 1) leads to the conclusion that a good choice of $K_{G,1}$ is $\geq -20dB$. As a matter of fact since $|P(j0.1)| = 60dB$ this choice guarantees that in any case $|G(j\omega)P(j\omega)| \geq 20dB$ since the anticipative actions increase the magnitude. Set $K_{G,1} = -40dB$.

Next, we choose the parameters τ_a, M_a. The choice of $K_{G,1}$ implies that each anticipative action msut introduce a magnitude increase of at most $20dB$. In order to obtain the largest phase margin as possible we will choose the normalized frequency $\omega_N = \omega\tau_a = 3.2$ rad/sec, to which it associated a maximum increase in phase equal to $54°$. Since the corresponding increase in magnitude is equal to $\approx 10dB$ we will colocate the phase increase at the frequency $\bar{\omega}$ such that $|K_{G,1}P(j\bar{\omega})| = -20dB$ in order to obtain a phase margin of approximately

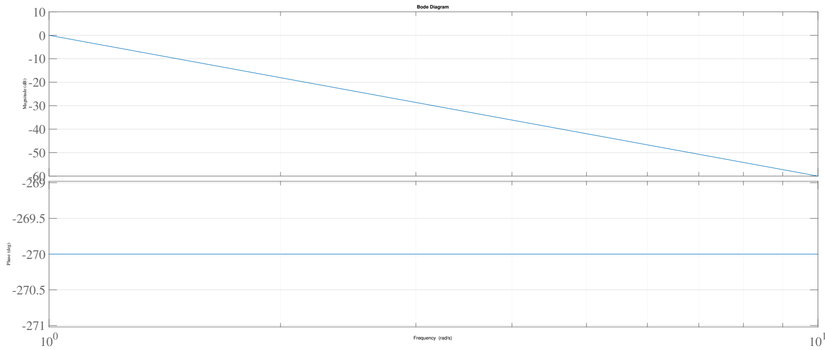

Figure 1: Bode plots of $P(s)$

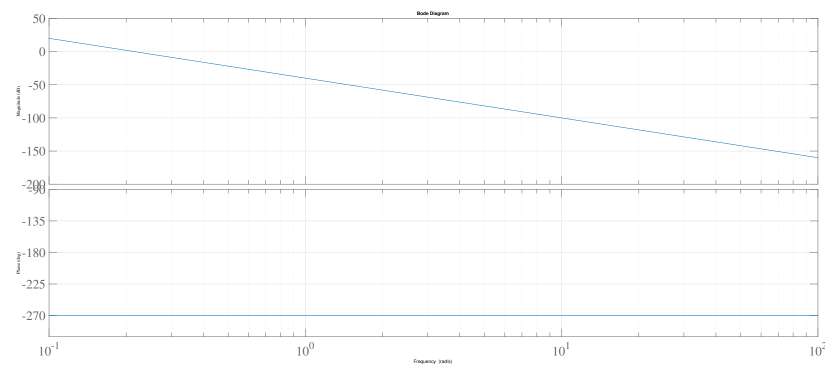

Figure 2: Bode plots of $K_{G,1}P(s)$

$-90° + 2 * 54° = 18°$. From inspection of the Bode plot of $K_{G,1}P(j\omega)$ (Fig. 2) we see that $\bar{\omega} = 0.5$rad/sec and therefore $\tau_a = 3.2/0.5$. The resulting controller $G(s)$ is

$$G(j\omega) = 0.01\Big(\frac{1 + \frac{3.2}{0.5}s}{1 + \frac{3.2}{5}s}\Big)^2$$

The Bode plots of $G(s)P(s)$ and its Nyquist plot are drawn in Fig. 3 and 4. The Nyquist plot shows that the closed-loop system is asymptotically stable (we have $-1+1=0$ counter-clockwise tours around the point $-1+0j$).

