Autonomous and Mobile Robotics
Solution of Midterm Class Test, 2017/2018

Solution of Problem 1

1. Calling $\boldsymbol{g}_1 = (q_2 \quad -q_1 \quad 0)^T$ and $\boldsymbol{g}_2 = (q_3 \quad 0 \quad -q_1)^T$ the two input vector fields, their Lie Bracket is easily computed as

$$\boldsymbol{g}_3 = [\boldsymbol{g}_1, \boldsymbol{g}_2] = \begin{pmatrix} 0 \\ q_3 \\ -q_2 \end{pmatrix}$$

Since $\det(\boldsymbol{g}_1 \; \boldsymbol{g}_2 \; \boldsymbol{g}_3) = 0$, we can conclude that $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is linearly dependent on $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. This implies that all higher-order brackets are also linearly dependent on $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, as can be easily checked. The accessibility rank condition is then violated and the system is not controllable.

2. Multiplying the first equation by q_1 one obtains

$$q_1\dot{q}_1 - q_1 q_2 u_1 - q_1 q_3 u_2 = 0$$

Now use the second and third equation to rewrite the constraint in Pfaffian form:

$$q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = 0 \tag{1}$$

Based on the previous controllability study, this constraint must be holonomic. Indeed, it is easily integrable as

$$q_1^2 + q_2^2 + q_3^2 = c \tag{2}$$

where c is a non-negative integration constant which depends on the initial condition $\boldsymbol{q}_0$ (in particular, $c = q_{10}^2 + q_{20}^2 + q_{30}^2$).

3. From a global viewpoint, constraint (2) implies that the motion of the system in configuration space is constrained to take place on the sphere centered at the origin and passing through $\boldsymbol{q}_0$ (i.e., the particular sphere centered at the origin on which motion is started). From a local viewpoint, constraint (1) entails that the configuration space velocity $\dot{\boldsymbol{q}}$ at a certain configuration $\boldsymbol{q}$ is always contained in the plane which is tangent to the above-mentioned sphere at $\boldsymbol{q}$.

Solution of Problem 2

Solution A (systematic — works for any q_i, q_f)

1. Since we are considering a path planning problem, it is appropriate to use the geometric version of the kinematic model of the unicycle:

$$\begin{aligned} x' &= \cos\theta\,\tilde{v} \\ y' &= \sin\theta\,\tilde{v} \\ \theta' &= \tilde{\omega} \end{aligned}$$

The problem may be solved by exploiting the fact that x and y are flat outputs. In particular, we can choose $x(s)$ and $y(s)$ as 3rd-order polynomials in s, with $s \in [0,1]$. Setting $\tilde{v}(0) = \tilde{v}(1) = k$, the boundary conditions are written as

$$x(0) = x_i = 0 \qquad x(1) = x_f = 1 \qquad x'(0) = k\cos\theta_i = k \qquad x'(1) = k\cos\theta_f = 0$$

and

$$y(0) = y_i = 0 \qquad y(1) = y_f = 1 \qquad y'(0) = k\sin\theta_i = 0 \qquad x'(1) = k\sin\theta_f = k$$

where $k = \tilde{v}(0) = \tilde{v}(1)$. A straightforward computation provides

$$\begin{aligned} x(s) &= (k-2)s^3 + (3-2k)s^2 + ks \\ y(s) &= (k-2)s^3 + (3-k)s^2 \end{aligned}$$

so that

$$\begin{aligned} x'(s) &= 3(k-2)s^2 + 2(3-2k)s + k \\ y'(s) &= 3(k-2)s^2 + 2(3-k)s \end{aligned}$$

and

$$\begin{aligned} x''(s) &= 6(k-2)s + 2(3-2k) \\ y''(s) &= 6(k-2)s + 2(3-k) \end{aligned}$$

For example, set $k = 1$, which implies forward motion. The associated evolution of θ is then computed as

$$\theta(s) = \text{Atan2}(y'(s), x'(s)) \qquad (3)$$

while the geometric inputs on the path are

$$\begin{aligned} \tilde{v}(s) &= \sqrt{(x'(s))^2 + (y'(s))^2} && (4) \\ \tilde{\omega}(s) &= \frac{y''(s)x'(s) - y'(s)x''(s)}{(x'(s))^2 + (y'(s))^2} && (5) \end{aligned}$$

2. Once a timing law $s(t)$ has been chosen, the actual velocity inputs are obtained as $v = \tilde{v}\,\dot{s}$ and $\omega = \tilde{\omega}\,\dot{s}$. This suggests letting $s(t) = \alpha\, t$ (i.e., $\dot{s} = \alpha$), with the following simple procedure for choosing α:

   Step 1. Compute[1]
   $$\tilde{v}_{\max} = \max_{s\in[0,1]} |\tilde{v}(s)| \qquad \tilde{\omega}_{\max} = \max_{s\in[0,1]} |\tilde{\omega}(s)|$$

   Step 2. Let
   $$\eta = \max\left\{\frac{\tilde{v}_{\max}}{v_{\max}}, \frac{\tilde{\omega}_{\max}}{\omega_{\max}}\right\}$$

   Step 3. If $\eta \leq 1$, let $\alpha = 1$ ($\tilde{v}_{\max}$ and $\tilde{\omega}_{\max}$ are already inside the bounds and therefore $\dot{s} = 1$ is an admissible choice); else, let $\alpha = 1/\eta$.

