that $x=0$ is

2.5 CONVERSE THEOREMS

Theorem 2.5.1 Let $x=0$ be an equilibrium of $\dot{x}=f(x,t)$, where $f:[0,\infty)\times D_r \to \mathbb{R}^n$ is continuously differentiable, $D_r \triangleq \{x\in\mathbb{R}^n \mid |x|<r\}$ and $\frac{\partial f}{\partial x}$ is bounded on D_r, uniformly in t. Let k, γ, r_0 be positive reals such that $r_0 < r/k$.

Let $D_0 = \{x \in \mathbb{R}^n \mid \|x\| < r_0\}$.

Assume

$$\|x(t)\| \le k\|x_0\| e^{-\gamma(t-t_0)}$$

$$\forall x_0 \in D_0, \ \forall t \ge t_0 \ge 0$$

Then there is a function $V: [0,\infty) \times D_0 \to \mathbb{R}$ that satisfies

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4 \|x\|$$

for some positive c_1, c_2, c_3, c_4. Moreover if $r = \infty$ and the origin is globally exponentially stable then $V(t,x)$ is defined and satisfies the above inequalities on $\mathbb{R}^n$. If the system is autonomous V can be chosen independent of t

<u>Proof</u>. Let $\psi(\tau, t, x)$ the solution of the system that starts at (t,x): $\psi(t,t,x) = x$.

For all $x \in D_0$, $\psi(\tau, t, x) \in D_\varepsilon$ for all $\tau \geq t$ (since $r_0 < \varepsilon/k$!).

Define
$$V(t,x) = \int_t^{t+T} \psi^T(\tau,t,x)\psi(\tau,t,x)\,d\tau$$
with $T > 0$ chosen later. By exponential decay
$$V(t,x) = \int_t^{t+T} \|\psi(\tau,t,x)\|^2 d\tau \leq$$
$$\leq \int_t^{t+T} k^2 e^{-2\gamma(\tau-t)} d\tau \|x\|^2 \leq \frac{k^2}{2\gamma}(1-e^{-2\gamma T})\|x\|^2$$

But $\left\|\frac{\partial f}{\partial x}\right\|$ is bounded on D_ε:
$$\left\|\frac{\partial f}{\partial x}\right\| \leq L \quad \forall x \in D_\varepsilon$$
The function $f(t,x)$ is Lipschitz on D with Lipschitz constant L.

Therefore,
$$\|\psi(\tau,t,x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}$$

Hence

$$v(t,x) \geq \int_t^{t+T} e^{-2L(\tau-t)} d\tau \, \|x\|^2$$

$$= \frac{1}{2L}\left(1-e^{-2LT}\right)\|x\|^2$$

Thus

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2$$

$$c_1 = \frac{1-e^{-2LT}}{2L} \qquad c_2 = \frac{k^2\left(1-e^{-2\gamma T}\right)}{2\gamma}$$

Define next

$$\psi_t(\tau,t,x) \triangleq \frac{\partial}{\partial t}\psi(\tau,t,x)$$

$$\psi_x(\tau,t,x) = \frac{\partial}{\partial x}\psi(\tau,t,x)$$

Then

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(t,x) = \psi^T(t+T,t,x)\psi(t+T,t,x)$$

$$-\psi^T(t,t,x)\psi(t,t,x) + \int_t^{t+T} 2\psi^T(\tau,t,x)\psi_t(\tau,t,x)\,d\tau$$

$$+ \int_t^{t+T} 2\psi^T(\tau,t,x)\,\psi_x(\tau,t,x)\,d\tau \cdot f(t,x)$$

$$= \psi^T(t+T,t,x)\,\psi(t+T,t,x) - \|x\|^2$$

$$+ \int_t^{t+T} 2\psi^T \underbrace{\left[\psi_t(\tau,t,x) + \psi_x(\tau,t,x) f(t,x)\right]} d\tau$$

It can be seen that

$$\overbrace{\psi_t(\tau,t,x) + \psi_x(\tau,t,x) f(t,x)} \equiv 0 \quad \forall \tau \geq t$$

(follows from $\psi(\tau,s,(\psi(s,t,x))) = x \;\; \forall s$)

