Control Systems
14/09/2018

Exercise 1 Denoting $L(s) = G(s)P(s)$, in the Laplace domain the error and output evolutions are described by

$$e(s) = W_e(s)v(s), \quad y(s) = W(s)v(s)$$

with $W(s) = \frac{L(s)}{1+L(s)}$ and $W_e(s) = \frac{1}{1+L(s)}$. We shall split the controller $G(s) = G_2(s)G_1(s)$ with $G_1(s)$ designed for steady-steady specifications (i.e., (ii)) and $G_2(s)$ for the remaining ones.

(ii) As $e(s) = W_e(s)v(s)$, one needs $W_e(0) = 0$ and $\left|\frac{\partial W_e}{\partial s}\right|_{s=0} \leq M$ with $M = 0.02$. As the plant possesses a pole at $s = 0$, the former condition (i.e., $W_e(0) = 0$) is verified whereas the latter one is ensured by the inequality above

$$|\frac{W_e(s)}{s}|_{s=0} \leq M.$$

Setting $G_1(s) = k_1$ and, for the time-being $G_2(s) = 1$, one has that the specification is fulfilled by setting

$$|\frac{1}{5k}| \leq 0.02 \implies |k_1| \geq 10.$$

Accordingly, we fix $k_1 = 10$ while constraining the gain of the outer loop of the controller to verify $G_2(0) > 1$.

(iii)-(iv) By inspecting the Bode plots (Figure 1) of

$$L_1(s) = \frac{50}{s(s+1)} \tag{1}$$

one notes that as $\omega \in [8, 14]$ rad/s

1. the magnitude is decreasing and $|L_1(j\omega)|_{dB} \in [-11.88, -2.21]$;
2. the phase is decreasing and $\angle L_1(j\omega) \in [-175.91^o, -172, 88^o]$ rad/s.

Accordingly, for increasing the phase margin $m_\varphi^* \geq 50^o$, an anticipative action is needed with phase contribution of at least 45.91^o at $\omega = 14$ rad/sec and 42.88^o at $\omega = 8$ rad/s. Moreover, specification (iv) sets a bound over the magnitude of the controller so that one has

$$|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} = |k_1|_{dB} + |G_2(j\omega)|_{dB} \leq 36$$

implying

$$|G_2(j\omega)|_{dB} \leq 16 \quad \text{and} \quad |G_2(0)|_{dB} > 1$$

where the latter bound comes from specification (ii). Accordingly, we set the outer control loop as composed of a one anticipating action and one proportional term so

