Control Systems
03/07/2018

Exercise 1 Denoting $L(s) = G(s)P(s)$, in the Laplace domain the input-output evolutions are described by

$$y(s) = W(s)v(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$.

(i)-(iii) By inspecting the Bode plots of

$$P(s) = \frac{s+1}{s^3} \quad (1)$$

(Figure 1), as $\omega \geq 5$ rad/s, one has that $|P(j\omega)|_{dB} \leq -27.79$ as $\omega \geq 5$ rad/s and $\angle P(j\omega \in [-180^o, -225^o]$. Accordingly, one has that for $\omega \geq 5$ rad/s, the controller $G(s)$ needs to be designed to increase the phase with limited magnitude effort bounded by $|G(j\omega)|_{dB} \leq 36$. By rewriting $G(s) = kG_a(j\omega)$ with

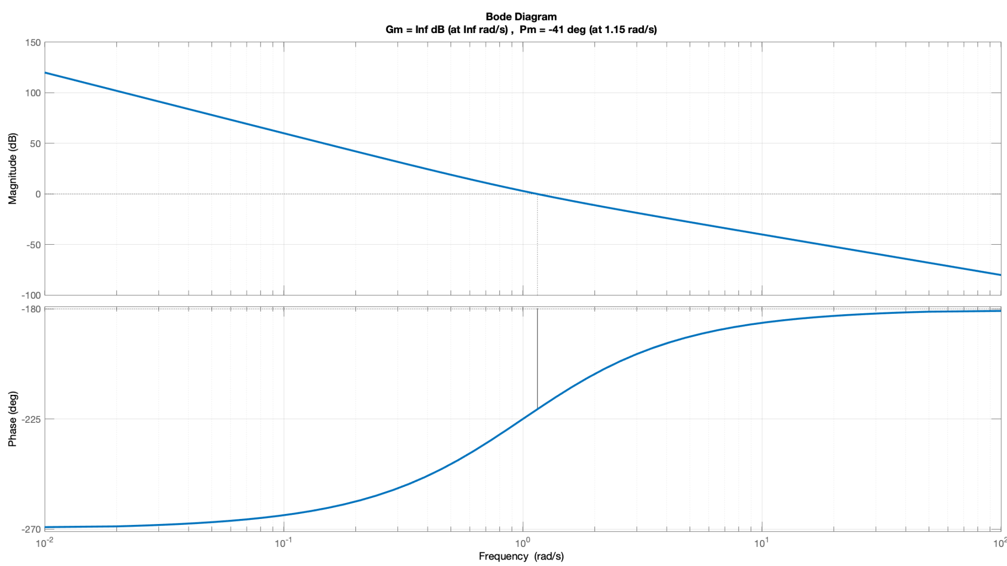

Figure 1: Bode plots of (1)

$$G_a(s) = \frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}$$

one hence gets that

$$|k|_{dB} + |G_a(j\omega)|_{dB} \leq 36 \implies |k|_{dB} \leq 36 - \max\{|G_a(j\omega)|_{dB}\}.$$

By setting $m_a = 16$ one has $\max\{|G_a(j\omega)|_{dB}\} \approx 24$ with $\max\{\angle G_a(j\omega)\} \approx 62^o$ corresponding to $\omega_n = 4$ rad/sec. Accordingly, to maximize the phase margin, one has to set ω_t^* as the desired cross-over frequency in such a way that

$$\begin{aligned} &|k|_{dB} + |G_a(j\omega_t^*)|_{dB} + |P(j\omega_t^*)|_{dB} = 0 \\ &\implies |k|_{dB} = -|G_a(j\omega_t^*)|_{dB} - |P(j\omega_t^*)|_{dB} \leq 36 - \max\{|G_a(j\omega)|_{dB}\} \\ &\implies |P(j\omega_t^*)|_{dB} \geq \max\{|G_a(j\omega)|_{dB}\} - |G_a(j\omega_t^*)|_{dB} - 36. \end{aligned}$$

Also, because for $\omega \geq 5$ rad/s $|P(j\omega)|_{dB} \leq -27.79$ then the above bound restitutes

$$\max\{|G_a(j\omega)|_{dB}\} - |G_a(j\omega_t^*)|_{dB} - 36 \leq -27.79 \implies \max\{|G_a(j\omega)|_{dB}\} - |G_a(j\omega_t^*)|_{dB} \leq 8.21.$$