Exercise 2 The root locus of $P(s)$ gives useful suggestions for the solution. The pole-zero excess is $n-m=3$ (number of poles minus number of zeroes) and the locus has at most $n-m+1=2$ singular points. The asymptote center is in

$$s_0 = -93 = -3.$$

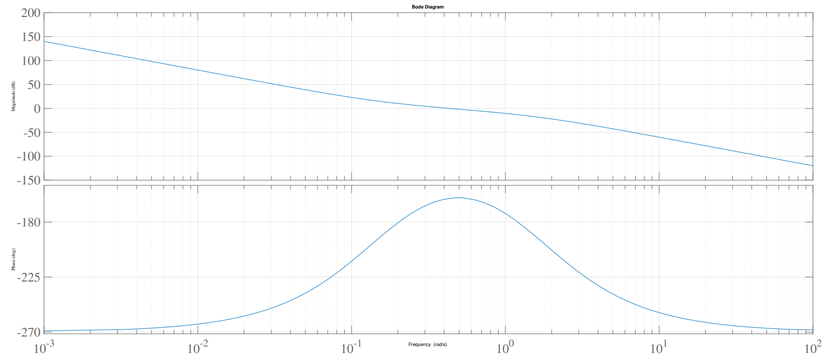

Figure 3: Bode plots of $G(s)P(s)$

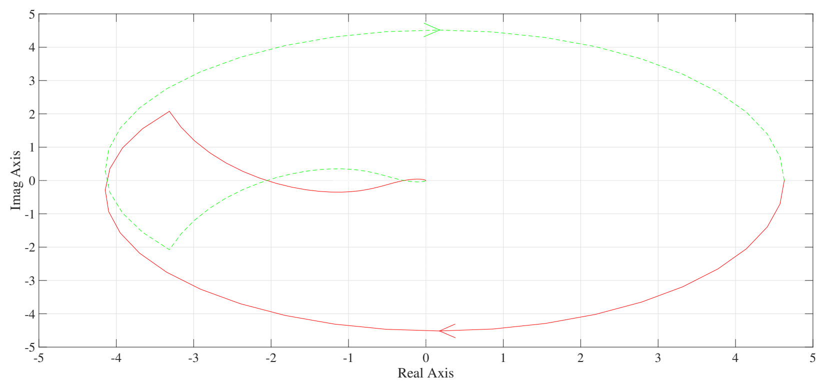

Figure 4: Nyquist plot of $G(s)P(s)$

The singular points equation is

$$3s^2 + 18s + a^2 = 0$$

and its roots are

$$s_{1,2} = \frac{-9 \pm \sqrt{9^2 - 3a^2}}{3}$$

These roots are real if and only if $a^2 \leq 27$. In particular, for $a^2 \in [0, 27]$ the position of the the first singular point s_1 varies from 0 (for $a = 0$) to -3 (for $a^2 = 27$). Both these points are in the negative locus. On the other hand, if $a^2 > 27$ the roots are complex conjugates and they are not points of the locus (as it can be verified by replacing these values in the locus equations). In the case $a^2 < 27$ there exist negative values of K for which the closed-loop system has three real negative poles while In the case $a^2 > 27$ we always have a real pole and two complex conjugate ones.

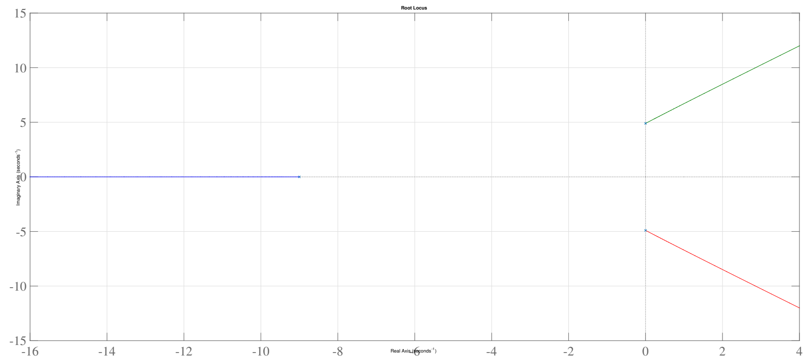

Figure 5: Positive root locus of $P(s)$

Set for example $a^2 = 24$ for which $s_1 = -2$ and $s_2 = -4$. The open loop transfer function is

$$KP(s) = \frac{K}{(s+9)(s^2+24)}$$

(see root locuses in Fiig. 5 and 6). For finding the values of K which satisfy the requirements of the exercise, it is sufficient to compute the values of K corresponding to the crossing of the imaginary axis and the two singular points s_1 and s_2 denoting K_0, K_1 and K_2 respectively these values we conclude that the closed-loop poles will be real and negative if