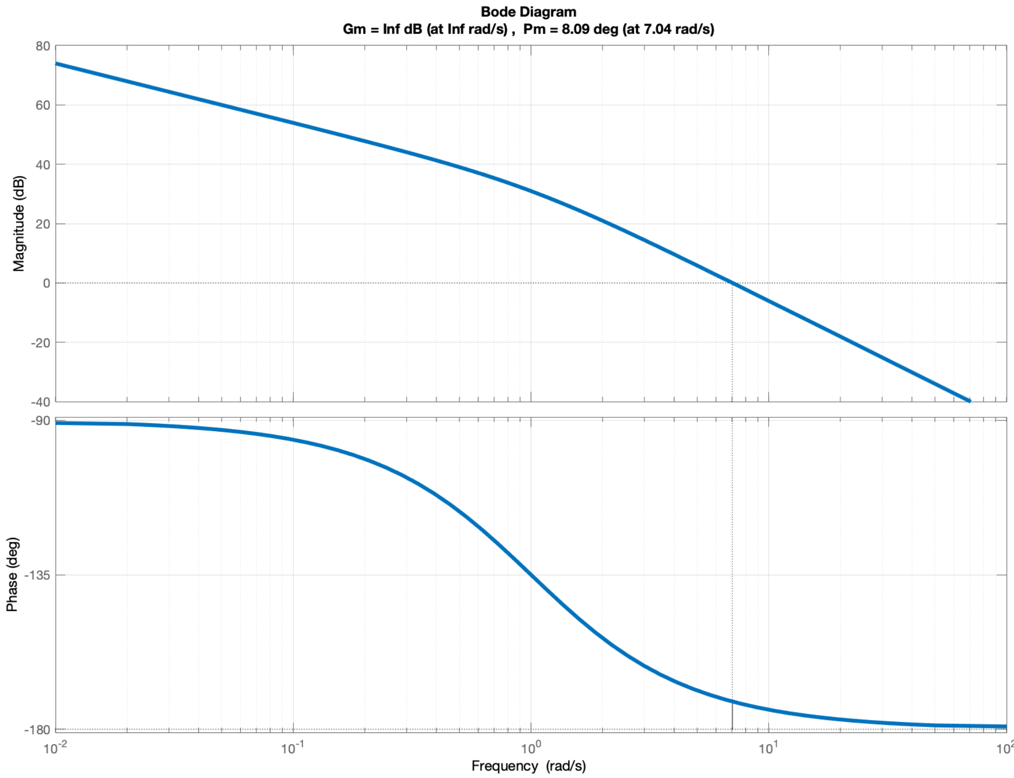

Figure 1: Bode plots of (1)

getting

$$G_2(s) = k_2 G_a(s), \quad G_a(s) = \frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}.$$

$$|k_2|_{dB} + \max_{\omega \geq 0} |G_a(j\omega)|_{dB} \leq 16 \quad \text{and} \quad |k_2|_{dB} > 0.$$

For saving the control effort of the controller, we shall set an anticipative function labeled by $m_a = 6$ and acting at $\omega_n = 2$ rad/s (that is $\tau_a = \frac{2}{\omega_t^*}$) so getting that $\angle G_a(j8) = 45^o$, $|G_a(j8)|_{dB} \approx 6.53$, $\max_{\omega \geq 0} |G_a(j\omega)|_{dB} \approx 16$. Accordingly, we shall set $\omega_t^* \in [8, 14]$ rad/s in such a way that

$$|L_1(j\omega_t^*)|_{dB} - 6.53 = 0$$

that is achieved for $\omega_t^* \approx 10.3$ rad/sec in correspondence of which $\angle L_1(j\omega) \approx -174.44^o$. Thus, setting $k_2 = 1$ specifications $(iii)-(iv)$ are satisfied with $m_\varphi^* \approx 50.6^o$ as confirmed by the Bode plots of

$$L(s) = G(s)P(s) = 10\frac{1+\tau_a s}{1+\frac{\tau_a}{m_a}s}\frac{5}{s(s+1)} \tag{2}$$

reported in Figure 2.