   As a consequence, the time duration of the final trajectory will be 1 for $\alpha = 1$ (no time scaling) and η for $\alpha = 1/\eta$ (time scaling).

Solution B (clever — only works for specific q_i, q_f)

1. Since the initial configuration is the origin and the final configuration is $(1, 1, \pi/2)$, a feasible path between them is a simple arc of circle centered at $(0, 1)$ and having radius 1. We can obtain the parametric description of this path by taking the 4th-quadrant quarter of the unit circle centered at the origin and translating it so that its center is at $(0, 1)$. The resulting path is
   $$\begin{aligned} x(s) &= \cos\left(s + \frac{3}{2}\,\pi\right) = \sin s \\ y(s) &= 1 + \sin\left(s + \frac{3}{2}\,\pi\right) = 1 - \cos s \end{aligned}$$
   with $s \in [0, \pi/2]$. On this path, we obviously have $\theta = s$, $\tilde{v} = 1$ and $\tilde{\omega} = 1$. This can also be verified using the flatness-based reconstruction formulas (3) and (4-5).

2. Since $\tilde{v}_{\max}$ and $\tilde{\omega}_{\max}$ are exactly at the limit value, the choice $s = t$ is admissible and leads to both control velocities being saturated along the whole trajectory.

[1] In general, the determination of these maximum values can be carried out numerically, by computing the values of $|\tilde{v}(s)|$ and $|\tilde{\omega}(s)|$ over a fine discretization of the [0,1] interval.

Solution of Problem 3

1. The kinematic model of the robot is readily obtained as a simple dynamic extension of the classical model with velocity inputs:

$$
\begin{aligned}
\dot{x} &= \cos\theta\, v \;=\; r\cos\theta\cdot\frac{\omega_R+\omega_L}{2}\\
\dot{y} &= \sin\theta\, v \;=\; r\sin\theta\cdot\frac{\omega_R+\omega_L}{2}\\
\dot{\theta} &= \omega \;=\; r\cdot\frac{\omega_R-\omega_L}{d}\\
\dot{\omega}_R &= a_R\\
\dot{\omega}_L &= a_L
\end{aligned}
$$

   where r is the wheel radius and d is the distance between the centers of the wheels. Note that the wheel angular velocities ω_L, ω_R are state variables in this model, whereas the control inputs are the wheel angular accelerations a_L, a_R.

2. Using Euler integration, a discrete-time model is written as

$$
\begin{aligned}
x_{k+1} &= x_k + r\,T_s\cos\theta_k\cdot\frac{\omega_{R,k}+\omega_{L,k}}{2}\\
y_{k+1} &= y_k + r\,T_s\sin\theta_k\cdot\frac{\omega_{R,k}+\omega_{L,k}}{2}\\
\theta_{k+1} &= \theta_k + r\,T_s\cdot\frac{\omega_{R,k}-\omega_{L,k}}{d}\\
\omega_{R,k+1} &= \omega_{R,k} + T_s\,a_{R,k}\\
\omega_{L,k+1} &= \omega_{L,k} + T_s\,a_{L,k}
\end{aligned}
$$

   where T_s is the sampling interval.

3. Since the position of the landmark is unknown, we are dealing with a SLAM problem. Denote the extended state vector to be estimated as $\boldsymbol{\chi} = (x\;\,y\;\,\theta\;\,\omega_R\;\,\omega_L\;\,x_l\;\,y_l)^T$, where (x_l, y_l) are the Cartesian coordinates of the landmark. The nonlinear discrete-time model describing the motion of the extended robot+landmark system is then

$$
\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} r\,T_s\cos\theta_k\cdot\dfrac{\omega_{R,k}+\omega_{L,k}}{2}\\ r\,T_s\sin\theta_k\cdot\dfrac{\omega_{R,k}+\omega_{L,k}}{2}\\ r\,T_s\cdot\dfrac{\omega_{R,k}-\omega_{L,k}}{d}\\ T_s\,a_{R,k}\\ T_s\,a_{L,k}\\ 0\\ 0 \end{pmatrix} + \begin{pmatrix} v_{1,k}\\ v_{2,k}\\ v_{3,k}\\ v_{4,k}\\ v_{5,k}\\ 0\\ 0 \end{pmatrix}
$$

   where $v_{i,k}$ is a white gaussian noise with zero mean and covariance $V_{i,k}$ $(i = 1,\ldots,5)$.

As for the output $\boldsymbol{y}$, we have a total of three measurements coming from the sensors at each sampling instant, i.e., the linear and angular velocity of the robot and the relative bearing of the landmark:

$$\boldsymbol{y}_k = \begin{pmatrix} r \cdot \dfrac{\omega_{R,k} + \omega_{L,k}}{2} \\ r \cdot \dfrac{\omega_{R,k} - \omega_{L,k}}{d} \\ \text{Atan2}(y_l - y_k, x_l - x_k) - \theta_k \end{pmatrix} + \begin{pmatrix} \boldsymbol{w}_{1,k} \\ \boldsymbol{w}_{2,k} \\ \boldsymbol{w}_{3,k} \end{pmatrix}$$

where $\boldsymbol{w}_{i,k}$ is a white gaussian noise with zero mean and covariance $\boldsymbol{W}_{i,k}$ $(i = 1, \ldots, 3)$.

The rest of the solution is straightforward: linearize the motion and measurement models (note that the last four equations of the former and the first two of the latter are already linear) and then write the EKF equations.