A suitable choice might hence be given by setting $m_a = 6$ acting at $\omega_n = 5$ rad/s so that

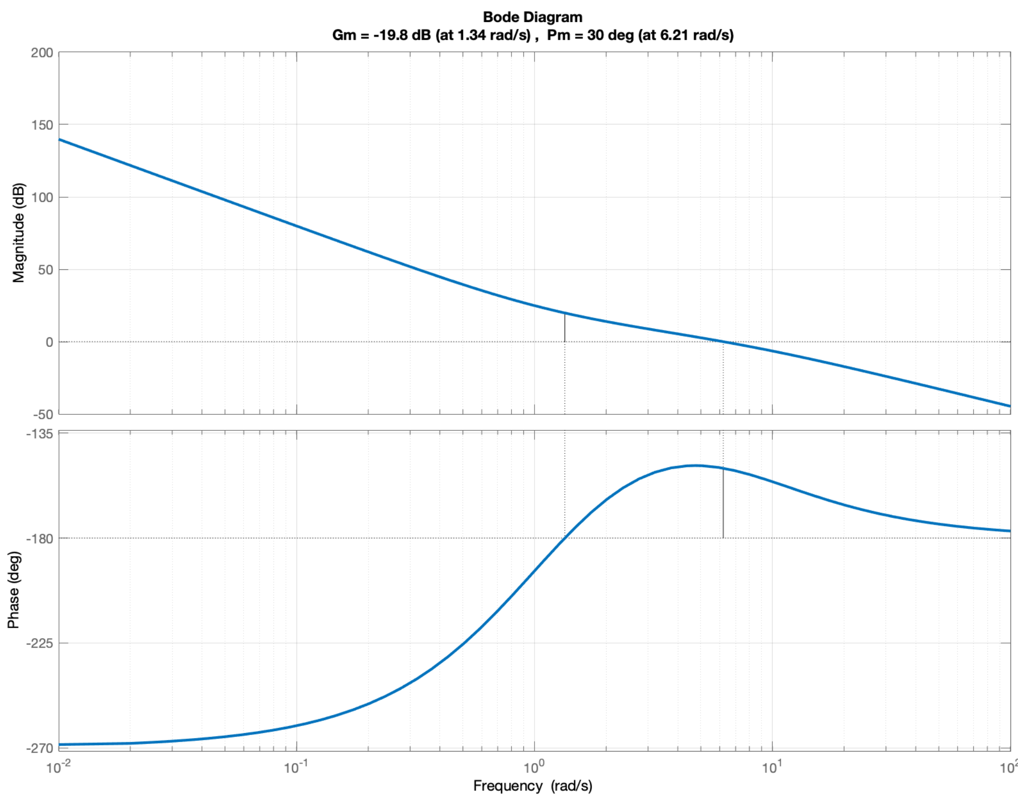

Figure 2: Bode plots of (2)

$|G_a(j\omega_t^*)|_{dB} \approx 11.86$ and $\angle G_a(j\omega_t^*) \approx 39^o$. Accordingly, the desired cross-over frequency is selected in such a way that $|k|_{dB} \leq 20$ is enough for assigning it so getting

$$|P(j\omega_t^*)|_{dB} \geq -31.86$$

that is ok for $\omega_t^* \approx 6.2$ rad/s so needing $|k|_{dB} = 20$ and thus $k = 10$.