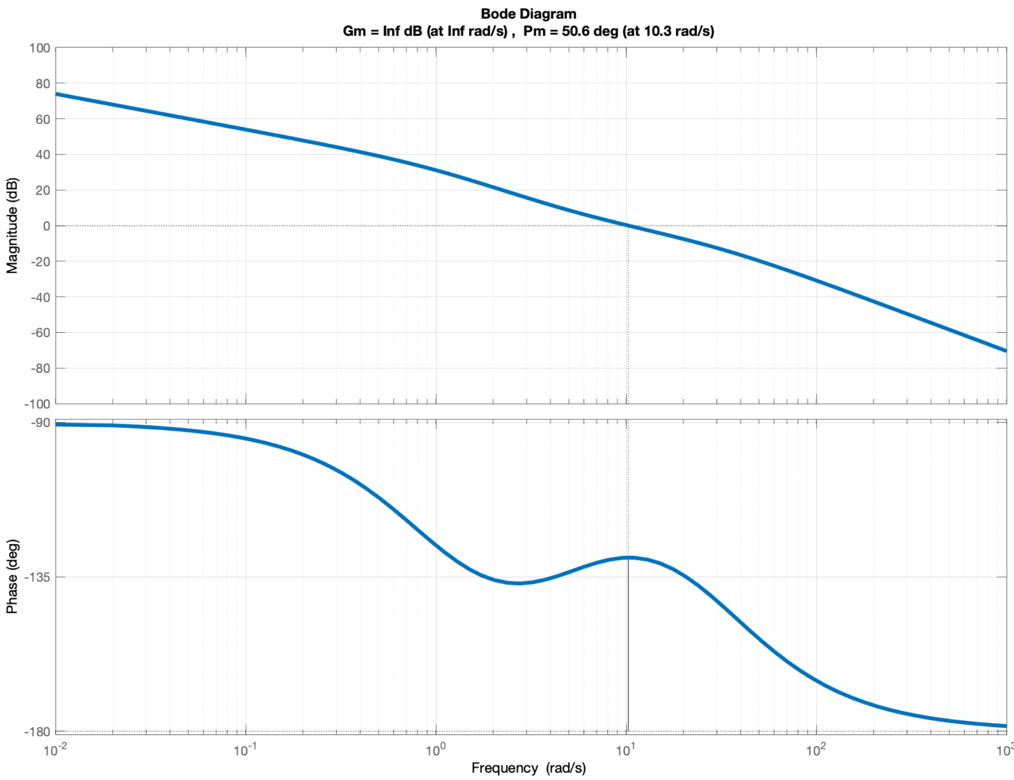

Figure 2: Bode plots of (3)

(i) The Nyquist plot of the open loop system

$$L(s) = kG_a(s)P(s) = 10\frac{1+0.7939s}{1+0.1323s}\frac{s+1}{s^3} \tag{3}$$

are reported in Figure 3. The number of counter-clockwise encirclements of $-1+j0$ on behalf of the extended Nyquist plot of $L(j\omega)$ is 0 as the number the open loop poles of L(s) with positive real part. Thus, the system is asymptotically stable in closed loop.

Exercise 2 Denoting $L(s) = G_2(s)P(s)$ with $P(s) = P_1(s)P_2(s)$, one has that in the Laplace domain the inputs-output evolutions are described by

$$y(s) = W_d(s)d(s) + W(s)v(s), \quad W_d(s) = \frac{1-G_1(s)P_2(s)}{1+L(s)}, \quad W(s) = \frac{L(s)}{1+L(s)}.$$

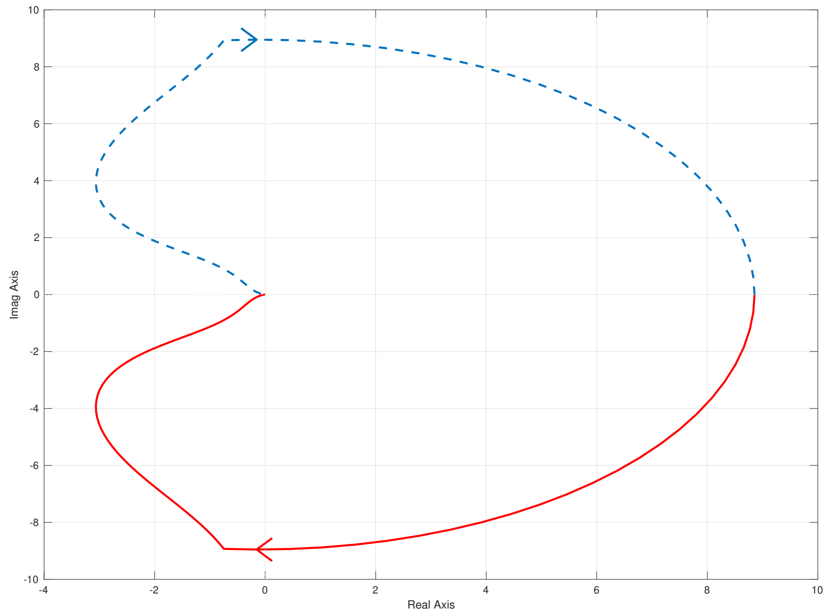

Figure 3: Nyquist plot of (3)

