Autonomous and Mobile Robotics
Solution of Class Test no. 2, 2010/2011

Solution of Problem 1

Let the robot configuration be $\boldsymbol{q} = (q_1, q_2)$, where q_1, q_2 are the relative joint angles. Define the following control points:

- $\boldsymbol{p}_1$, located on the first link at a distance $\alpha < 1$ from the first joint;
- $\boldsymbol{p}_2$, located on the second link at a distance $\beta < 1$ from the second joint;
- $\boldsymbol{p}_3$, the manipulator tip.

The required torque will be

$$\tau(\boldsymbol{q}) = -\boldsymbol{J}_1^T(\boldsymbol{q})\nabla U_r(\boldsymbol{p}_1(\boldsymbol{q})) - \boldsymbol{J}_2^T(\boldsymbol{q})\nabla U_r(\boldsymbol{p}_2(\boldsymbol{q})) - \boldsymbol{J}_3^T(\boldsymbol{q})\nabla U_r(\boldsymbol{p}_3(\boldsymbol{q})) - \boldsymbol{J}_3^T(\boldsymbol{q})\nabla U_a(\boldsymbol{p}_3(\boldsymbol{q})) \tag{1}$$

where:

- $\boldsymbol{J}_i(\boldsymbol{q})$, $i = 1, 2, 3$, is the Jacobian matrix of the forward kinematics function associated to the control point $\boldsymbol{p}_i$;
- $U_r(\boldsymbol{p}_i)$, $i = 1, 2, 3$, is the repulsive potential acting on the control point $\boldsymbol{p}_i$ due to the point obstacle;
- $U_a(\boldsymbol{p}_3)$ is the attractive potential acting on the control point $\boldsymbol{p}_3$ (the manipulator tip) due to the goal.

One easily obtains the coordinates of the control points

$$\boldsymbol{p}_1(\boldsymbol{q}) = (\alpha \cos q_1, \alpha \sin q_1) \tag{2}$$

$$\boldsymbol{p}_2(\boldsymbol{q}) = (\cos q_1 + \beta \cos(q_1 + q_2), \sin q_1 + \beta \sin(q_1 + q_2)) \tag{3}$$

$$\boldsymbol{p}_3(\boldsymbol{q}) = (\cos q_1 + \cos(q_1 + q_2), \sin q_1 + \sin(q_1 + q_2)) \tag{4}$$

and the corresponding Jacobian matrices to be used in (1):

$$\boldsymbol{J}_1(\boldsymbol{q}) = \begin{pmatrix} -\alpha \sin q_1 & 0 \\ \alpha \cos q_1 & 0 \end{pmatrix}$$

$$\boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} -\sin q_1 - \beta \sin(q_1 + q_2) & -\beta \sin(q_1 + q_2) \\ \cos q_1 + \beta \cos(q_1 + q_2) & \beta \cos(q_1 + q_2) \end{pmatrix}$$

$$\boldsymbol{J}_3(\boldsymbol{q}) = \begin{pmatrix} -\sin q_1 - \sin(q_1 + q_2) & -\sin(q_1 + q_2) \\ \cos q_1 + \cos(q_1 + q_2) & \cos(q_1 + q_2) \end{pmatrix}$$

Let us now compute the artificial force fields at the generic Cartesian point $\boldsymbol{p} = (x, y)$. The repulsive field ($\gamma = 2$) generated by the point obstacle at $(-1, 0)$ is

$$-\nabla U_r(\boldsymbol{p}) = \begin{cases} \dfrac{k_r}{\eta^2(x,y)} \left(\dfrac{1}{\eta_0} - \dfrac{1}{\eta(x,y)} \right) \nabla\eta(x,y) & \text{if } \eta(x,y) \le \eta_0 \\ \\ 0 & \text{if } \eta(x,y) > \eta_0 \end{cases} \tag{5}$$

where:

- k_r is a positive constant;
- η_0 is the obstacle range of influence, chosen to be smaller than 1 to preserve a global minimum at the assigned goal;
- $\eta(x,y) = \sqrt{(x-1)^2 + y^2}$ is the distance between $\boldsymbol{p}$ and the obstacle in $(1, 0)$, and $\nabla\eta(x,y)$ is its gradient, readily obtained as

$$\nabla\eta(x,y) = \begin{pmatrix} \dfrac{x-1}{\sqrt{(x-1)^2+y^2}} \\ \dfrac{y}{\sqrt{(x-1)^2+y^2}} \end{pmatrix}$$

With the goal at $(1, -1)$, the attractive field at $\boldsymbol{p}$ is

$$-\nabla U_a(\boldsymbol{p}) = k_a \begin{pmatrix} 1-x \\ -1-y \end{pmatrix} \tag{6}$$

where k_a is a positive constant.

At this point, one simply computes the repulsive fields at $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$, using (2–4) in (5), and the attractive field at $\boldsymbol{p}_3$, using (4) in (6), and finally plugs these expressions in (1).