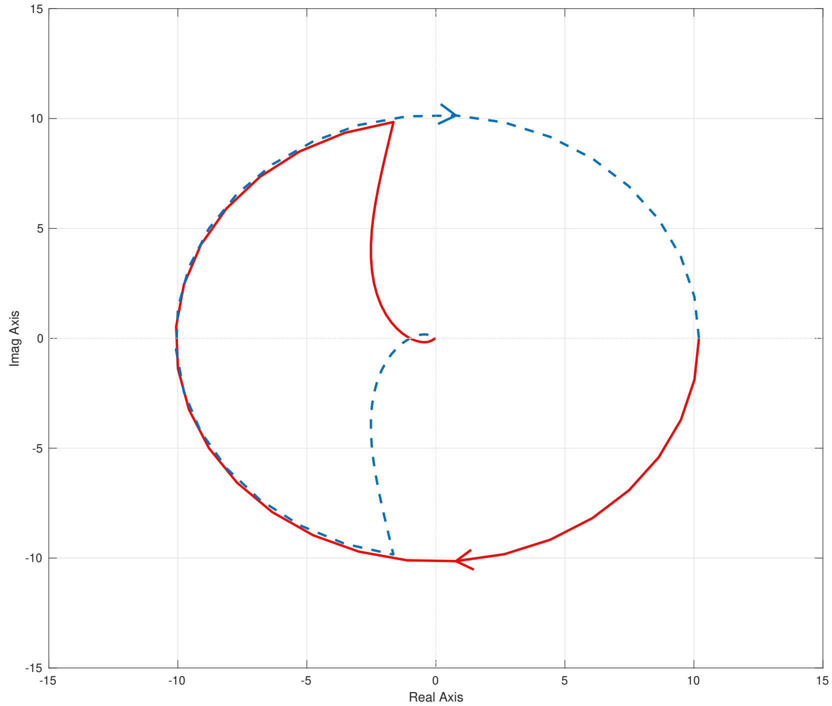

Figure 3: Nyquist plot of (2)

(iii) The Nyquist plot of the open loop system

$$L(s) = kG_a(s)P(s) = 10\frac{1+0.7939s}{1+0.1323s}\frac{s+1}{s^3} \tag{2}$$

are reported in Figure 3. The number of counter-clockwise encirclements of $-1+j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 $(1-1)$ as the number the open loop poles of L(s) with positive real part. Thus, the system is asymptotically stable in closed loop.

Exercise 2 The transfer functions of the dynamical systems involved in the interconnection are given by

$$\begin{aligned} y_1(s) =& P(s)u_1(s), \quad P(s). = \frac{1}{s(s-2)} \\ y_2(s) =& H(s)d(s), \quad H(s) = \frac{1}{s+3}. \end{aligned}$$

Accordingly, the output evolutions in the Laplace domain are described by

$$y(s) = W(s)v(s) + W_d(s)d(s), \quad W(s) = \frac{L(s)}{1+L(s)}, \quad W_d(s) = \frac{H(s)}{1+L(s)}$$

with $L(s) = G(s)P(s)$. For ensuring zero steady state output response to a constant disturbance $d(t)$, one needs $W_d(0) = 0$. As the plant $P(s)$ possesses an open loop pole at $s = 0$, no further action is needed so that (after making the system asymptotically stable) the specification is already satisfied by the plant.

For making the system asymptotically stable, we need to design $G(s)$ so to assign all poles of $W(s)$ with negative real part. As $P(s)$ has relative degree $r = n - m$ but positive center of the asymptotes, a static controller $G(s) = k$ with $k \in \mathbb{R}$ is not enough. Thus, we set

$$G(s) = k\frac{s+z}{s+p}$$

and set $z, p \in \mathbb{R}$ in such a way that the new center of the asymptote is negative; namely,

$$s_0' = \frac{-p+2+z}{2} < 0.$$

Thus, let us set $p = 26$ and $z = 4$ so getting $s_0' = -10$. Accordingly, $k \in \mathbb{R}$ can be now fixed by invoking the Routh criterion and compute the Routh table of the closed-loop pole polynomial

$$p(s,k) = s(s+26)(s-2) + k(s+4) = s^3 + 24s^2 + (k-52)s + 4k$$

so getting

r^3	1	$k-52$
r^2	6	k
r^1	$k-312/5$	
r^0	k	

so getting that the closed-loop system is asymptotically stable for $k > \frac{312}{5}$.
The root locus of $G(s)P(s)$ is equivalent to the one of $L(s) = \frac{1}{k}G(s)P(s) = \frac{s+4}{s(s-2)(s+26)}$ (that is when discarding the gain). The center of the asymptotes has been already computed and is $s_0' = -10$. Moreover, the locus possesses one singularity of multiplicity $\mu = 2$ given by the solution to the equations

$$\begin{aligned} p(s,k) =& s^3 + 24s^2 + (k-52)s + 4k = 0 \\ \frac{\partial p(s,k)}{\partial s} =& 3s^2 + 48s + k - 52 \end{aligned}$$

and provided by the couple $(s^*, k^*) \approx (0.9175, 5.4366)$. Moreover, the locus is intersecting the imaginary axis in correspondance of $(s,k) \in \mathbb{C} \times \mathbb{R}$ making the Routh table not regular. In this case, the locus is intersecting the imaginary axis at $s_1^* = -24$ and $s_2^* = 0$ corresponding to, respectively, $k_1^* = \frac{312}{5}$ and $k_2^* = 0$. The locus is reported in Figure 4.