(i) For ensuring that $y(t) \equiv 0$ for all disturbances, one has to define $G_1(s)$ in such a way that $W_d(s) = 0$. This is achieved by setting $G_1(s)P_2(s) = 1$ that is

$$G_1(s) = \frac{s-2}{s+4}.$$

(ii) As $e(s) = W_e(s)v(s)$ with $W_e(s) = \frac{1}{1+L(s)}$, for ensuring that $|e(t)| \leq 1$ for ramp references $v(t) = 1$, one needs

$$W_e(0) = 0, \quad |\frac{\partial W_e}{\partial s}|_{s=0} \leq 1$$

so that one needs $G_2(s) = \frac{k_1}{s}\hat{G}_2(s)$ with $k_1 \in \mathbb{R}$. Setting for the time-being $\hat{G}_2(s) = 1$, one gets that the specification is verified if k_1 is such that

$$|\frac{W_e(s)}{s}|_{s=0} \leq 1$$

that is implied by $k_1 \geq \frac{1}{2}$. Accordingly, we fix $k_1 = \frac{1}{2}$ and set $\hat{G}_2(0) \geq 1$.

For stabilizing the closed-loop system (that is assignining all poles of $W(s)$ with negative real part) setting $\hat{G}_2(s) = \hat{k}$ with $\hat{k} > 1$ is not enough. Indeed, setting

$$F(s) = \frac{\hat{k}}{2s}P(s) = \underbrace{\frac{\hat{k}}{2}}_{:=k}\frac{s+4}{s(s-1)(s-2)} \tag{4}$$

one has that the closed-loop pole polynomial is $p_F(s,\hat{k}) = s^3 - 3s^2 + (2+k)s + 4k$ exhibiting two sign variations in the coefficients for all $\hat{k} \in \mathbb{R}$. Accordingly, by invoking the necessary condition of the Routh criterion, there exists no $\hat{G}(s) = \hat{k} \in \mathbb{R}$ making the closed-loop system asymptotically stable. Thus, one can set

$$\hat{G}(s) = \frac{s+z}{s+p}$$

with $p \in \mathbb{R}$ chosen in such a way that

$$\frac{-p+3+4+z}{2} < 0 \implies p - z > 7. \tag{5}$$

Hence, setting $z = 4$, the above necessary condition is satisfied for $p = 21$. Thus, one has that

$$L(s) = \underbrace{\frac{\hat{k}}{2}}_{:=k} \frac{(s+4)^2}{s(s-1)(s-2)(s+21)}$$

so that the stabilizing $\hat{k}$ can be computed by invoking the Routh criterion and computing the Routh table associated to the closed-loop pole polynomial

$$p_L(s,k) = s^4 + 18s^3 + (k-61)s^2 + (8k+42)s + 16k$$

that is given by

$$\begin{array}{c|ccc} r^4 & 1 & k-61 & 16k \\ r^3 & 9 & 4k+21 & \\ r^2 & \frac{5k-570}{9} & 16k & \\ r^1 & \frac{20k^2-2319k-11970}{5k-570} & & \\ r^0 & k & & \end{array}$$

Thus, one has that for $k > 120.9$ (and thus $\hat{k} > 241.8$) the closed-loop system is asymptotically stable under the controller

$$G_2(s) = \frac{\hat{k}}{2}\frac{s+4}{s(s+21)}.$$

The root locus of $P(s)G_1(s)G_2(s) = \frac{\hat{k}}{2}\frac{s+4}{s(s-1)(s+21)}$ is equivalent to the one of

$$K(s) = \frac{s+4}{s(s-1)(s+21)}.$$

Accordingly, denoting by n and m the number of poles and zeros and $r = n - m = 2$ as the relative degree, one has that positive locus possesses one vertical asymptote centered at

$$s_0 = \frac{-21+1+4}{2} = -8.$$

Moreover, the positive locus possesses one sigularity of order $\mu = 2$ at $(s^*, \tilde{k}^*) \approx (0.48, 1.2)$

