Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2019/2020

Solution of Problem 1

1. The two input vector fields are $\boldsymbol{g}_1 = (-\cos\gamma \quad \frac{\sin\gamma}{\rho} \quad \frac{\sin\gamma}{\rho})^T$ and $\boldsymbol{g}_2 = (0 \quad -1 \quad 0)^T$. Their Lie Bracket is easily computed as

$$\boldsymbol{g}_3 = [\boldsymbol{g}_1, \boldsymbol{g}_2] = -\begin{pmatrix} * & \sin\gamma & * \\ * & \frac{\cos\gamma}{\rho} & * \\ * & \frac{\cos\gamma}{\rho} & * \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \frac{\cos\gamma}{\rho} \\ \frac{\cos\gamma}{\rho} \end{pmatrix}$$

   Since $\det(\boldsymbol{g}_1 \; \boldsymbol{g}_2 \; \boldsymbol{g}_3) = 1/\rho$, the accessibility rank condition is satisfied and the system is controllable.

2. Since $v = \bar{v}$, the second equation becomes

$$\dot{\gamma} = \frac{\sin\gamma}{\rho}\,\bar{v} - \omega$$

   where the first term now represents a drift and ω is the only control input. This equation can be easily linearized letting

$$\omega = \frac{\sin\gamma}{\rho}\,\bar{v} - u$$

   In fact, we obtain

$$\dot{\gamma} = \frac{\sin\gamma}{\rho}\,\bar{v} - \frac{\sin\gamma}{\rho}\,\bar{v} + u = u$$

   i.e., a simple integrator dynamics from the new input u to γ. The desired set-point $\gamma = \pi/2$ can be made globally exponentially stable by a proportional feedback:

$$u = k(\pi/2 - \gamma) \quad k > 0$$

   The original velocity input is computed as

$$\omega = \frac{\sin\gamma}{\rho}\,\bar{v} - k(\pi/2 - \gamma)$$

   At steady state, it will be $\gamma = \pi/2$ so that $\dot{\rho} = 0$ (from the first unicycle equation), i.e., $\rho =$ const. This obviously indicates that the unicycle is a moving along a circle centered at the origin.

Solution of Problem 2

Consider the kinematic model of the front-wheel-drive bicycle:

$$\begin{aligned}\dot{x}_R &= v_F \cos\theta \cos\phi \\ \dot{y}_R &= v_F \sin\theta \cos\phi \\ \dot{\theta} &= v_F \frac{\sin\phi}{\ell} \\ \dot{\phi} &= \omega\end{aligned}$$

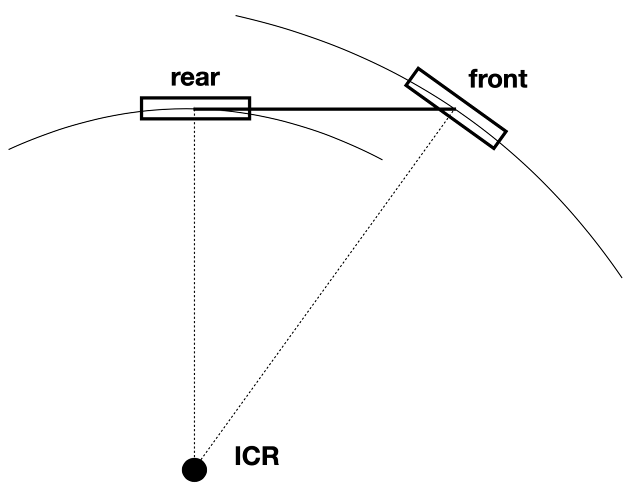
where x_R, y_R are the coordinates of the rear wheel and v_F denotes the velocity of the front wheel. For the velocity v_R of the rear wheel, we can write

$$v_R^2 = \dot{x}_R^2 + \dot{y}_R^2 = v_F^2 \left(\cos^2\theta \cos^2\phi + \sin^2\theta \cos^2\phi\right) = v_F^2 \cos^2\phi$$

which confirms that the velocity of the rear wheel is never larger than the velocity of the front wheel. Clearly, $v_R = v_F$ when $\phi = 0$ (the robot is moving on a straight line).

The same exact result is obtained starting from the kinematic model of the rear-wheel-drive bicycle, where v_R appears as a velocity input, and computing the velocity v_F of the front wheel.

The geometric interpretation of this result is straightforward. As shown in the figure below, both the front wheel and the rear wheel move instantaneously along an arc of circle, but the radius of the rear wheel circle is smaller than the radius is of the front wheel circle; the velocity of the rear wheel must therefore be smaller or at most equal (when $\phi = 0$) to that of the front wheel.



Solution of Problem 3

1. The kinematic model of the robot is readily obtained as a simple dynamic extension of the classical model with velocity inputs:

$$\begin{aligned}\dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \\ \dot{v} &= a_v \\ \dot{\omega} &= a_\omega\end{aligned}$$

   Note that the driving and steering velocities v and ω are state variables in this model, whereas the control inputs are the accelerations a_v, a_ω.

2. Using Euler integration, a discrete-time motion model is written as

$$\begin{aligned}x_{k+1} &= x_k + T_s\, v_k \cos\theta_k \\ y_{k+1} &= y_k + T_s\, v_k \sin\theta_k \\ \theta_{k+1} &= \theta_k + T_s\, \omega_k \\ v_{k+1} &= v_k + T_s\, a_{v,k} \\ \omega_{k+1} &= \omega_k + T_s\, a_{\omega,k}\end{aligned}$$

   where T_s is the sampling interval.

3. For the measurement model, we have a total of three readings coming from the sensors at each sampling instant, i.e., the distance d to the upper wall and the robot orientation and its velocity along the x axis. These quantities are expressed as a function of the system states as follows:

$$\boldsymbol{h}_k = \begin{pmatrix} a - y_k \\ \theta_k \\ v_k\cos\theta_k \end{pmatrix}$$

   The rest of the solution is straightforward: linearize the motion and measurement models and then write the EKF equations. Note that in this case all measurements will be used for the correction stage of the filter, whereas the prediction stage assumes that the control inputs $a_{v,k}$ and $a_{\omega,k}$ are known (in any case, it would not be possible to reconstruct them from the available measurements).