$$max(K_2, K_0) \leq K \leq K_1$$

For computing K_0, K_1 and K_2V we replace in the locus equation

$$NUM(1 + KP(s)) = (s^2 + 24)(s + 9) + K = 0$$

the corresponding values of s. We find

$$K_0 = -216, K_1 = -196, K_2 = -200.$$

For $a^2 = 27$ the admissible values of K are therefore $[-200, -196]$

As to the final part of the exercise, the steady state output response for a disturbance $d(t) = A \sin at$ is

$$y_{ss}(t) = |W_d(ja)| A \sin(at + \text{Arg}(W(ja))$$

where $W_d(s)$ is the disturbance-to-output transfer function of the closed-loop

$$W_d(s) = \frac{1}{1 + KP(s)} = \frac{(s+9)(s^2+a^2)}{(s+9)(s^2+a^2)+K}$$

Since $W_d(s)$ has two zeroes at $s = \pm ja$ the forced response tends to zero as $t \to +\infty$ so that the steady state output response is 0.

Exercise 3. First, let us study the controllability of the open loop $\dot{x} = Ax + Bu$.

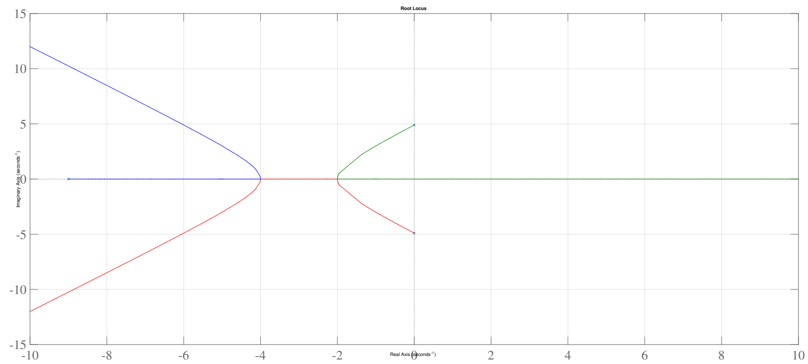

Figure 6: Negative root locus of $P(s)$

The controllability matrix

$$R = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and $detR = 0$. Therefore, the system is not controllable for all values of β and α.

We must discuss the values of α and β for which the eigenvalues of the controlled process can be moved (by state feedback) with real part $\leq -\gamma$. Therefore, we must discuss the values of α and β for which the invariant spectrum $\mathcal{F}_R$ of $A + BF$ has real part $\leq -\gamma$. For this we use the Hautus tests. The eigenvalues of A are $\{-1, -\alpha\}$ (i.e. roots of det $(\lambda I - A) = \lambda^2 + (1+\alpha)\lambda + \alpha = 0$).

1) Controllability. Case $\beta = 1$. Hautus test gives: for eigenvalue $\lambda = -1$ if $\alpha \neq 1$

$$rank \begin{pmatrix} A - \lambda I & B \end{pmatrix} = \begin{pmatrix} 1 & 1 & \beta \\ -\alpha & -\alpha & -\beta \end{pmatrix} = 2 \Rightarrow \lambda = -1 \notin \mathcal{F}_R$$

and if $\alpha = 1$

$$rank \begin{pmatrix} A - \lambda I & B \end{pmatrix} = \begin{pmatrix} 1 & 1 & \beta \\ -\alpha & -\alpha & -\beta \end{pmatrix} = 1 \Rightarrow \lambda = -1 \in \mathcal{F}_R$$

Therefore, for satisfying the requirement that the real parts of the eigenvalues of $A + BF$ be $\leq -\gamma$ we must have $-1 \leq -\gamma \Rightarrow \gamma \leq 1$.

On the other hand, for eigenvalue $\lambda = -\alpha$

$$rank \begin{pmatrix} A - \lambda I & B \end{pmatrix} = \begin{pmatrix} \alpha & 1 & \beta \\ -\alpha & -1 & -\beta \end{pmatrix} = 1 \Rightarrow \lambda = -\alpha \in \mathcal{F}_R, \forall \alpha$$

Therefore, for satisfying the requirement that the real parts of the eigenvalues of $A + BF$ be $\leq -\gamma$ we must have $-\alpha \leq -\gamma \Rightarrow \gamma \geq \alpha$.