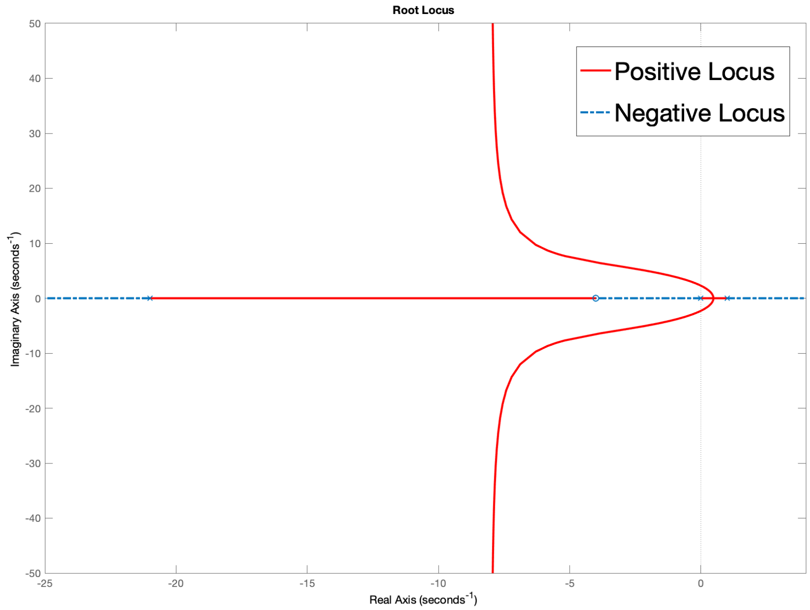

Figure 4: Root Locus of $P(s)G_1(s)G_2(s) = \frac{\hat{k}}{2}\frac{s+4}{s(s-1)(s+21)}$.

as solutions to the equalities

$$\begin{aligned} p_K(s,\tilde{k}) =& s^3 + 20s^2 + (\tilde{k}-21)s + 4\tilde{k} = 0 \\ \frac{\partial p_K(s,\tilde{k})}{\partial s} =& 3s^2 + 40s + \tilde{k} - 21. \end{aligned}$$

The locus is reported in Figure 4.

Exercise 3 The closed-loop denominator of the input-output transfer function is given by

$$p(s) = NUM(1 + P(s)) = s^3 + 3s^2 + (3+z)s + 1 - z.$$

(i) For the roots of $p(s)$ to possess real part smaller or equal to $-\frac{1}{2}$ it is necessary and sufficient, by the Routh criterion, that the polynomial

$$p_{\frac{1}{2}}(s) = p(s - \frac{1}{2}) = s^3 + \frac{3}{2}s^2 + (z + \frac{3}{4})s - \frac{3}{2}z + \frac{1}{8}$$

is Hurwitz. By computing the Routh table

$$\begin{array}{c|cc} r^3 & 1 & z+\frac{3}{4} \\ r^2 & \frac{3}{2} & -\frac{3}{2}z+\frac{1}{8} \\ r^1 & 3z+1 & \\ r^0 & -3z+\frac{1}{4} & \end{array}$$

the closed-loop system has all poles with real part smaller than $-\frac{1}{2}$ for all $z \in (-\frac{1}{3}, \frac{1}{12})$.

(ii) It is evident that for $z = 0$, the roots of $S^* := \{s \in \mathbb{C} \text{ s.t. } p(s) = 0\} = \{-1\}$. For determining, more in general, all roots of $p(s)$ it is enough for the discriminant of the polynomial $p(s)$ given by

$$\Delta^* = -4z^2(z + 27)$$

to be non-negative. This is the case for $z \in \{0\} \cup (-\infty, -27)$