Exercise 3 As the transfer function $P(s)$ describes the input-output behavior, for computing the state response a state-space realization is needed. To this end, we consider the controllable state-space realtization of $P(s) = \frac{s+2}{s^3-2s^2-3s}$ being provided by

$$\begin{aligned} \dot{x} =& Ax + Bu \\ y =& Cx \end{aligned}$$

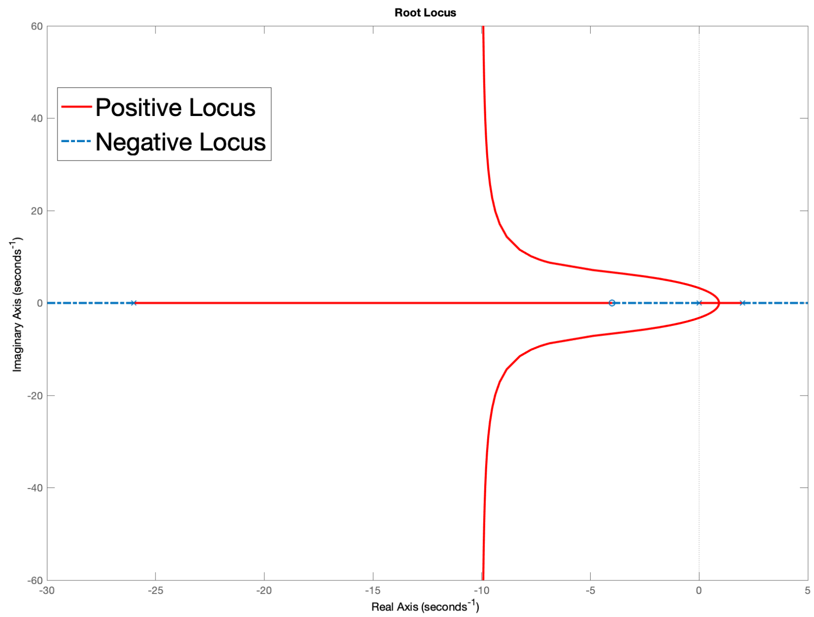

Figure 4: Root Locus of $L(s) = \frac{s+4}{s(s-2)(s+26)}$.

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

As in this case poles and eigenvalues coincide, the system possesses three aperiodic modes corresponding to $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = 3$. Thus, the state evolution ensuing from $x_0 = (1\ 1\ 0)^\top$ is provided as a linear combination of the aperiodic modes as

$$x(t) = e^{At}x_0 = c_1 z_1 + c_2 e^{-t} z_2 + c_3 e^{3t} z_3 \tag{3}$$

with

$$z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad z_3 = \begin{pmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

being the eigenvectors corresponding to the eigenvalues and

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}^{-1} x_0 = \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

so getting $c_1 = \frac{5}{3}$, $c_2 = -\frac{3}{4}$ and $c_3 = \frac{3}{4}$. To this end, each aperiodic natural modes evolves over an invariant subspace spanned by the corresponding eigenvectors, it is enough to set

$x_0 \in \text{span}\{z_2\}$ so directly implying $c_1 = c_3 = 0$.

From (3), it is clear that for making $x(t) \to 0$ as $t \to \infty$, the initial condition needs to be set so to annihilate all non-convergent modes, that is so to get $c_1 = c_3 = 0$.

For computing the output response to $u(t) = c_+$ with $c \in \mathbb{R}$ being constant it is enough to compute

$$y(t) = c\mathcal{L}^{-1}\Big(\frac{P(s)}{s}\Big)[t].$$

In particular, one gets

$$\frac{P(s)}{s} = \frac{s+2}{s^2(s+1)(s-3)} = \frac{R_{11}}{s} + \frac{R_{12}}{s^2} + \frac{R_2}{s+1} + \frac{R_3}{s-3}$$

with

$$R_{11} = \frac{1}{9}, \quad R_{12} = -\frac{2}{3}, \quad R_2 = -\frac{1}{4}, \quad R_3 = \frac{5}{36}$$

so that the output response for $x_0 = 0$ is given by

$$y(t) = \frac{c}{9}_+ - \frac{2c}{3}t_+ - \frac{c}{4}e^{-t}_+ + \frac{5c}{36}e^{3t}_+.$$