Solution of Problem 2

Using Euler integration and including noise, a discrete-time nonlinear model of the UAV dynamics is derived as

$$\boldsymbol{q}_{k+1} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ \psi_{k+1} \\ \phi_{k+1} \end{pmatrix} = \begin{pmatrix} x_k + v_k\, T_s \cos\psi_k \\ y_k + v_k\, T_s \sin\psi_k \\ \psi_k - \dfrac{g}{v_k}\, T_s \tan\phi_k \\ \phi_k + u_{\phi,k}\, T_s \end{pmatrix} + \begin{pmatrix} w_{p1,k} \\ w_{p2,k} \\ w_{p3,k} \\ w_{p4,k} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}_k, \boldsymbol{u}_k) + \boldsymbol{w}_{p,k}$$

where T_s is the sampling interval, $\boldsymbol{u}_k = (v_k \;\; u_{\phi,k})^T$ is the input vector in $[t_k, t_{k+1}]$, and $\boldsymbol{w}_{p,k} = (w_{p1,k} \;\; \dots \;\; w_{p4,k})^T$ is a white gaussian process noise with zero mean and covariance matrix $\boldsymbol{W}_{p,k}$.

At the k-th step, the Cartesian coordinates of the sensor are exactly (x_k, y_k). The output equation, which represents a measure of the bearing angle between the sensor and each landmark, is therefore

$$\boldsymbol{z}_k = \begin{pmatrix} \text{atan2}(y_k - y_1, x_k - x_1) - \psi_k \\ \text{atan2}(y_k - y_2, x_k - x_2) - \psi_k \end{pmatrix} + \begin{pmatrix} w_{m1,k} \\ w_{m2,k} \end{pmatrix} = \boldsymbol{h}(\boldsymbol{q}_k) + \boldsymbol{w}_{m,k}$$

where $\boldsymbol{w}_{m,k} = (w_{m1,k} \;\; w_{m2,k})^T$ is a white gaussian measurement noise with zero mean and covariance matrix $\boldsymbol{W}_{m,k}$. Note that we have assumed for simplicity an identity association map, i.e., that the sensor readings are exactly in the same order of the landmarks.

The linearization of the process and output equations, respectively evaluated at the previous estimate $\hat{\boldsymbol{q}}_k$ and at the prediction $\hat{\boldsymbol{q}}_{k+1|k}$, gives

$$\boldsymbol{F}_k = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k = \hat{\boldsymbol{q}}_k} = \begin{pmatrix} 1 & 0 & -v_k T_s \sin\hat{\psi}_k & 0 \\ 0 & 1 & v_k T_s \cos\hat{\psi}_k & 0 \\ 0 & 0 & 1 & -\dfrac{g\,T_s}{v_k \cos^2\hat{\phi}_k} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\boldsymbol{H}_{k+1} = \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k = \hat{\boldsymbol{q}}_{k+1|k}} = \begin{pmatrix} \dfrac{-(\hat{y}_{k+1|k} - y_1)}{(\hat{x}_{k+1|k} - x_1)^2 + (\hat{y}_{k+1|k} - y_1)^2} & \dfrac{\hat{x}_{k+1|k} - x_1}{(\hat{x}_{k+1|k} - x_1)^2 + (\hat{y}_{k+1|k} - y_1)^2} & -1 & 0 \\ \dfrac{-(\hat{y}_{k+1|k} - y_2)}{(\hat{x}_{k+1|k} - x_2)^2 + (\hat{y}_{k+1|k} - y_2)^2} & \dfrac{\hat{x}_{k+1|k} - x_2}{(\hat{x}_{k+1|k} - x_2)^2 + (\hat{y}_{k+1|k} - y_2)^2} & -1 & 0 \end{pmatrix}$$

The EKF equations are therefore obtained as follows.

1. State and covariance prediction:

$$\begin{aligned} \hat{\boldsymbol{q}}_{k+1|k} &= \boldsymbol{f}(\hat{\boldsymbol{q}}_k, \boldsymbol{u}_k) \\ \boldsymbol{P}_{k+1|k} &= \boldsymbol{F}_k \boldsymbol{P}_k \boldsymbol{F}_k^T + \boldsymbol{W}_{p,k} \end{aligned}$$

2. Correction:

$$\begin{aligned} \hat{\boldsymbol{q}}_{k+1} &= \hat{\boldsymbol{q}}_{k+1|k} + \boldsymbol{R}_{k+1} \boldsymbol{\nu}_{k+1} \\ \boldsymbol{P}_{k+1} &= \boldsymbol{P}_{k+1|k} - \boldsymbol{R}_{k+1} \boldsymbol{H}_{k+1} \boldsymbol{P}_{k+1|k} \end{aligned}$$

where the innovation is

$$\boldsymbol{\nu}_{k+1} = \boldsymbol{z}_{k+1} - \begin{pmatrix} \text{atan2}(\hat{y}_{k+1|k} - y_1, \hat{x}_{k+1|k} - x_1) - \hat{\psi}_k \\ \text{atan2}(\hat{y}_{k+1|k} - y_2, \hat{x}_{k+1|k} - x_2) - \hat{\psi}_k \end{pmatrix}$$

and the Kalman gain matrix

$$\boldsymbol{R}_{k+1} = \boldsymbol{P}_{k+1|k} \boldsymbol{H}_{k+1}^T (\boldsymbol{H}_{k+1} \boldsymbol{P}_{k+1|k} \boldsymbol{H}_{k+1}^T + \boldsymbol{W}_{m,k+1})^{-1}$$

is a 4×2 matrix.

The covariance of the estimate $\boldsymbol{P}_k$ will be initialized at a certain value $\boldsymbol{P}_0$ reflecting the uncertainty on the initial estimate $\hat{q}_0$.