Therefore

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(t,x) = \psi^T(t+T,t,x)\,\psi(t+T,t,x)$$

$$-\|x\|^2 \leq -(1-k^2 e^{-2\gamma T})\|x\|^2$$

Choose $T = \ln(2k^2)/2\gamma \Rightarrow$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f(t,x) \leq -c_3\|x\|^2$$

with $c_3 \hat{=} 1/2$

Notice

$$\frac{\partial}{\partial \tau}\psi_x(\tau,t,x) = \frac{\partial f}{\partial x}(\tau,\psi(\tau,t,x))\,\psi_x(\tau,t,x)$$

$$\psi_x(t,t,x) = I$$

Since

$$\left\|\frac{\partial f}{\partial x}\right\| \le L \quad \text{on} \quad D_2$$

then

$$\|\psi_x(\tau,t,x)\| \le e^{L(\tau-t)}$$

Hence

$$\left\|\frac{\partial V}{\partial x}\right\| = \left\|\int_t^{t+T} 2\psi^T(\tau,t,x)\,\psi_x(\tau,t,x)\,d\tau\right\|$$

$$\le \left\|\int_t^{t+T} 2\,\|\psi(\tau,t,x)\|\,\|\psi_x(\tau,t,x)\|\,d\tau\right.$$

$$\le \int_t^{t+T} 2k e^{-\gamma(\tau-t)} e^{L(\tau-t)}\,d\tau\,\|x\|$$

$$= \underbrace{\frac{2k}{\gamma-L}\left(1-e^{-(\gamma-L)T}\right)}\,\|x\|$$

$$\Rightarrow \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

If all the assumptions hold globally $\Rightarrow$ r_0 can be taken arbitrarily large. If the system is autonomous then $\psi(\tau,t,x)$ depends only on $\tau - t$:

$$\psi(\tau,t,x) = \psi(\tau - t, 0, x)$$

Then

$$V(t,x) = \int_t^{t+T} \psi^T(\tau - t, 0, x)\,\psi(\tau - t, 0, x)\,d\tau$$

$$= \int_0^T \psi^T(s,0,x)\,\psi(s,0,x)\,ds$$

Theorem 25.2 Let $x=0$ be an equilibrium point of $\dot{x} = f(t,x)$, $f : [0,\infty) \times D_2 \to \mathbb{R}^n$ continuously differentiable, $D_2 \triangleq \{x \in \mathbb{R}^n \mid \|x\| < r\}$, $\frac{\partial f}{\partial x}$ bounded and Lipschitz on D_2, uniformly in t. Let

$$A(t) \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

Then the origin is exponentially stable if and only if it is exponentially stable for $\dot{x} = A(t)x$

Proof. (If) part follows from Theorem 2.4.3.

(Only if part) Write the linear system as

$$\dot{x} = f(t,x) - [f(t,x) - A(t)x]$$

$$= f(t,x) - g(t,x)$$

Recall that

$$\|g(t,x)\| \leq L\|x\| \quad \forall x \in D_2, \ \forall t \geq 0$$

Since $x=0$ is ES for $\dot{x} = f(t,x)$ there are $k, \gamma, c > 0$ such that

$$\|x(t)\| \leq k\|x_0\| e^{-\gamma(t-t_0)}$$

$$\forall t \geq t_0 \geq 0$$

$$\forall \|x_0\| < c$$

Choosing $r_0 < \min\{c, \frac{c}{k}\}$ all conditions of Theorem 2.5.1 are satisfied. Let $V(t,x)$ be as in the proof of Theorem 2.5.1:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) - \frac{\partial V}{\partial x} g(t,x) \le \underbrace{-c_3 \|x\|^2 + c_4}_{\text{from Theorem 2.5.1}} \underbrace{L \|x\|^3}_{\text{from above}}$$

The choice $\rho < \min\{r_0, c_3/c_4 L\}$ ensures $\dot{V}(t,x)$ negative definite in $\|x\| < \rho$. Conditions of Corollary 2.3.2 are satisfied and $x=0$ is ES for $\dot{x} = A(t)x$

EX. $\dot{x} = -x^3$

- $x=0$ is AS
- the linearization around the origin is $\dot{x}=0$ $\Rightarrow$ A is not Hurwitz! using theorem 2.5.2 $\dot{x}=-x^3$ ■

$\Rightarrow$ $x=0$ is not AS for

<u>Theorem 2,5,3</u> Same assumptions as theorem 2.5.1 and 2.5.2.
Let $\beta \in \mathcal{KL}$ and $r_0>0$ such that $\beta(r_0,0)<r$, with $D_0 \triangleq \{x\in\mathbb{R}^n \mid \|x\|<r_0\}$. Assume

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) \quad \forall x_0 \in D_0 \quad \forall t \geq t_0 \geq 0$$

then there exists a continuously differentiable $V: [0,\infty)\times D_0 \longrightarrow \mathbb{R}$ such that

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$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(\|x\|)$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_4(\|x\|)$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}$ defined on $[0, r_0]$. If the system is autonomous V can be chosen independent of t ◀